NOTAS DE FÍSICA

VOLUME X

Nº 16

ON BARGMANN-WIGNER THEORY FOR PARTICLES OF SPIN 2

by

Alberto Vidal and S. W. Mac Dowell

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

RIO DE JANEIRO

1963

Notas de Física - Volume X - Nº 16

ON BARGMANN-WIGNER THEORY FOR PARTICLES OF SPIN 2

Alberto Vidal
Centro Latinoamericano de Física
and Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brasil

S. W. Mac Dowell
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brasil

(Received April 24, 1963)

Introduction

The Bargmann-Wigner wave function for spin-2-particles (hereafter refered to as BW-wave function [1] is a 4th rank symmetric spinor with 35 independent components [2]. In this note, however, we introduce new wave functions in terms of which we develop a formulation of the BW-theory for the case of particles of spin 2. This formulation of the BW-theory enables us to compare the theories of Bargmann-Wigner and Fierz-Pauli [3] and show that for free particles they are equivalent.

BW-Wave Function

We proceed in the same way as in a previous paper [7] forming covariant tensors out of the product of the ten symmetric Dirac matrices and the BW-wave function. We obtain:[4]

$$\phi_{ij} = (C^{-1}\gamma_i)_{\alpha\beta}\,(C^{-1}\gamma_j)_{\sigma\eta}\,\psi_{\eta\sigma\beta\alpha} \tag{1}$$

$$F_{[ij]} = (C^{-1}\gamma_i\,\gamma_j)_{\alpha\beta}(C^{-1}\gamma_k)_{\sigma\eta}\,\psi_{\eta\sigma\beta\alpha} \tag{2.1}$$

$$F_{[ij][k\ell]} = (C^{-1}\gamma_i\,\gamma_j)_{\alpha\beta}\,(C^{-1}\gamma_k\,\gamma_\ell)_{\sigma\eta}\,\psi_{\eta\sigma\beta\alpha} \tag{2.2}$$

where ϕ_{ij} is a second rank symmetric tensor and $F_{[ij]k}$ and $F_{[ij][k\ell]}$ are a 3rd and 4th rank antisymmetric tensors with respect to the indices in brackets respectively. They satisfy the following independent relations:

$$g^{ik}\,F_{[ij]k} = 0 \tag{3.1}$$

$$g^{j\ell}\,g^{ik}\,F_{[ij][k\ell]} = 0 \tag{3.2}$$

$$\varepsilon^{ijk\ell}\,F_{[ij]k} = 0 \tag{3.3}$$

$$\varepsilon^{ijk\ell}\,F_{[ij][k\ell]} = 0 \tag{3.4}$$

$$\phi_{j\ell} = g^{ik}\,F_{[ij][k\ell]} \tag{4}$$

In the derivations of these conditions we have used the well known relation[5]

$$J_{\alpha\sigma}\,G_{\eta\beta} = \frac{1}{4}\sum_A \gamma^A_{\eta\sigma}(J\,\gamma^A\,G)_{\alpha\beta}$$

where $J_{\alpha\sigma}$ and $G_{\eta\beta}$ are arbitrary. The conditions (3) and (4) reduce to 35 the number of independent components of the tensors ϕ_{ij}, $F_{[ij]k}$, $F_{[ij][k\ell]}$. Because of (4) one can take the $F_{[ij]k}$ and $F_{[ij][k\ell]}$ subject to the subsidiary conditions (3) as an equivalent representation of the BW-wave function.

The inverse transformation of (1) and (2) is:

$$\psi_{\alpha\beta\sigma\eta} = \frac{S}{8}\Big[\phi_{ij}(\gamma^i C)_{\alpha\beta}(\gamma^j C)_{\sigma\eta} - F_{[ij]k}(\gamma^i\gamma^j C)_{\sigma\eta} +$$

$$+ \frac{1}{4} F_{[ij][k\ell]}(\gamma^i\gamma^j C)_{\alpha\beta}(\gamma^k\gamma^\ell C)_{\sigma\eta}\Big] \qquad (5)$$

where S is the symmetrization operator acting on the spinor indices. It is interesting to note that this is a generalization of the Leite Lopes wave function for particles of spin 1. [7]

BW-Wave Equations

Let us now obtain the force-free wave equations for the wave function defined by (2).

The BW-equations for spin-2-particles is:

$$(i\,\gamma^i_{\alpha'\alpha}\,\partial_i - m\,\delta_{\alpha\alpha'})\,\psi_{\alpha\beta\sigma\eta} = 0 \qquad (6)$$

Then from (6) and (2) one obtains the following two sets of equations:

$$F_{[ij]k} = -\frac{1}{m^2}\,\partial^\ell\Big[\partial_i F_{[\ell k]j} - \partial_j F_{[\ell k]i}\Big] \qquad (7.1)$$

$$(\Box + m^2)\,F_{[ij]k} = 0 \qquad (7.2)$$

$$F_{[ij][k\ell]} = \frac{i}{2m}\Big[\partial_\ell F_{[ij]k} - \partial_k F_{[ij]\ell} + \partial_j F_{[k\ell]i} - \partial_i F_{[k\ell]j}\Big] \qquad (7.3)$$

$$(\Box + m^2)\,F_{[ij][k\ell]} = 0 \qquad (8.1)$$

$$F_{[ij][k\ell]} = -\frac{1}{2m^2}\,\partial^r\Big[\partial_i F_{[rj][k\ell]} + \partial_j F_{[ir][k\ell]} + \partial_k F_{[ij][r\ell]} +$$

$$+ \partial_\ell F_{[ij][kr]}\Big] \qquad (8.2)$$

$$F_{[ij]k} = \frac{1}{m} \partial^{\ell} F_{[ij][k\ell]} \tag{8.3}$$

These two sets together with the supplementary conditions (3) are the tensor form of the BW-equations (6). One can show that equations (7.1, 2,3) and (3.1,3) may be obtained from (8.1, 2,3) and (3.2,4). Reciprocally the former set can be obtained from the later. Therefore the two sets are equivalente. Furthermore either set plus relation (4) implies equation (6) for the BW-wave function defined by (5). Thus the BW-theory contains two independent but equivalent theories for particles of spin 2.

Equivalence of the Bargmann-Wigner and Fierz-Pauli Equations

The Fierz-Pauli's equations for a particle of spin 2 are: [3]

$$\partial^{j} \phi_{j\ell} = 0 \tag{9.1}$$

$$(\Box + m^2) \phi_{j\ell} = 0 \tag{9.2}$$

where $\phi_{j\ell}$ is a traceless second rank symmetric tensor with 9 independent components, so that:

$$g^{j\ell} \phi_{j\ell} = 0 \tag{9.3}$$

One can verify that these equations may be obtained from equations (8) or (7) and the subsidiary conditions (3) if $\phi_{j\ell}$ is defined by (4). Thus the BW-equations imply the Fierz-Pauli's equations (9).

On the other hand, introducing (4) into (8) or (7) one obtains after some computation:

$$F_{[ij][k\ell]} = \frac{1}{m^2} (\partial_i \partial_\ell \phi_{jk} - \partial_j \partial_\ell \phi_{ik} + \partial_k \partial_j \phi_{i\ell} - \partial_k \partial_i \phi_{j\ell}) \tag{10}$$

$$F_{[ij]k} = -\frac{1}{m} (\partial_i \phi_{jk} - \partial_j \phi_{ik}) \tag{11}$$

which in Fierz-Pauli's formalism may be considered as definitions of $F_{[ij][k\ell]}$ and $F_{[ij]k}$ in terms of $\phi_{j\ell}$. But $F_{[ij]k}$ and $F_{[ij][k\ell]}$ as defined by (11) and (12) satisfy the equations (7) and (8) and the subsidiary conditions (3) if $\phi_{j\ell}$ satisfies the equations (9).

Therefore, we conclude that the Bargmann-Wigner and Fierz-Pauli equations are equivalent for free particles.

* * *

Acknowledgements

The authors are very greatful to Professor J. Leite Lopes and to Professor J. Tiomno for useful discussions.

* * *

REFERENCES

1. V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. 34, 211 (1948).

2. We denote the spinor and vector indices by greek and italic letters respectively.

3. M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A173, 211 (1939).

4. For the γ-matrices the Schweber-Bethe-de Hoffmann's notation, Mesons and Fields, vol. I (1956), is used.
The C-matrix is such that

$$C^T = - C \qquad (T = \text{transpose})$$

and

$$C^{-1} \gamma_m C = - \gamma_m^T$$

The metric is

$$g_{ik} = - \delta_{ik} \qquad (i, k = 1, 2, 3)$$

$$g_{oo} = 1$$

5. H. Umezawa, Quantum Field Theory, page 41-2, North-Holland Publishing Co., 1956.

6. J. Leite Lopes, Lectures on Relativistic Wave Equations, Monografias de Física, Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, page 70 (1960).

7. C. G. Oliveira and A. Vidal, BW-theory for spin-3/2-particles, Notas de Física IX, 225 (1962).

* * *