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ON MANDELSTAM'S PROGRAM IN POTENTIAL SCATTERING

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ABSTRACT Mandelstam's program for constructing the scattering amplitude from its analytic properties and unitarity is analysed in the case of non-relativistic scattering by a cut-off potential or by a hard sphere. The asymptotic behaviour of the scattering amplitude in the momentum transfer plane is obtained, leading to a double dispersion representation for the amplitude. The usefulness of this representation is limited by an essential singularity at infinity in the momentum transfer plane. An infinite system of dispersion relations, connecting each partial wave with all succeeding ones, is derived from the dispersion relation for fixed momentum transfer. The partial-wave amplitudes must be constructed from this system together with the unitarity condition. Possible ambiguities in the solution of this problem are investigated. It is shown that ambiguities in the exact solution affecting only a finite number of partial waves (Castillejo, Dalitz and Dyson ambiguities) do not exist. They would arise, however, in approximate solutions and it would be very hard, in practice, to eliminate them from the exact solution. The ambiguities can be formulated in terms of the positions of the poles of the S-matrix. A series of sum rules which must be fulfilled by the poles is derived. The solution of the system is investigated in the particular case of scattering by a hard sphere. In this case, if one assumes that the exact solution is known for angular momenta larger than some (arbitrarily given) value, each partial-wave dispersion relation for smaller values of the angular momentum can be exactly solved, and it follows from the sum rules that the solution is unique.

I. Introduction

Since the beginning of the work on dispersion relations, two essentially different standpoints have been taken concerning their applications. The first one is to regard them as consistency relations, which can be applied to test whether the general principles from which they have been shown or are expected to follow are compatible with the experimental data. They could then be considered as broad restrictions, which might be fulfilled by a large class of physical theories, rather than leading to a unique theory. The other and much more ambitious standpoint is to regard dispersion relations, combined with unitarity, as the basis for a complete theory of strong interactions. They are then treated as integral equations which can be solved, in principle, in terms of a small number of fundamental constants. This program has been proposed by Mandelstam (1), together with a conjecture on the validity of the double dispersion relations upon which the program is based.

Assuming the correctness of this conjecture, it is still very difficult to test the program in elementary-particle physics, on account of the complexity of possible intermediate states. In practice, the approximation of neglecting multi-particle intermediate states is always applied. It is far from clear to what extent the results derived by means of this approximation are to be trusted and regarded as an effective test of the validity of the program.

In view of this difficulty, several investigations have been made in connection with simpler models, specially for non-relativistic potential scattering. The only case which has been considered so far is that of a superposition of Yukawa potentials:

$$r\,V(r) = \int_m^\infty \sigma(\mu)\ \exp(-\mu r)\,dr\ . \qquad (1.1)$$

The validity of Mandelstam's representation in this case has been proved by several authors (2, 3, 4). According to Blankenbecler, Goldberger, Khuri and Treiman (3), the scattering amplitude can be uniquely constructed, in principle, given the Mandelstam representation (including Born's approximation and the energies of the bound states) and the unitarity condition. However, there remained the problem of determining the number of subtractions, which is related with the behaviour of the scattering amplitude at infinity in the momentum transfer plane.

This behaviour was investigated by Regge and collaborators (2, 5, 6), who have shown, with some further restrictions on the potential, that it is given by a (generally complex) power of the momentum transfer. The power depends on the energy, but upper bounds for it were determined and related with the number of bound states and resonances.

The actual construction of the scattering amplitude has not been carried out. The weight function of the representation is to be determined as the limit of a sequence of polynomials in the coupling constant, but the rate of convergence of this sequence is unknown. Moreover, it would be very hard to test the result, because the behaviour of the solution is not well known for this class of potentials.

The analytic properties of scattering amplitudes are simplest and most fully known in the case of a cut-off potential, i.e. a potential which vanishes identically beyond a certain radius. There are several examples, belonging to this class, in which the exact partial wave amplitudes are known and have a simple analytic form. A cut-off potential, however, cannot be represented as a superposition of Yukawa potentials. The analytic behaviour of the amplitudes is entirely different in the two cases. It may appear strange that such a radical difference should exist, for one would expect that the effect of introducing a cut-off in the potential at arbitrarily large distances should be very small. However, arguments of this kind cannot be applied to analytic continuation.

The purpose of the present work is to investigate Mandelstam's program in the case of a cut-off potential. We shall also consider a singular limiting case of such a potential, namely, a hard sphere.

The analytic properties of the scattering amplitude which are required for deriving double dispersion relations are almost completely known for cut-off potentials. The only exception is the behaviour at infinity in the momentum transfer plane, which will be derived in Section II. It is given by an energy-dependent essential singularity, such that a double dispersion representation is not valid in the usual sense, but only as a limit, in which interchange of the order of the integrals is not allowed.

However, in the case of a cut-off potential, the double dispersion representation is not required: one can apply directly the partial wave expansion and the dispersion relation for fixed momentum transfer. This relation will be employed in Section III, where we shall project the partial waves out of the total scattering amplitude. The result is an infinite system of dispersion relations, coupling each partial wave to all succeeding ones.

In Section IV, we shall investigate to what extent this system, together with the unitarity condition, suffices to determine the solution. It will be shown that there are no solutions differing from the physical solution by a finite number of partial waves only. Ambiguities involving an infinite number of partial waves cannot be excluded, but they would be much harder to construct, in view of the complicated form taken by the unitarity condition.

The possible ambiguities arising in each partial wave dispersion relation can be expressed in terms of the positions of the poles of the corresponding S-funtion. However, these positions are not completely arbitrary: they must fulfill a series of sum rules, which will be derived in Section V.

In Section VI, we shall consider in more detail the case of a hard sphere. It will be shown that, in this case, if the exact partial wave amplitudes are given beyond a certain value of the angular momentum (no matter how large) all lower-order partial waves can be uniquely determined from the system of dispersion relations, with the help of the sum rules.

II. The double dispersion representation

A. Summary of known results

Let $V(r)$ be a potential which vanishes for $r > a$ and satisfies the condition

$$\int_0^a r|V(r)|dr < \infty . \tag{2.1}$$

Let $f(k,\tau)$ be the scattering amplitude, expressed as a function of the wave number k and the momentum transfer τ. The following properties of f are known in this case:

(a) $f(k, \tau)$ is an analytic function of both variables, regular in the topological product of the τ plane and the upper half of the k plane, except for a finite number of simple poles on the imaginary axis, $k = i\kappa_n$, which are associated with the bound states.

(b) For fixed τ, and $|k| \longrightarrow \infty$ in the upper half-plane,

$$f(k, \tau) \longrightarrow f_B(\tau), \tag{2.2}$$

where

$$f_B(\tau) = -\frac{m}{2\pi h^2} \int \exp(-i\vec{\tau}\cdot\vec{r})\, V(r) d\vec{r} \tag{2.3}$$

is Born's approximation ($\tau = |\vec{\tau}|$).

Properties (a) and (b), not including the analyticity in τ, have been proved by Khuri (7) and Klein and Zemach (8). The analyticity in τ follows, as a limiting case, from the results of (3) and from the work of Hunziker (9).

Property (a) has also been proved under more general assumptions about the scatterer (10), which apply, in particular, to the case of a hard sphere. In this case, however, property (b) must be replaced by

$$(b') \; f(k, \tau) = O(k) \tag{2.4}$$

for $|k| \longrightarrow \infty$ in the upper half-plane.

It follows from (a) and (b) that, in the case of a cut-off potential, $f(k, \tau)$ satisfies the following dispersion relation for fixed momentum transfer (7):

$$f(k,\tau) = f_B(\tau) + \sum_n \frac{\Gamma_n(\tau)}{k^2 + \kappa_n^2} + \frac{2}{\pi}\int_0^\infty \frac{g(k',\tau)}{k'^2 - k^2}\, k'\, dk' \quad (\Im m\, k > 0), \tag{2.5}$$

where $\Gamma_n(\tau)$ is a polynomial in τ of degree ℓ_n, the angular momentum of the nth bound state, and

$$g(k',\tau) = \frac{1}{2i}\left[f(k',\tau) - f(-k',\tau)\right]. \qquad (2.6)$$

The relation (2.5) is valid for real or complex values of τ. For real τ, the symmetry relation $f(-k',\tau) = f^*(k',\tau)$ implies

$$g(k',\tau) = \mathcal{I}m\, f(k',\tau) \qquad \text{(real } \tau). \qquad (2.7)$$

In the hard sphere case, the following dispersion relation follows from (a) and (b'):

$$f(k,\tau) = f(0,\tau) + \frac{2}{\pi}k^2 \int_0^\infty \frac{g(k',\tau)}{k'(k'^2-k^2)}\,dk' \quad (\mathcal{I}m\, k > 0). \qquad (2.8)$$

It was shown in (10), by direct summation of the partial wave expansion, that

$$f(0,\tau) = - a\cos(\tau a), \qquad (2.9)$$

where a is the radius of the sphere.

In order to derive a double dispersion representation for $f(k,\tau)$, we must still determine the behaviour of $g(k,\tau)$ for fixed, real k and $|\tau| \longrightarrow \infty$. Unfortunately, this is a completely unphysical limit, so that it is hard to foretell the result from physical arguments.

For a cut-off potential, one might expect that (2.2) would remain valid. This would imply

$$f(k, \tau) = \underline{O}\left[\exp(-i\,\tau a)\right] \qquad (2.10)$$

for $|\tau| \longrightarrow \infty$ in the upper half-plane.

For the hard sphere, the behaviour of $f(k, \theta)$, the scattering amplitude expressed as a function of k and the scattering angle θ, is known for fixed, physical θ and $|k| \longrightarrow \infty$ in the upper half-plane. According to classical causality arguments, which can be applied in this case, we have, under these conditions (11),

$$f(k, \theta) = \underline{O}\left[\exp(-2\,ik\,a\,\sin\frac{\theta}{2})\right]. \qquad (2.11)$$

Since $\tau = 2k\sin\frac{\theta}{2}$, this leads again to (2.10), for $|\tau| \longrightarrow \infty$ and $|k| \longrightarrow \infty$ simultaneously, in such a way that the above relation between τ and k is preserved. One might expect this result to remain valid for fixed k and $|\tau| \longrightarrow \infty$; this seems to be confirmed by (2.9).

However, we shall see that (2.10) is not the right answer for real τ, although it will turn out to be correct when $|\tau| \longrightarrow \infty$ in the upper half-plane, in directions away from the real axis.

In order to determine the behaviour of $g(k, \tau)$ for $|\tau| \longrightarrow \infty$, we shall employ the partial wave expansion

$$f(k,\tau) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2\,ik}\left[S_\ell(k) - 1\right] P_\ell\left(1 - \frac{\tau^2}{2k^2}\right), \qquad (2.12)$$

where $S_\ell(k)$ is the S-function for the ℓth partial wave and P_ℓ is the ℓth Legendre polynomial. It follows from (2.6), (2.12) and the unitarity condition that

$$g(k,\tau) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4k} \left|S_\ell(k) - 1\right|^2 P_\ell \left(1 - \frac{\tau^2}{2k^2}\right). \qquad (2.13)$$

The behaviour of P_ℓ for $|\tau| \to \infty$ follows from (12,I,p. 189)

$$P_\ell(z) \approx (2\ell-1)!! \quad z^\ell/\ell! \; (|z| \to \infty; \; \ell = 0, 1, 2, \ldots), \qquad (2.14)$$

where $(2\ell-1)!! = 1.3.5\ldots(2\ell-1)$. According to (2.14), (2.13) behaves like a power series in τ for $|\tau| \to \infty$. It then follows from the theory of entire functions (13) that the behaviour of $g(k,\tau)$ for $|\tau| \to \infty$ is determined by the behaviour of the terms of the partial wave expansion for $\ell \to \infty$.

We shall consider first the case of a hard sphere, and then we shall extend the results to cut-off potentials.

B. The hard sphere

In this case, we have

$$S_\ell(k) - 1 = -2\, j_\ell(ka)/h_\ell^{(1)}(ka), \qquad (2.15)$$

where j_ℓ and $h_\ell^{(1)}$ are the spherical Bessel function and the spherical Hankel function of the first kind, respectively, of order ℓ.

The behaviour of $g(k,\tau)$ for $|\tau| \longrightarrow \infty$, derived from (2.13) and (2.15), is given by Eq. (A13) of Appendix A. It corresponds to an energy-dependent essential singularity which is quite different from (2.10), although (2.10) is correct away from the real axis, as will be shown in Appendix A.

Let us now apply these results to derive a dispersion relation for g. It is convenient, for this purpose, to consider it as a function of the variables $s = k^2$, $t = \tau^2$. For fixed $s > 0$, the function

$$G(s,t) = \exp\left[-2\, e^{-i\pi/4}\, a(st)^{1/4}\right] g(s,t) \qquad (2.16)$$

is analytic in the t plane cut from 0 to ∞ ($0 \leqslant \arg t < 2\pi$) and, according to (A13),

$$G(s,t) = \underline{O}(t^{-\frac{1}{2}}) \qquad (|t| \longrightarrow \infty). \qquad (2.17)$$

Taking into account (2.7), this leads to the dispersion relation

$$G(s,t) = \frac{1}{\pi}\int_0^\infty \exp\left[-a(4st')^{1/4}\right] \sin\left[a(4st')^{1/4}\right] \mathscr{I}m\, f(s,t')\, \frac{dt'}{t'-t}$$

$$(\mathscr{I}m\, t \neq 0). \qquad (2.18)$$

Since we already know that both sides of (2.8) are analytic in the topological product of the cut s and t planes, we can now substitute (2.18) in the right-hand side of (2.8) to get a double dispersion representation for $f(s,t)$.

It must be remembered, however, that the asymptotic behaviour (A13), which led to (2.18), is valid for fixed a and $|t| \to \infty$, whereas the values of $s' = k'^2$ in the integral (2.8) range from 0 to ∞. The limit for $s \to \infty$, $|t| \to \infty$, depends on the manner in which both variables approach their limiting values. Thus, in order to substitute (2.18) in (2.8), we must interpret the integral in (2.8) as the limit of an integral with finite upper limit of integration. Taking into account (2.9), we finally get

$$f(s,t) = -a\cos(at^{\frac{1}{2}}) + s \lim_{\sigma \to \infty} \int_0^{\sigma} \frac{ds'}{\pi} \frac{\exp[2e^{-i\pi/4}a(s't)^{1/4}]}{s'(s'-s)} \cdot$$

$$\cdot \int_0^{\infty} \frac{dt'}{\pi} \frac{\exp[-a(4s't')^{1/4}] \sin[a(4s't')^{1/4}]}{(t'-t)} \mathcal{I}m\, f(s', t')$$

$$(\mathcal{I}m\, s \neq 0, \quad \mathcal{I}m\, t \neq 0) \quad (2.19)$$

This is the double dispersion representation for $f(s,t)$ in the hard sphere case. It differs from the usual kind of double dispersion relations by the exponential factors which are required to remove the essential singularity at $|t| \to \infty$ and by the fact that the integral in s' must be understood as a limit. Interchange of the order of integration after proceeding to the limit $\sigma \rightarrow \infty$ is not allowed. This is due to the non-uniformity of the asymptotic behaviour in t for $s \rightarrow \infty$.

C. Extension to a cut-off potential

In order to extend the above results to the case of a cut-off potential, it suffices to determine the behaviour of $S_\ell(k) - 1$ for $\ell \rightarrow \infty$. If the potential is sufficiently regular, the centrifugal term in the radial equation must predominate in this limit, so that $S_\ell(k) - 1$ should tend to Born's approximation:

$$S_\ell(k) - 1 \approx - 2\, ik \int_0^a r^2\, j_\ell^2(kr)\, U(r)\, dr, \qquad (2.20)$$

where $U(r) = 2m\, V(r)/\hbar^2$. It has been shown by Carter (14) that the absolute value of the right-hand side of (2.20) is always an upper bound to the left-hand side for large enough ℓ. We shall restrict ourselves to potentials for which (2.20) is valid for large ℓ.

If we take also $\ell \gg (ka)^2$, (2.20) becomes

$$S_\ell(k) - 1 \approx - \frac{2\, ik^{2\ell+1}}{[(2\ell+1)!!]^2} \int_0^a r^{2\ell+2}\, U(r)\, dr \; . \qquad (2.21)$$

Let $U^{(m)}(a-0)$ be the first non-vanishing derivative of $U(r)$ at $r = a - 0$, with $U^{(0)}(a-0) = U(a-0)$. Then, by repeated partial integration, (2.21) becomes

$$S_\ell(k) - 1 \approx (-1)^{m+1} \frac{2i\, a^{m+2}\, U^{(m)}(a-0)(ka)^{2\ell+1}}{\left[(2\ell+1)!!\right]^2 (2\ell+3)(2\ell+4)\ldots(2\ell+3+m)} \,. \quad (2.22)$$

This result can easily be checked in the case of a rectangular potential.

Comparing (2.22) with (A3) and (A4), we see that the only difference in $|S_\ell - 1|^2$ is a slowly-varying factor

$$\frac{a^{2m+4}|U^{(m)}(a-0)|^2}{(2\lambda+1)^2\,(2\lambda+3)^2\,(2\lambda+4)^2\ldots(2\lambda+3+m)^2}$$

which must be incorporated to the integrand of (A6). Taking this factor at the saddle points (A12), we find, in the place of (A13),

$$g(k,\tau) \approx \frac{ka}{2^{5/2}}\,|U^{(m)}(a-0)|^2 \left\{ \frac{\exp\left[2a(ik\tau)^{\frac{1}{2}}\right]}{(ik\tau)^{m+\frac{5}{2}}} + \frac{\exp\left[2a(-ik\tau)^{\frac{1}{2}}\right]}{(-ik\tau)^{m+\frac{5}{2}}} \right\} \quad (|\tau| \to \infty) \,. \quad (2.23)$$

Taking into account (2.5), we get from (2.23), in complete analogy with (2.19), the double dispersion representation.

$$f(s,t) = f_B(t) + \sum_n \frac{\Gamma_n(t)}{s+s_n} + \lim_{\sigma\to\infty} \int_0^{\sigma} \frac{ds'}{\pi} \frac{\exp\left[2e^{-i\pi/4}\, a(s't)^{1/4}\right]}{(s'-s)} \cdot$$

$$\cdot \int_0^{\infty} \frac{dt'}{\pi} \frac{\exp\left[-a(4s't')^{1/4}\right] \sin\left[a(4s't')^{1/4}\right]}{(t'-t)} \quad \mathcal{I}m\, f(s', t')$$

$$(\mathcal{I}m\, s \neq 0, \;\; \mathcal{I}m\, t \neq 0). \quad (2.24)$$

The remark made in connection with (2.19) on the non-interchangea bility of the order of integration also applies to (2.24).

The radius of the potential does not appear in the dispersion relation for fixed momentum transfer (2.5). This relation holds also for potentials of the type (1.1). In this case, however, $f(k,\tau)$ has branch cuts in the τ plane running from $2im$ to $i\infty$ and from $-2im$ to $-i\infty$. The effect of cutting off the potential is to remove these branch cuts to infinity, so that one gets an entire function. The radius of the potential reappers in the behaviour at infinity, where there is an essential singularity, given by (2.23). This behaviour also depends on how smoothly the potential approaches zero at $r = a$ (this gives the value of m in (2.23)).

In contrast with the case of a superposition of Yukawa potentials, there is no subtraction problem for a cut-off potential: (2.24) is valid irrespective of the strength of the potential. However, on account of the exponential factors and the corresponding lack of freedom in the order of integration, it is not so useful as Mandelstam's representation.

The unitarity condition takes the form (3)

$$\mathcal{I}m\, f(k,\tau) = \frac{k}{4\pi}\int f^*(k,|\vec{k}'-\vec{k}''|)\, f(k,|\vec{k}''-\vec{k}|)\, d\Omega'' \,, \qquad (2.25)$$

where $\tau = |\vec{k}' - \vec{k}|$ and $\vec{k}'^2 = \vec{k}''^2 = k^2$. According to (2.7), this condition can be rewritten as follows:

$$g(k, \cos\theta) = \frac{k}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\pi} f(-k, \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\varphi')\,.$$

$$\cdot f(k, \cos\theta') \sin\theta' \, d\theta' \,, \qquad (2.26)$$

where $\cos\theta = 1 - (\tau^2/2\,k^2)$. The unitarity condition has physical significance only for real values of k and for $-1 \leqslant \cos\theta \leqslant 1$. However, for real k, both sides of (2.26) are analytic functions of $\cos\theta$, so that (2.26) remains valid for arbitrary values of $\cos\theta$ or τ by analytic continuation.

If we replace f by (2.19) or (2.24), taken for real values of τ, in the right-hand side of (2.26), we get, according to (2.7), an integral equation for $\mathcal{I}m\, f(k,\tau)$. However, the iterative solution proposed by Mandelstam and applied to the case of a superposition of Yukawa potentials (3) cannot be employed here, because the weight function differs from zero in the entire first quadrant of the (s,t) plane.

III. The infinite system of partial wave dispersion relations

The unitarity condition takes its simplest form for the individual partial waves. For this reason, the procedure usually followed in the applications of Mandelstam's program is to project the partial waves out of the dispersion relation for the total amplitude.

For this purpose, we shall apply the dispersion relation for fixed momentum transfer (2.5) or (2.8), together with (2.13). For potentials of the type (1.1), the partial wave expansion (2.13) converges only for $\tau^2 \leqslant 4\,m^2$. For a cut-off potential or in the hard sphere case, however, we can apply it for all values of τ.

The partial wave amplitude $f_\ell\,(k)$ is given by

$$f_\ell(k) = \frac{S_\ell(k)-1}{2ik} = \frac{1}{2}\int_0^\pi f(k,\tau = 2k\sin\frac{\theta}{2})P_\ell(\cos\theta)\sin\theta\,d\theta. \tag{3.1}$$

Substituting f by (2.5) or (2.8) and taking into account (2.13), we encounter, in both cases, the integral

$$C_{\ell\ell'}(x) = \frac{1}{2}(2\ell'+1)\int_0^{\pi} P_{\ell'}(1-x+x\cos\theta)\,P_{\ell}(\cos\theta)\sin\theta\,d\theta. \quad (3.2)$$

This integral is evaluated in Appendix B. The results are given by Eqs. (B3), (B5) and (B8).

According to (2.9) and (3.1), we also encounter the integral

$$\frac{1}{2}\int_0^{\pi}\cos(2ka\sin\frac{\theta}{2})\,P_{\ell}(\cos\theta)\sin\theta\,d\theta = \frac{d}{dk}\left[k\,j_{\ell}^2(ka)\right], \qquad (3.3)$$

which has been evaluated by differentiating the Clebsch-Heine expansion (12, II, p. 316).

Taking into account (3.3) and the results of Appendix B, we get from (2.8), (2.9), (2.13) and (3.1), in the hard sphere case,

$$\mathcal{R}e\, f_{\ell}(k) = -\frac{d}{dk}\left[ka\,j_{\ell}^2(ka)\right] + \frac{k^{2\ell+2}}{\pi}\,P\int_{-\infty}^{\infty}\frac{\mathcal{I}m\, f_{\ell}(k')}{k'^{2\ell+2}(k'-k)}\,dk' +$$

$$+\frac{k^{2\ell+2}}{\pi}\sum_{m=1}^{\infty}\sum_{s=0}^{m-1}(-1)^s\,(\ell;m;s)\,k^{2s}\int_{-\infty}^{\infty}\frac{\mathcal{I}m\, f_{\ell+m}(k')}{k'^{2\ell+2s+3}}\,dk'$$

$$(\ell = 0, 1, 2, \ldots), \quad (3.4)$$

where P denotes Cauchy's principal value and the numerical coefficients $(\ell;\, m;\, s)$ are defined in (B9).

Similarly, in the case of a cut-off potential, let $i\rho_{\ell n}$ be the residue of $S_\ell(k)$ at the pole $i\kappa_{\ell n}$, corresponding to the nth bound state of angular momentum ℓ ($\rho_{\ell n}$ is real). Then, according to (2.5) and (2.12),

$$\Gamma_{\ell n}(\tau) = \text{residue } f(k,\tau)\Big|_{k=i\kappa_{\ell n}} = (2\ell+1)\,\rho_{\ell n}\, P_\ell\left(1+\frac{\tau^2}{2\kappa_{\ell n}^2}\right). \tag{3.5}$$

Taking into account (3.5), we get from (2.5), in complete analogy with (3.4),

$$\mathcal{R}e\, f_\ell(k) = f_{\ell B}(k) + (-1)^\ell k^{2\ell} \sum_n \frac{\rho_{\ell n}}{\kappa_n^{2\ell}(k^2+\kappa_{\ell n}^2)} + \frac{k^{2\ell}}{\pi} P\int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_\ell(k')}{k'^{2\ell}(k'-k)}\,dk' +$$

$$+ (-1)^\ell k^{2\ell} \sum_{m\geq 1} \sum_n \frac{\rho_{\ell+m,n}}{\kappa_{\ell+m,n}^{2\ell+2}} \sum_{s=0}^{m-1} (\ell;m;s)\, k^{2s}/\kappa_{\ell+m,n}^{2s} +$$

$$+ \frac{k^{2\ell}}{\pi} \sum_{m=1}^{\infty} \sum_{s=0}^{m-1} (-1)^s (\ell;m;s)\, k^{2s} \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_{\ell+m}(k')}{k'^{2\ell+2s+1}}\,dk', \tag{3.6}$$

where the sums over m and n in the fourth term are both finite sums (the total number of bound states is finite), and

$$f_{\ell B}(k) = -\frac{2m}{\hbar^2} \int_0^a V(r)\, j_\ell^2(kr)\, r^2\, dr \tag{3.7}$$

is Born's approximation.

Let us write explicitly the first few terms in the first few equations of (3.6) (assuming for simplicity that there are no bound states):

$$\mathcal{R}e\, f_0(k) = f_{0B}(k) + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_0(k')}{k'-k}\, dk' + \frac{1}{\pi}\Bigg[3\int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_1(k')}{k'} dk' + 5\int_{-\infty}^{\infty}\left(1-2\frac{k^2}{k'^2}\right)\frac{\mathcal{I}m\, f_2(k')}{k'} dk' + \dots \Bigg],$$

$$\mathcal{R}e\, f_1(k) = f_{1B}(k) + \frac{k^2}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_1(k')}{k'^2(k'-k)}\, dk' + \frac{k^2}{\pi}\Bigg[5\int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_2(k')}{k'^3} dk' + 7\int_{-\infty}^{\infty}\left(2-3\frac{k^2}{k'^2}\right)\frac{\mathcal{I}m\, f_3(k')}{k'^3} dk' + \dots \Bigg],$$

$$\mathcal{R}e\, f_2(k) = f_{2B}(k) + \frac{k^4}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_2(k')}{k'^4(k'-k)}\, dk' + \frac{k^4}{\pi}\Bigg[7\int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_3(k')}{k'^5} dk' + \dots \Bigg],$$

. . . .

This infinite system of partial wave dispersion relations was first considered in the relativistic case by MacDowell (15). It was also mentioned by Goldberger (16) in the non-relativistic case.

The relations (3.4) or (3.6) couple the real part of each partial wave amplitude with the imaginary part of the same amplitude and the amplitudes of all subsequent partial waves, so that we have a "triangular" system of equations. The coupling to higher-order partial waves appears in each equation through a series of polynomials in k^2.

In addition to the system of equations (3.4) or (3.6), the partial wave amplitudes must satisfy the unitarity condition,

$$\mathcal{I}m\, f_\ell(k) = k|f_\ell(k)|^2 . \tag{3.8}$$

Notice that both (3.4) and (3.6) automatically give the correct low-energy behaviour of the amplitudes:

$\mathcal{R}e\, f_\ell = \underline{O}(k^{2\ell})$, $\mathcal{I}m\, f_\ell = \underline{O}(k^{4\ell+1})$ for $k \to 0$ (the latter follows from the former and (3.8)).

The dispersion relation (3.6) can be rewritten in a more transparent way by introducing the function

$$g_\ell(k) = f_\ell(k)/k^{2\ell} = \left[S_\ell(k) - 1\right]/(2ik^{2\ell+1}), \tag{3.9}$$

which is regular at the origin. According to well-known properties of the $\underline{S}$-matrix for a cut-off potential, g_ℓ is a meromorphic function of k. It follows from the work of Humblet (17) that, for $|k| \to \infty$ on the real axis,

$$S_\ell(k) - 1 = \underline{O}(k^{-1}) \tag{3.10}$$

and, for $|k| \to \infty$ in the lower half-plane,

$$S_\ell(k) \approx (-1)^\ell \, (2k/i)^{m+2} \exp(-2ika)/U^{(m)}(a-0), \tag{3.11}$$

where $U^{(m)}(a-0)$ is the quantity that appears in (2.22).

Let $k_{\ell n}$ be the poles of $S_\ell(k)$ in the lower half-plane, which are symmetrically placed with respect to the imaginary axis. It was shown by Humblet (17, pp. 45, 71) that $\mathcal{R}e\, k_{\ell n} = O(n)$ and $\mathcal{I}m\, k_{\ell n} = O(\log n)$ for large n. Let

$$R_{\ell n} = \text{residue } g_\ell(k)\Big|_{k = k_{\ell n}}$$

The following dispersion relation is then verified by $g_\ell(k)$:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, g_\ell(k')}{k' - k}\, dk' = - \mathcal{R}e\, g_\ell(k) + 2\, \mathcal{R}e \sum_n \frac{R_{\ell n}}{k - k_{\ell n}}, \tag{3.12}$$

where the sum is extended over all the poles in the lower half-plane, taken in the order of increasing modulus. This relation, which generalizes a result due to Lee (18), is obtained by considering the integral on the left-hand side taken over a sequence of contours closed by half-circles passing halfway between the poles in the lower half-plane. It follows from (3.9), (3.10) and (3.11) that the integrals over half-circles tend to zero when their radii tend to infinity, leading to (3.12).

According to (3.9) and (3.12), (3.6) can be rewritten as follows:

$$\mathcal{R}e\left[g_\ell(k)-\sum_n \frac{r_{\ell n}}{k-i\kappa_{\ell n}}-\sum_n \frac{R_{\ell n}}{k-k_{\ell n}}\right]-\frac{1}{2}\ g_{\ell B}(k)=$$

$$=\frac{1}{2}(-1)^\ell \sum_{m\geq 1}\sum_n \frac{\rho_{\ell+m,n}}{\kappa^2_{\ell+m,n}}\sum_{s=0}^{m-1}(\ell;m;s)\,k^{2s}/\kappa^{2s}_{\ell+m,n}+$$

$$+\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{s=0}^{m-1}(-1)^s\,(\ell;m;s)\,k^{2s}\int_{-\infty}^{\infty}k'^{\,2m-2s-1}\,\mathcal{I}m\ g_{\ell+m}(k')dk', \tag{3.13}$$

where

$$r_{\ell n}=\text{residue } g_\ell(k)\Big|_{k=i\kappa_{\ell n}}=(-1)^\ell\ \rho_{\ell n}/(2i\kappa^{2\ell+1}_{\ell n})\ .$$

The expression within square brackets in the left-hand side of (3.13) is the entire part of the meromorphic function $g_\ell(k)$, and $g_{\ell B}(k)=f_{\ell B}(k)/k^{2\ell}$ is, according to (3.7), an entire function of k. Thus, the right-hand side of (3.13) is the expansion of an entire function, which explains why it is valid for all k, in contrast with the case of a superposition of Yukawa potentials.

According to Humblet (17, p. 53), the entire part of $f_\ell(k)$ is associated with potential scattering, whereas the pole terms are associated with resonance scattering. Thus, we see from (3.13) that the coupling of each partial wave to the higher-order ones is related with potential scattering of that partial wave.

IV. Ambiguities in the solution

A dispersion relation involving a single partial wave is usually verified by a large class of functions, rather than having a unique solution. This was first shown by Castillejo, Dalitz and Dyson (19), in connection with Low's equation for meson-nucleon scattering. The resulting ambiguities are known as CDD ambiguities.

In the present problem, the partial wave amplitudes must verify the infinite system (3.4) or (3.6), in which each partial wave is coupled to all the others, together with the unitarity condition (3.8). The total amplitude, given by (2.12), must fulfill conditions (a) and (b) (or (b')) of Section IIA and have the correct behaviour at infinity in the Υ plane.

The question which will now be investigated is: are these properties sufficient to determine $f(k,\Upsilon)$ or do additional solutions exist, besides the physical one ? If they do, Mandelstam's program cannot be carried out in this form unless supplementary conditions are given to select the physical solution.

We shall restrict ourselves, for simplicity, to the hard sphere case and to the case of potentials without bound states. The extension of the results to include bound states is straight forward. With this restriction, it follows from condition (a) and (3.1) that

(i) $S_\ell(k)$ is a regular analytic function in the upper half-plane.

According to (3.1), the behaviour of $S_\ell(k)$ for $|k| \longrightarrow \infty$ in the upper half-plane depends on the behaviour of $f(k, \tau = 2k \sin \frac{\theta}{2})$ for $|k| \longrightarrow \infty$. In the hard sphere case, this behaviour is given by (2.11), so that we get

$$S_\ell(k) \exp(2ika) = \underline{O}(k)$$

for $|k| \longrightarrow \infty$ in the upper half-plane. Since $|S_\ell(k) \exp(2ika)| = 1$ on the real axis, it then follows from the Phragmén-Lindelöf theorem (20) that

(ii) $|S_\ell(k) \exp(2ika)| \leqslant 1$ in the upper half-plane.

This result is also known to be true for a cut-off potential without bound states (17, 21).

The unitarity condition

(iii) $S_\ell(k)\, S_\ell^*(k) = 1$ (real k)

enables us to extend the definition of $S_\ell(k)$ to the lower half-plane, with the help of the Schwarz reflection principle, by (22)

$$S_\ell(k^*) = \left[S_\ell^*(k)\right]^{-1} . \tag{4.1}$$

According to (4.1), the only possible singularities of S_ℓ in the lower half-plane are poles, corresponding to the zeros in the upper half-plane, so that

(iv) $S_\ell(k)$ is a meromorphic function.

The symmetry relation $f(-k, -\tau) = f^*(k,\tau)$ (real k), together with (3.1), implies

(v) $S_\ell(-k) = S_\ell^*(k)$ (real k).

Conditions (i) to (v) are well-known properties of the S-function in the present problem. Let us now introduce Wigner's R-function (23)

$$R_\ell(k^2) = \frac{1}{ik}\left[\frac{S_{\ell a}(k)-1}{S_{\ell a}(k)+1}\right], \tag{4.2}$$

where

$$S_{\ell a}(k) = (-1)^\ell \exp(2ika)\, S_\ell(k) . \tag{4.3}$$

It was shown by Van Kampen (22, 24) that (i) to (v) imply the following properties of the R-function:

(i') R_ℓ is a meromorphic function of k^2;
(ii') R_ℓ is real for real values of k^2;
(iii') all the poles of R_ℓ lie on the real axis;
(iv') $\Im m\, R_\ell$ has the same sign as $\Im m\, k^2$;
(v') the Mittag-Leffler expansion of R_ℓ is

$$R_\ell(k^2) = \sum_n \frac{\gamma_n^2}{\mu_n - k^2}, \tag{4.4}$$

where γ_n and μ_n are real and the poles μ_n cannot have an accumulation point at finite distance.

It follows from (v') that any ambiguities in the solution can be expressed in terms of changes in the parameters γ_n, μ_n, just like the CDD ambiguities (19).

Let $f(k,\tau)$ be the physical solution, and let us assume that there exists another solution $f'(k,\tau)$ differing from f in a single partial wave, S_ℓ being replaced by S_ℓ' and R_ℓ by R_ℓ'. Then, according to (2.12), (4.2) and (4.3),

$$f'(k,\tau) - f(k,\tau) = \frac{1}{2}(-1)^\ell (2\ell+1) \frac{(S_{\ell a}+1)^2 (R_\ell' - R_\ell)}{\left[2 - ik(S_{\ell a}+1)(R_\ell' - R_\ell)\right]} \cdot$$

$$\cdot \; P_\ell\left(1 - \frac{\tau^2}{2k^2}\right) \exp(-2ika). \quad (4.5)$$

It follows from (ii), (4.2), (4.3) and (4.4) that the right-hand side of (4.5) has an essential singularity of the type $\exp(-2ika)$ for $|k| \to \infty$ in the upper half-plane. Thus, $f'(k,\tau)$ violates condition (b) or (b') of Section IIA, so that it is not an acceptable solution. The same is obviously true for any function differing from f only by a finite number of partial waves.

Thus, <u>it is not possible to construct an extra solution by modifying any finite number of partial waves: there are no CDD ambiguities</u>. This contradicts a result due to Barut and Ruei (25). Their argument, however, is incorrect[1]. The unitarity condition played an important role in the derivation of this result, by allowing us to extend the definition of $S_\ell(k)$ to the lower half-plane. The analytic properties of $f(k,\tau)$ as a function of both variables were also required in the derivation of (i). However, the behaviour for $|\tau| \rightarrow \infty$ entered only in the form (2.11), i.e. when $|k| \rightarrow \infty$ simultaneously.

At least in the hard sphere case, the physical reason for the absence of CDD ambiguities can be explained by causality considerations. In fact, in this case, the amplitude is identical to that for scattering of a classical massless scalar field by a totally reflecting sphere, so that signals with sharp fronts can be built.

The exponential factor exp(-2ika), which dominates the behaviour of $S_\ell(k)$ in the upper half-plane, represents the phase advancement of a spherical multipole wave upon reflection at the surface of the scatterer. This factor cannot appear in the forward scattering amplitude because it would lead to instantaneous transmission of signals across the sphere. It can be shown that the same condition also prevents the appearance of this factor for fixed non-zero momentum transfer (10).

Thus, although each term of the partial wave expansion (2.12) blows up exponentially for $|k| \rightarrow \infty$ in the upper half-plane, the phases of the partial waves are coupled by causality in such a way that the full amplitude has at most a linear divergence. This result remains true for non-relativistic particles, although the classical causality condition can no longer be applied in this case.

It is not possible to modify a finite number of partial waves without destroying the phase relationships that are responsible for the elimination of the exponential factor. This explains why CDD ambiguities cannot exist.

Possible ambiguities, if any, must therefore involve modification of an infinite number of partial waves. Partial wave analysis is then no longer of any use: one can work directly with the total amplitude. However, any extra solutions must satisfy the unitarity condition in the form (2.26), and it is very hard to see how they could be constructed.

The above arguments allow us to eliminate CDD ambiguities in principle, but not in practice. In fact, one would have to sum the whole series of partial waves in order to make sure of the cancellation of the exponential factor in the asymptotic behaviour in the upper half-plane, and this would be extremely difficult by analytical means. The same applies, *a fortiori*, to approximate solutions.

V. Sum rules for the poles of the S-matrix

It has been shown by Van Kampen (22) that a function $S_\ell(k)$ satisfying conditions (i) to (v) of Section IV can be represented by a canonical product expansion

$$S_\ell(k) = \pm \exp(-2ik\alpha) \prod_n \frac{(k_n + k)}{(k_n - k)},$$

where k_n are the poles of $S_\ell(k)$, taken in order of increasing modulus, and $\alpha \leqslant a$. In the present case, according to (3.4) and (3.6), we must take the + sign, because $S_\ell(0) = 1$. Moreover, it has been shown by Regge (26) that $\alpha = a$ for a cut-off potential. The same is true for the hard sphere (27). Thus,

$$S_\ell(k) = \exp(-2ika) \prod_n \frac{(k_n + k)}{(k_n - k)}. \qquad (5.1)$$

This expansion is a counterpart of (4.4), and the CDD ambiguities can also be expressed in terms of the positions of the poles k_n, instead of the parameters (γ_n, μ_n). However, in addition to conditions (i) to (v), the infinite system of partial wave dispersion relations also imposes a condition on the low-energy behaviour of S_ℓ(cf. (3.1), (3.4) and (3.6)):

$$S_\ell(k) - 1 = iC_\ell(ka)^{2\ell+1} + \underline{O}(k^{2\ell+2}) \qquad (k \rightarrow 0), \qquad (5.2)$$

where C_ℓ is a real constant. According to (3.4),

$$C_\ell = -\frac{2}{(2\ell-1)!!\ (2\ell+1)!!} \qquad (5.3)$$

in the hard sphere case.

It will now be shown that (5.2) gives rise to additional restrictions on the positions of the poles k_n. In the first place, by taking the logarithmic derivative of (5.1) at $k = 0$, we get

$$\sum_n \frac{1}{k_n} = i\,a + \frac{1}{2}\; S'_\ell(0) = i\,a \left(1 + \frac{1}{2}\, C_\ell\, \delta_{\ell,o}\right), \qquad (5.4)$$

where $\delta_{\ell,o}$ is Kronecker's delta and the last equality follows from (5.2). The "sum rule" (5.4) was derived by Van Kampen (22).

Now let us take $\ell \geqslant 1$ and let us consider the integral

$$I = \int_C \log S_\ell(k)\; \frac{dk}{k^{2p+2}} \quad (1 \leqslant p \leqslant \ell), \qquad (5.5)$$

where C is a contour consisting of the real axis, from $-R$ to $-\varepsilon$ and from ε to R, a half-circle γ of radius $\varepsilon(\varepsilon \longrightarrow 0)$ and another one Γ of radius $R(R \longrightarrow \infty)$, centered at the origin (figure 1).

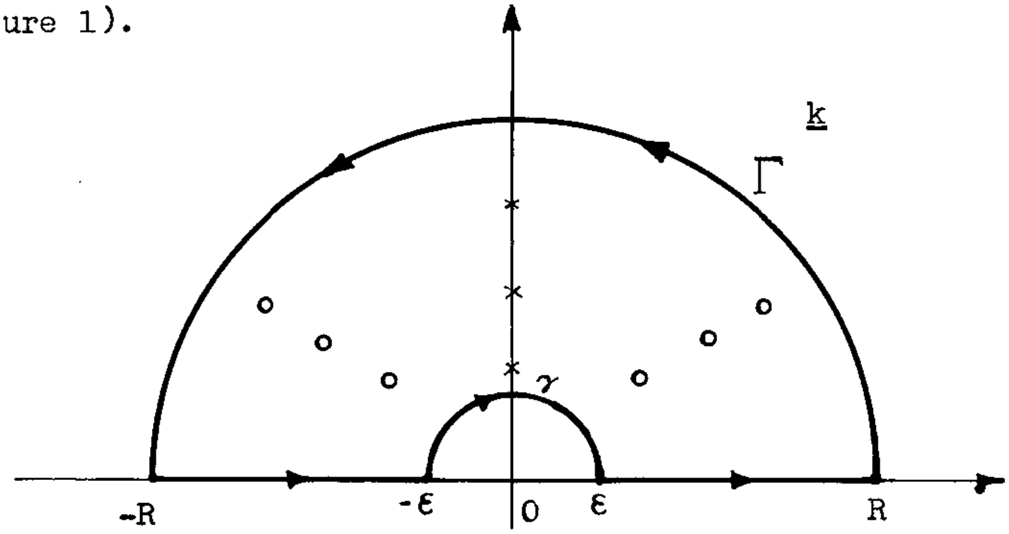

Fig. 1. Contour of integration in the k plane.
××× Poles of $S_\ell(k)$; o o o Zeros of $S_\ell(k)$.

We have

$$\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} = \int_{\varepsilon}^{R} \log \left[S_\ell(-k)\, S_\ell(k) \right] \frac{dk}{k^{2p+2}} = 0 \,, \tag{5.6}$$

because $S_\ell(-k)\, S_\ell(k) = 1$. It follows from property (ii) of section IV that

$$\int_{\Gamma} = \underline{O}\,(R^{-2p}) \longrightarrow 0 \qquad \text{for } R \longrightarrow \infty \,. \tag{5.7}$$

Finally, according to (5.2) and (5.5),

$$\lim_{\varepsilon \to 0} \int_{\gamma} = \pi\, C_\ell\, a^{2\ell+1}\, \delta_{p,\ell} \,. \tag{5.8}$$

It follows from (5.6), (5.7) and (5.8) that

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} I = \pi\, C_\ell\, a^{2\ell+1}\, \delta_{p,\ell} \,. \tag{5.9}$$

On the other hand, by partial integration,

$$I = -\Delta_C \left[\frac{\log S_\ell(k)}{(2p+1)\, k^{2p+1}} \right] + \frac{1}{(2p+1)} \int_C \frac{S_\ell'(k)}{S_\ell(k)} \frac{dk}{k^{2p+1}} \,, \tag{5.10}$$

where $\Delta_C f$ denotes the variation of f round the contour C.

According to a well-known formula (28), we have

$$\Delta_C \left[\log S_\ell(k)/k^{2p+1}\right] = 2\pi i\ (N-n)/R^{2p+1} ,$$

$$\int_C \frac{S_\ell'(k)}{S_\ell(k)} \frac{dk}{k^{2p+1}} = 2\pi i \left[\sum_j \frac{1}{k_j'^{2p+1}} - \sum_m \frac{1}{(i\kappa_m)^{2p+1}}\right] ,$$

where N and n are, respectively, the number of zeros and the number of poles of $S_\ell(k)$ contained within the contour C, k_n' are the zeros and $i\kappa_m$ are the poles in the upper half-plane (bound states). According to Humblet (17, pp. 45, 71), $N = O(R)$ and, according to (2.1), $n = O(1)$ for $R \to \infty$, so that the first term of (5.10) vanishes in this limit. On the other hand, to each zero k_j' in the upper half-plane corresponds a pole $k_j = - k_j'$ in the lower half-plane, so that we finally get

$$\lim_{\substack{\epsilon \to 0 \\ R \to \infty}} I = - \frac{2\pi i}{(2p+1)} \sum_n \frac{1}{k_n^{2p+1}} , \tag{5.11}$$

where the summation is extended over all the poles of $S_\ell(k)$, both in the upper and in the lower half-plane.

Equating (5.9) to (5.11), we get

$$\sum_n \frac{1}{k_n^{2p+1}} = \frac{1}{2}(2p+1)\ i\ a^{2\ell+1}\ C_\ell\ \delta_{p,\ell} = \frac{1}{2} \lim_{k \to 0}\left[k^{-2\ell} S_\ell'(k)\right]\delta_{p,\ell}$$

$$(p = 1, 2, \ldots, \ell). \tag{5.12}$$

According to (5.4) and (5.12), we have, for $\ell = 0$,

$$\sum_n \frac{1}{k_n} = i\,a + \frac{1}{2} S_o'(0) = i\,a \left(1 + \frac{C_o}{2} \right) \qquad (5.13)$$

and, for $\ell \geq 1$,

$$\sum_n \frac{1}{k_n} = i\,a\,, \qquad (5.14)$$

$$\sum_n \frac{1}{k_n^{2p+1}} = 0 \qquad (p = 1, 2, \ldots \ell - 1)\,, \qquad (5.15)$$

$$\sum_n \frac{1}{k_n^{2\ell+1}} = \frac{1}{2} \lim_{k \to 0} \left[\frac{S_\ell'(k)}{k^{2\ell}} \right] = \frac{1}{2} (2\ell + 1)\, i\, a^{2\ell+1}\, C_\ell . \qquad (5.16)$$

These relations, which give the sums of the inverses of odd powers of the poles of the S-matrix, will be called sum rules. For the ℓth partial wave, $\ell + 1$ sum rules must be fulfilled. Any modifications in the positions of the poles (CDD ambiguities) must be compatible with these rules.

VI. The hard sphere case

We have seen in Section III that the infinite system of partial wave dispersion relations is a "triangular" system. This suggests trying to solve it backwards, starting at some very high value of ℓ to find the asymptotic form of the solution for large ℓ, and then going back step by step to lower values of ℓ. For each ℓ, one then has to solve a partial wave dispersion relation with known inhomogeneous term. Since f_ℓ should tend to Born's approximation $f_{\ell B}$ for $\ell \to \infty$, one could take as first approximation, for sufficiently large ℓ,

$$\mathcal{R}e\, f_\ell \approx f_{\ell B}\ , \qquad \mathcal{I}m\, f_\ell \approx 0\ ,$$

which would effectively reduce (3.6) to a finite system. However, one would encounter CDD ambiguities at each step of the solution.

In order to investigate further this "backwards" method of solution, let us now consider the hard sphere case, assuming that the exact solution of (3.4) is known for $\ell > \ell_0$. Substituting the solution in the right-hand side of (3.4) for $\ell = \ell_0$, one gets an equation of the form

$$\mathcal{R}e\, f_\ell(k) = \frac{k^{2\ell+2}}{\pi}\, P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, f_\ell(k')}{k'^{2\ell+2}(k'-k)}\, dk' + F_\ell(k), \qquad (6.1)$$

where $F_\ell(k)$ must be computed by summing the series of polynomials in the right-hand side of (3.4). We shall assume that this summation has also been carried out. In order to compute the result, let us consider the exact solution (2.15). We shall denote the corresponding partial wave amplitude by

$$\tilde{f}_\ell(k) = \frac{\tilde{S}_\ell(k)-1}{2ik} = \frac{i j_\ell(ka)}{k\, h_\ell^{(1)}(ka)} \,. \tag{6.2}$$

By the same method which led to (3.12), we find [2]

$$-\frac{k^{2\ell+2}}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\, \tilde{f}_\ell(k')}{k'^{2\ell+2}(k'-k)}\, dk' = \mathcal{R}e\, \tilde{f}_\ell(k) -$$

$$- 2\, \mathcal{R}e \sum_n \left(\frac{k}{\tilde{k}_n}\right)^{2\ell+2} \frac{\rho_n}{(k-\tilde{k}_n)} - \frac{1}{2} C_\ell\, k^{2\ell}\, a^{2\ell+1}, \tag{6.3}$$

where C_ℓ is given by (5.3), $\tilde{k}_n$ are the poles of $\tilde{f}_\ell$ in the lower half-plane, and

$$\rho_n = \text{residue } \tilde{f}_\ell(k)\Big|_{k=\tilde{k}_n} \,. \tag{6.4}$$

The poles are the roots of: $h_\ell^{(1)}(ka) = 0$. There are exactly ℓ poles (27). It follows from (6.1) and (6.3) that

$$F_\ell(k) = 2\, \mathcal{R}e\left[\tilde{f}_\ell(k) - \sum_{n=1}^{\ell} \left(\frac{k}{\tilde{k}_n}\right)^{2\ell+2} \frac{\rho_n}{(k-\tilde{k}_n)}\right] - \frac{1}{2} C_\ell\, k^{2\ell}\, a^{2\ell+1}. \tag{6.5}$$

Let us consider the function

$$g_\ell(k) = \left[f_\ell(k) - F_\ell(k)\right]/k^{2\ell+2} = \left[f_\ell(k) - \tilde{f}_\ell(k)\right]/k^{2\ell+2} \;+$$

$$+ \left\{ \frac{i j_\ell(ka)}{k^{2\ell+3} h_\ell^{(2)}(ka)} + \sum_{n=1}^{\ell} \frac{\rho_n}{\tilde{k}_n^{2\ell+2}(k-\tilde{k}_n)} + \right.$$

$$\left. + \sum_{n=1}^{\ell} \frac{\rho_n^*}{(\tilde{k}_n^*)^{2\ell+2}(k-\tilde{k}_n^*)} + \frac{c_\ell}{2k^2} a^{2\ell+1} \right\} . \qquad (6.6)$$

In terms of this function, (6.1) can be rewritten as follows:

$$\mathcal{R}e\ g_\ell(k) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}m\ g_\ell(k')}{k'-k} dk' . \qquad (6.7)$$

According to Titchmarsh's theorem (29), (6.7) implies that $g_\ell(k)$ has an analytic continuation which is regular in the upper half plane and tends to zero for $|k| \longrightarrow \infty$. It is readily seen that the expression within curly brackets in (6.6) also has these properties. Thus, the same must be true for the remaining term $\left[S_\ell(k) - \tilde{S}_\ell(k)\right]/(2ik^{2\ell+3})$.

The function

$$\Delta_\ell(k) = S_\ell(k) - \tilde{S}_\ell(k) = S_\ell(k) - \exp(-2ika) \prod_{n=1}^{\ell} \frac{(\tilde{k}_n + k)}{(\tilde{k}_n - k)} \qquad (6.8)$$

is therefore regular in the upper half-plane and $\mathcal{O}(k^{2\ell+3})$ for $|k| \longrightarrow \infty$. Since $|\Delta_\ell(k)| \leqslant 2$ on the real axis, it follows from the Phragmén-Lindelöf theorem (20) that

$$|\Delta_\ell(k)| \leqslant 2 \quad \text{in the upper half-plane.} \tag{6.9}$$

Thus, according to (6.8),

$$S_\ell(k) = (-1)^\ell \exp(-2ika)\left[1 + \underline{O}(k^{-1})\right] + \underline{O}(1) \tag{6.10}$$

for $|k| \longrightarrow \infty$ in the upper half-plane.

It follows from the above results that $S_\ell(k)$ satisfies conditions (i) to (v) of Section IV. Furthermore, its behavior for $k \longrightarrow 0$ is given by (5.2) and (5.3). According to Section V, this implies

$$S_\ell(k) = \exp(-2ika) \prod_n \frac{(k_n + k)}{(k_n - k)}, \tag{6.11}$$

where the poles k_n in the lower half-plane must verify the sum rules (5.13) to (5.16).

It follows from (6.10) and from the unitarity condition on the real axis, $|S_\ell(k)| = 1$, that there cannot be an accumulation point of zeros of $S_\ell(k)$ at infinity in the upper half-plane. Since the poles in the lower half-plane cannot have an accumulation point at finite distance (22), the total number of poles k_n must be finite: we shall call it m.

This implies that $k_1, k_2, \ldots, k_m$ are roots of an algebraic equation of degree m. Let us write this equation in the form

$$a_m k^m + a_{m-1} k^{m-1} + \ldots + a_1 k + 1 = 0 . \tag{6.12}$$

Then, the inverses of the poles, $x_1 = 1/k_1, \ldots, x_m = 1/k_m$, are roots of

$$x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m = 0 , \qquad (6.13)$$

and (5.13) to (5.16) give the sums of the first $\ell + 1$ odd powers of the roots of (6.13).

It is well known that the coefficients of an algebraic equation of degree m can be expressed in terms of the sums of the first m powers of the roots. It was shown by Vahlen (30) that they can also be expressed in terms of the sums of the first m odd powers.

Let

$$S_k = \sum_{n=1}^{m} x_n^k . \qquad (6.14)$$

Then, according to Vahlen,

$$\frac{a_1 Z + a_3 Z^3 + \ldots}{1 + a_2 Z^2 + \ldots} = - \tanh \left[\sum_{n=1}^{m} \tanh^{-1} (x_n Z) \right] =$$

$$= - \tanh \left(S_1 Z + S_3 \frac{Z^3}{3} + S_5 \frac{Z^5}{5} + \ldots \right) . \qquad (6.15)$$

The first member of (6.15) is the ratio of the odd-power terms of (6.12) to the even-power ones, i.e. it is a rational fraction of order m. This implies that the continued fraction expansion of the last member,

$$- \tanh\left(S_1 Z + S_3 \frac{Z^3}{3} + S_5 \frac{Z^5}{5} + \ldots \right) = \cfrac{C_o Z}{1 + \cfrac{C_1 Z^2}{1 + \cfrac{C_2 Z^2}{1 + \ddots \cfrac{}{1 + C_{m-1} Z^2}}}} \qquad (6.16)$$

terminates with the term $C_{m-1} Z^2$.

By identifying the coefficients of the power-series expansions of the first and second members of (6.16), one can express C_k as a rational function of S_1, S_3, ..., S_{2k+1}. For instance,

$$C_o = - S_1 ,$$

$$C_1 = \frac{S_1^3 - S_3}{3 S_1} .$$

On the other hand, by identifying the first member of (6.15) with the second member of (6.16), one can express $a_1, a_2, \ldots, a_m$ in terms of C_o, C_1, ..., C_{m-1}. In this way one gets expressions for a_1, a_2, ..., a_m as rational functions of S_1, S_2, ..., S_{2m-1}.

Any symmetric function of the roots, in particular S_{2m+1}, is therefore a rational function of S_1, S_2, ..., S_{2m-1}:

$$S_{2m+1} = \frac{Q_m(S_1, S_3, \ldots, S_{2m-1})}{P_m(S_1, S_3, \ldots, S_{2m-1})} , \qquad (6.17)$$

where P_m and Q_m are rational entire functions without a common factor. The function

$$R_{m+1}(S_1, S_3, \ldots, S_{2m+1}) = S_{2m+1} P_m(S_1, S_3, \ldots, S_{2m-1}) - \\ - Q_m(S_1, S_3, \ldots, S_{2m-1}) \qquad (6.18)$$

is therefore an irreducible entire function of $S_1, S_3, \ldots S_{2m+1}$, which vanishes if $S_1, S_3, \ldots, S_{2m+1}$ are sums of odd powers of the roots of an algebraic equation of degree m. To each value of m there corresponds a (uniquely defined) function R_{m+1}.

It was shown by Vahlen (30) that

$$C_k = \frac{1}{(2k-1)(2k+1)} \frac{R_{k+1} R_{k-2}}{R_k R_{k-1}} . \qquad (6.19)$$

Now let us apply these results to the sum rules (5.13) to (5.16). These relations are fulfilled by the poles $\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_\ell$ of (6.2), which are roots of an algebraic equation of degree ℓ. According to the above, the first ℓ sum rules suffice to determine the coefficients of this equation, and therefore its roots $\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_\ell$. The last sum rule, which gives the value of $S_{2\ell+1}$, must therefore be a consequence of the first ℓ rules, so that, according to (6.18), we must have

$$R_{\ell+1} = 0 . \qquad (6.20)$$

It follows from (6.19) and (6.20) that

$$C_\ell = 0 \qquad (6.21)$$

in (6.16). Therefore, the equation

$$\cfrac{C_o Z}{1+\cfrac{C_1 Z^2}{1+\ddots\cfrac{}{1+C_{\ell-1} Z^2}}} = -1 \qquad (6.22)$$

is an algebraic equation of degree ℓ which, according to (6.15) and (6.16), must follow from the original equation of degree m (6.12):

$$\frac{a_1 Z + a_3 Z^3 + \ldots}{1 + a_2 Z^2 + \ldots} = -1 \,. \qquad (6.23)$$

This leads to the following alternative possibilities:

(A) $m = \ell$, so that $k_n = \tilde{k}_n$.

(B) $m > \ell$. In this case, the numerator and denominator of (6.23) must have a common factor, which is a polynomial in Z^2, and can be factored out in the left-hand side of (6.12). Then, in addition to the poles $\tilde{k}_1, \ldots, \tilde{k}_\ell$, there would exist pairs of equal and opposite poles $(k_n, - k_n)$. Clearly, such pairs would not alter the value of $S_1, S_3, \ldots, S_{2\ell+1}$.

Alternative (B), however, is excluded by the fact that $S_\ell(k)$ cannot have any poles in the upper half-plane. Thus, (A) must be valid and, according to (6.8) and (6.11), this implies

$$S_\ell(k) = \tilde{S}_\ell(k) \ . \tag{6.24}$$

The solution of (6.1) is therefore unique and it is given by (6.2). Substituting the result in (3.4) for $\ell = \ell_o - 1$, we get another equation of the type (6.1); the same procedure can therefore be applied to all remaining equations of the system.

Thus, if we know the exact solution of (3.4) for $\ell > \ell_o$, the solution for $\ell \leqslant \ell_o$ is uniquely determined and follows from the sum rules.

VII. Conclusion

Although a double dispersion relation in the usual sense does not exist for a cut-off potential, we have seen that one can obtain an infinite system of coupled partial wave dispersion relations by projecting the partial waves out of the dispersion relation for fixed momentum transfer.

It is generally meaningless to speak of the solution of a dispersion relation involving a single partial wave, because there is a wide class of possible solutions. However, in the case of the infinite system, the analyticity requirements in both variables for the full amplitude (including the behaviour at infinity), together with the unitarity condition, imply the non-existence of alternative solutions differing from the physi cal one only by a finite number of partial waves. On the other hand, it seems very hard to modify an infinite number of partial waves in a way compatible with the unitarity condition.

If one considers an isolated dispersion relation taken from the infinite system, its solution still involves CDD ambiguities, which can be expressed as ambiguities in the position of the poles of the S-matrix in the lower half of the k-plane. However, the low-energy behaviour of the amplitude, which follows from the dispersion relation, leads to a series of sum rules which must be fulfilled by the poles.

In the particular case of a hard sphere, if one assumes that the exact solution is known for angular momenta larger than some (arbitrarily given) value, the remaining finite system of partial wave dispersion relations can be explicitly solved with the help of the sum rules, and the solution is unique. This is probably due to the specially simple structure of the S-matrix in this case: there is a finite number of poles for each value of the angular momentum, whereas the number of poles is infinite in the general case of a cut-off potential.

The above results suggest that Mandelstam's program can lead, in principle, to a unique solution in the present case. In practice, however, the elimination of ambiguities is an extremely difficult problem, and it seems to be practically impossible when approximation methods are employed.

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Appendix A

Asymptotic behaviour of $g(k,\tau)$ for $|\tau| \to \infty$.

In order to determine the behaviour of the scattering amplitude for $|\tau| \to \infty$, Regge (2) applied Watson's transformation (31) to the partial wave expansion (2.12), reducing it to the integral

$$f(k,\tau) = \int_C \frac{(2\lambda+1)}{4k} \left[S_\lambda(k) - 1\right] P_\lambda\left(\frac{\tau^2}{2k^2} - 1 \right) \frac{d\lambda}{\sin(\pi\lambda)}, \qquad (A1)$$

where C is the contour shown in figure 2. The contour is then deformed onto the imaginary axis. In this process, it sweeps across the poles of $S_\lambda(k)$, which are located in the first quadrant. For a suitably restricted class of potentials of the type (1.1), Regge showed that the number of poles is finite.

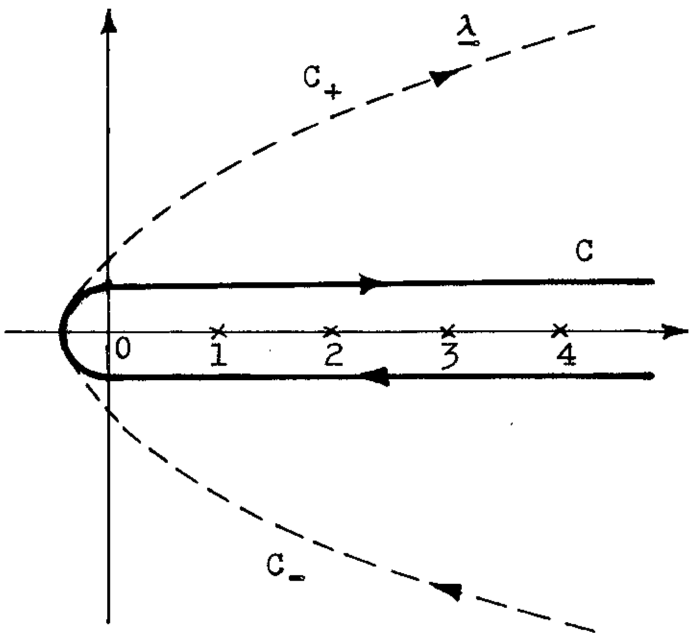

Fig. 2. Contour of integration in the λ plane.

The behaviour of (A1) for $|\tau| \longrightarrow \infty$ is then determined by the residue of the integrand at the pole having the largest real part.

In the case of a hard sphere (2.15), this method cannot be employed, because $S_\lambda(k)$ has an infinite set of poles, which are the zeros of $h_\lambda^{(1)}(ka)$ (31). We shall therefore follow a different procedure, which combines Watson's transformation with the saddle-point method.

It follows from (2.15) that

$$|S_\ell(k) - 1|^2 = \frac{4\, j_\ell^2(ka)}{j_\ell^2(ka) + n_\ell^2(ka)}, \qquad (A2)$$

where n_ℓ is the spherical Neumann function of order ℓ. For $\ell \gg (ka)^2$, we can apply the expansions (32)

$$\left[\frac{j_\ell(ka)}{n_\ell(ka)}\right]^2 = \frac{(ka)^{4\ell+2}}{[(2\ell-1)!!]^2\,[2\ell+1)!!]^2} \left\{1 - \frac{4\ell+2}{(2\ell-1)(2\ell+3)}(ka)^2 + \underline{0}\left[\frac{(ka)^4}{\ell^2}\right]\right\} \qquad (A3)$$

and

$$|S_\ell(k) - 1|^2 = 4[j_\ell(ka)/n_\ell(ka)]^2 - 4[j_\ell(ka)/n_\ell(ka)]^4 + \dots \qquad (A4)$$

Although the expansions (A3) and (A4) are valid only for sufficiently large ℓ, we can replace them in (2.13) for all ℓ, because the difference affects only a finite number of terms, which contribute at most a polynomial in τ. Restricting ourselves to the main terms of these expansions and employing also (2.14), we find that, up to a polynomial in τ, the behaviour of $g(k,\tau)$ for $|\tau| \to \infty$ is the same as that of the function

$$\gamma(k,\tau) = k\, a^2 \sum_{\ell=0}^{\infty} (-1)^{\ell} \left[\frac{\ell!}{(2\ell)!}\right]^2 \frac{(2\, a^2\, k\tau)^{2\ell}}{(2\ell+1)!} . \qquad (A5)$$

Just as in (A1), this series can be rewritten as a contour integral,

$$\gamma(k,\tau) = \frac{1}{2}\, i\, k\, a^2 \int_C \left[\frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}\right]^2 \frac{(2\, a^2\, k\,\tau)^{2\lambda}}{\Gamma(2\lambda+2)} \frac{d\lambda}{\sin(\pi\lambda)} , \qquad (A6)$$

where C is the contour shown in figure 2, $\Gamma(Z)$ is the gamma function, and the integrand is rendered single-valued by restricting ourselves to $k \geqslant 0$ and to the first quadrant of the τ plane:

$$0 \leqslant \arg(\tau a) \leqslant \frac{\pi}{2} . \qquad (A7)$$

The behaviour of g in other quadrants follows from the symmetry relations

$$g(k,\tau) = g^*(k,-\tau^*) = g(k,-\tau) = g^*(k,\tau^*), \qquad (A8)$$

which are an immediate consequence of (2.13).

To determine the asymptotic behaviour of (A6) for $|\tau| \to \infty$, we shall apply the saddle-point method. For this purpose, let us deform the upper half of the contour C into the upper half-plane and the lower half into the lower half-plane, away from the real axis. The main contributions to the integral arise from large values of $|\lambda|$, for which the gamma functions can be replaced by Stirling's approximation. Moreover, if $|\mathcal{I}m\,\lambda|$ is sufficiently large,

$$[\sin(\pi\lambda)]^{-1} \approx \mp 2\,i\,\exp(\pm i\lambda\pi)\ , \tag{A9}$$

where the upper signs correspond to $\mathcal{I}m\ \lambda > 0$ and the lower ones to $\mathcal{I}m\ \lambda < 0$. Thus, we find

$$\gamma(k,\tau) \sim \frac{k}{2^3\,\pi^{\frac{1}{2}}} \left\{ \int_{C_+} \exp\left[F_+(\lambda,k,\tau)\right] d\lambda - \int_{C_-} \exp\left[F_-(\lambda,k,\tau)\right] d\lambda \right\}, \tag{A10}$$

where C_+ and C_-, shown in dashed line in figure 2, are the upper and the lower half of the deformed contour, and

$$F_{\pm}(\lambda,k,\tau) = -4\lambda\log\lambda + \Big[2\log(2\,a^2\,k\,\tau) -$$

$$-2(3\log 2-2) \pm i\,\pi\Big]\lambda - \frac{3}{2}\log\lambda + \underline{O}(\lambda^{-1})\ . \tag{A11}$$

The saddle points of $F_{\pm}$ are located at

$$\lambda_{\pm} = \frac{a}{2}\,(\pm\,i\,k\,\tau)^{\frac{1}{2}}\ , \tag{A12}$$

respectively. According to (A7), λ_+ belongs to the first quadrant and λ_- to the fourth quadrant, as it ought to be.

Evaluating the contribution from the saddle points, we find that the asymptotic behaviour of γ, and therefore also of g, is given by

$$g(k,\tau) \approx \frac{ka}{2^{5/2}} \left\{ \frac{\exp\left[2a(ik\tau)^{\frac{1}{2}}\right]}{(ik\tau)^{\frac{1}{2}}} + \frac{\exp\left[2a(-ik\tau)^{\frac{1}{2}}\right]}{(-ik\tau)^{\frac{1}{2}}} \right\} (|\tau| \longrightarrow \infty). \tag{A13}$$

Although this expression has been derived only for values of τ belonging to the first quadrant, it is readily seen, with the help of (A8), that it remains valid in the other quadrants as well, provided that we take: $-\pi < \arg(\tau a) \leqslant \pi$.

The first term within curly brackets dominates the asymptotic behavior in the lower half of the τ plane, whereas the second term dominates in the upper half-plane. On the real axis, both terms are of the same order, and we find

$$g(k,\tau) \approx 2^{-3/2}\, ka(k\tau)^{-\frac{1}{2}} \exp\left[(2k\tau)^{\frac{1}{2}}\, a\right] \cos\left[(2k\tau)^{\frac{1}{2}}\, a - \frac{\pi}{4}\right]. \tag{A14}$$

Thus, we have an oscillation with exponentially increasing amplitude.

The contribution from the second term within brackets in (A3), which was neglected above, is of the order $(ka)^{3/2}\,(\tau a)^{-\frac{1}{2}}$ relative to the first term. Thus, the conditions for the validity of (A13) are: $|k\tau|^{\frac{1}{2}}\, a \gg 1$, $|\tau a| \gg (ka)^3$. Similarly, the second term of the expansion (A4) gives rise to contributions of the order of $\exp\left[4a(\pm\, i\, k^3\tau)^{1/4}\right]$, which are negligible in comparison with (A13) for $|\tau| \longrightarrow \infty$.

This method can also be applied directly to $f(k,\tau)$ by writing (2.12) in the form

$$f(k,\tau) \approx \sum_{\ell} \frac{(2\ell+1)}{k} \left[\frac{j_\ell(ka)}{n_\ell(ka)} + i \frac{j_\ell^2(ka)}{n_\ell^2(ka)} - \right.$$

$$\left. - \frac{j_\ell^3(ka)}{n_\ell^3(ka)} + \ldots \right] P_\ell \left(1 - \frac{\tau^2}{2k^2} \right). \qquad \text{(A15)}$$

The sum of terms containing even powers of j_ℓ / n_ℓ in the above expansion is equal to $ig(k,\tau)$. The first term within square brackets gives rise to a contribution behaving asymptotically like $- a \cos(\tau a)$ (cf. (2.9)). The contribution from the third term is of the order of $\exp\left[3a(\pm ik^2 \tau)^{1/3}\right]$, and so on: the nth term gives contributions in $\exp\left[na(\pm\, i\, k^{n-1} \tau)^{1/n}\right]$. Thus, while (2.10) is indeed correct for $|\tau| \longrightarrow \infty$ along any direction in the upper half-plane, it is not valid on the real axis, where the imaginary part of f, given by (A14), dominates the asymptotic behaviour.

Notice that, although the first term within square brackets in (A15) is much larger than the second one for $\ell \gg (ka)^2$, it is the latter which determines the asymptotic behaviour on the real axis. This is due to the alternating character of the series (A5). Since this might give rise to some doubts concerning the validity of the approximations which were employed in the derivation of (A13), it is worthwhile to establish some inequalities which confirm this result.

It was shown in Ref. 10 that, if $\ell \geqslant 2\, ka \geqslant 0$, $(\log \ell)^{\frac{1}{2}} \gg 1$,

$$\left| \frac{j_\ell(ka)}{h_\ell^{(1)}(ka)} \right| \leqslant \left(\frac{e\,k\,a}{2\ell+1} \right)^{2\ell+1} . \tag{A16}$$

On the other hand, according to Picone's inequality (12, II, p. 276),

$$|P_\ell(Z)| \leqslant \frac{(2\ell-1)!!}{\ell!} (1+|Z|)^\ell . \tag{A17}$$

Substituting these results in (2.13) and (2.15), and employing Stirling's approximation for the factorials, we get, for $|\tau| \gg k$,

$$|g(k,\tau)| \lesssim \frac{e\,k\,a}{|\tau k|^{\frac{1}{2}}} \sum_\ell \frac{1}{(\pi\ell)^{\frac{1}{2}}} \left(\frac{e\,a|\tau k|^{\frac{1}{2}}}{2\ell+1} \right)^{4\ell+1} \lesssim$$

$$\lesssim \frac{2^{3/2}\,e\,k\,a}{|\tau k|^{\frac{1}{2}}} \sum_\ell \frac{1}{\left[2\pi(4\ell+1)\right]^{\frac{1}{2}}} \left(\frac{e}{4\ell+1} \right)^{4\ell+1} (2a|\tau k|^{\frac{1}{2}})^{4\ell+1} <$$

$$< 2^{3/2}\,e\,k\,a \, \frac{\exp(2a|\tau k|^{\frac{1}{2}})}{|\tau k|^{\frac{1}{2}}} . \tag{A18}$$

An inequality in the opposite sense can be obtained on the imaginary axis, $\tau = \pm i|\tau|$, because (2.13) then becomes a series of positive terms, the sum of which is certainly larger than any one of its terms. For sufficiently large $|\tau|$, the largest term of the series corresponds to $\ell \approx a|k\tau|^{\frac{1}{2}}/2$, leading to

$$|g(k, \pm i|\tau|)| > \frac{k\,a^{\frac{1}{2}}}{2^{3/4}\,\pi^{\frac{1}{2}}} \; \frac{\exp(2a|\tau k|^{\frac{1}{2}})}{|\tau k|^{3/4}} . \qquad \text{(A19)}$$

The inequalities (A18) and (A19) confirm the result (A13).

* * *

Appendix B

Evaluation of the integral $C_{\ell'\ell}(x)$

To compute the integral (3.2), let us employ the result (12, I, p. 15):

$$P_{\ell'}(1-x+x\cos\theta) = \sum_{p=0}^{\ell'} (-1)^p \frac{(\ell'+p)!}{(p!)^2\,(\ell'-p)!}\, x^p \left(\frac{1-\cos\theta}{2}\right)^p . \qquad \text{(B1)}$$

We have (12, II, p. 219)

$$\frac{1}{2}\int_{-1}^{1} \left(\frac{1-\mu}{2}\right)^p P_\ell(\mu)d\mu = 0 \qquad \text{if} \quad p < \ell ,$$

$$= (-1)^\ell \frac{(p!)^2}{(p-\ell)!\,(p+\ell+1)!} \quad \text{if} \quad p \geqslant \ell . \qquad \text{(B2)}$$

Substituting (B1) and (B2) in (3.2), we get

$$C_{\ell'\ell}(x) = 0 \qquad \text{if} \quad \ell' < \ell, \tag{B3}$$

$$C_{\ell'\ell}(x) = (-1)^{\ell}(2\ell'+1) \sum_{p=\ell}^{\ell'} \frac{(-1)^p (\ell'+p)!}{(\ell'-p)!(p-\ell)!(p+\ell+1)!} x^p \text{ if } \ell' \geqslant \ell . \tag{B4}$$

In particular,

$$C_{\ell\ell}(x) = x^{\ell} . \tag{B5}$$

It is readily seen that, for $m \geqslant 1$,

$$C_{\ell+m,\ell}(x) = \frac{(-1)^m (2\ell+2m+1)!}{m!(2\ell+m+1)!} x^{\ell+m} F(-2\ell-m-1, -m; -2\ell-2m; \frac{1}{x}), \tag{B6}$$

where $F(a, b; c; Z)$ is the confluent hypergeometric function. We have (33)

$$F(a, b; c; Z) = (1-Z)^{c-a-b} F(c-a, c-b; c; Z) . \tag{B7}$$

It follows that

$$C_{\ell+m,\ell}(x) = x^{\ell}(1-x) \sum_{s=0}^{m-1} (-1)^s (\ell;m;s) x^s \quad (m = 1, 2, \ldots), \tag{B8}$$

where

$$(\ell;m;s) = \frac{(2\ell+2m+1)}{m(2\ell+m+1)} \frac{(2\ell+m+s+1)!}{s!(m-s-1)!(2\ell+s+1)!} . \tag{B9}$$

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Footnotes

[1] The argument is based on the assertion that the expression $y_\ell(k)\, f_\ell^2(k) \left[1 + y_\ell(k)\, f_\ell(k)\right]^{-1}$ (Ref. 25, Eq. (19)) tends to zero for $|k| \to \infty$ in the upper half-plane if f_ℓ behaves like $\exp(-2\,i\,C\,k)$ $(C > 0)$ and y_ℓ behaves like $k^{2\ell} \exp(k^2)$ for $|k| \to \infty$. However, this is not true below the first or second bisector, where the above expression behaves like $\exp(-2\,i\,Ck)$, so that it has an essential singularity at infinity.

[2] The following treatment applies to $\ell \geqslant 1$; it must be slightly modified for $\ell = 0$.

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