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RADIATIVE CORRECTION TO PION DECAY

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RADIATIVE CORRECTION TO PION DECAY*

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Summary: The angular distributions and energy distributions of the photon are studied in the radiative decay of Pion. For pion decay described as secondary process the Fermi interaction is assumed to be Vector - Axial Vector. Second order radiative corrections, when pion-lepton interaction is assumed to be elementary, are also studied. The corrected branching ratio of the radiative to the non-radiative decay is found to be $(3.8) \times 10^{-2}$ for the Pion decaying into muon and $(2.7) \times 10^{-2}$ for the electron mode of decay. The radiative correction to the decays is 5 to 6 percent.

I. INTRODUCTION

Pion is known to decay into muon and neutrino with a lifetime of 2.4×10^{-8} sec. The corresponding radiative decay is also observed with the branching ratio of $3.3 \pm 1.3 \times 10^{-4}$ for radiative to

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non-radiative mode[1]. The corresponding electron modes have been observed, conclusively, only very recently[2,3]. The ratio for the electron mode to the muon mode, found rare earlier, seems to be in agreement with theoretical predictions. The theoretical estimates are given by 1.3×10^{-4} and $\sim 10^{-6}$ for the non-radiative and radiative electron decays respectively[4].

The present investigation was undertaken to see if the earlier discrepancy of extreme rarity of electron decay could be explained by the radiative correction, due to its much smaller mass compared to that of the muon. Our results are in agreement with the present findings[3]. The radiative correction both for the electron and the muon modes is only 5 to 6 percent. In the next section the expressions for the probability distribution of photon in the radiative decay are described. The expressions are given for the decay regarded as an elementary process as well as for the case when the decay mechanism is a secondary effect. For the latter case we assume the Two Component neutrino theory with left handed neutrino[5] and the Fermi interaction to be Vector minus Axial vector[6]. In the third section the radiative corrections to the decay are discussed assuming a direct interaction linear in pion, muon and neutrino fields.

II. RADIATIVE DECAY OF PION

The interaction Hamiltonian (apart from the electromagnetic interaction terms) is of the form

$$\mathcal{H}_I (x) = \frac{g'}{\sqrt{2}} (\bar{\mu} \gamma_5 \nu) \, \pi (x) + \text{h.c.}$$

$$\gamma_5 \nu = + \nu$$

when pion decay is an elementary process. For the decay regarded as a secondary process, the relevant part of the interaction Hamiltonian is given by

$$\mathcal{H}_I(x) = g(\bar{p}\gamma_5 n)\,\pi(x) + \frac{G}{\sqrt{2}}\left[\bar{p}\gamma_\mu(1+\lambda\gamma_5)\,\bar{n}\right]\left[\bar{\mu}\gamma_\mu\nu\right] + \text{h.c.}$$

$$\gamma_5\nu = +\nu$$

π, μ, ν, n and p represent the pion, muon, neutrino, neutron and proton fields respectively. g, g' and G represent the coupling constants. λ is the ratio of the Axial Vector to Vector coupling constants in the Fermi interaction[7].

The matrix element when the pion decay is elementary, is easily shown to be

$$\mathcal{M} = \bar{u}(p)\,\frac{i}{(k\cdot p)}\left[\frac{1}{2}(\epsilon\cdot\gamma)(k\cdot\gamma) + \epsilon\cdot p\right]\gamma_5\, v(p_\nu)$$

For the decay regarded as secondary effect, it is

$$\mathcal{M} = \mathcal{M}_V + \mathcal{M}_A$$

$$\mathcal{M}_V = -G_V\,\bar{u}(p)\,(v\cdot\gamma)\gamma_5 v(p_\nu)$$

$$\mathcal{M}_A = \bar{u}(p)\left[G_A\,(k\cdot p)\,(\epsilon\cdot\gamma) + G_A'\,m_\pi^2\,\frac{im}{(k.p)}\left(\frac{1}{2}(\epsilon\cdot\gamma)(k.\gamma) + \epsilon\cdot p\right)\right]\gamma_5 v(p_\nu)$$

where

$$G_A = \frac{2i}{3}\,\frac{\pi^2}{M}\lambda\,, \qquad G_A' = 4i\,\frac{\pi^2}{m_\pi^2} \qquad \text{and} \qquad G_V = i2\,\frac{\pi^2}{M}$$

$$= i\frac{2\pi^2}{3}\lambda\left(\frac{1}{M}\right) \qquad\qquad = i4\pi^2\lambda\,L\left(\frac{M}{m_\pi^2}\right) \qquad\qquad = i2\pi^2\left(\frac{1}{M}\right)$$

m_π is the mass of pion and M is the mass of the virtual baryons through which the decay takes place. For simplicity in calculations the difference in masses of the intermediate baryons is neglected. It can be taken into account by some averaging process over all possible baryon pairs. v_μ is a dual four-vector defined by

$$v_\mu = \delta_{\lambda\delta\sigma\mu} P_\lambda \ \epsilon_\delta k_\sigma$$

L is the logarithmic divergent integral[8]

$$L = \frac{1}{i\pi^2} \int \frac{d^4q}{(q^2+M^2)\ [\ (-q+P)^2+M^2]}$$

P is four-momentum of pion, k that for photon with polarization vector ϵ and u(p) and v(p) denote the Dirac spinors for muon and neutrino with four momenta p and p_ν respectively. In deriving the above matrix elements we have made the choice of gauge so that $(\epsilon . p) = 0$. It may be remarked that the inner bremsstrahlung diagrams give rise to similar infrared divergent terms for the direct and the indirect decay. We will see that this term is an important term for the indirect decay as well.

The probability distributions, summed over spins of the leptons and the polarization of the photon, for the emission of photon making an angle θ with the direction of neutrino are given in the center of mass frame of pion by (we assume $\hbar = c = m_\pi = 1$)

$$\mathcal{P}(\omega,\theta) = \text{const.}\ W(\omega,\theta).\ d|\underline{k}|.\text{Sin}\ \theta\ d\theta$$

where

$$W(\omega,\theta) = \frac{|\underline{k}|^2}{\omega} \ \frac{2(\omega_m-\omega)}{(\omega-|\underline{k}|\cos\ \theta-1)^2}\ F(\omega,\theta)$$

where $\omega_m = (1-m^2+\lambda^2)/2$ is the maximum energy of photon of mass λ and m is the mass of muon.

For direct interaction

$$F(\omega,\theta) = \frac{(\omega_m-\omega)}{[\omega-\omega_m(\omega-|\underline{k}|\cos\theta)]} \cdot \left\{ \left[1 + \frac{1}{2}\ (\frac{\lambda}{\omega})^2\right] (\omega-|\underline{k}|\cos\theta) + \right.$$

$$\left. + \frac{1}{\omega^2}\,\omega_m(\omega_m-\omega)\,\frac{(|\underline{k}|^2\sin^2\theta + \lambda^2}{[\omega-\omega_m(\omega-|\underline{k}|\cos\theta)]}\right\}$$

We have given a mass λ to photon to avoid infrared divergence. The sum over polarization is then made over all three polarizations of a neutral vector meson. Energy ω of photon is then given by $\omega = \sqrt{|\underline{k}|^2+\lambda^2}$.

For the decay via Fermi interaction we have[9] ($\lambda = 0$):

$$F(\omega,\theta) = F_v + F_A + F_{A\cdot v}$$

where

$$F_v = -|G_v|^2\, 2\omega^2\, (\omega_m-\omega) \cdot \left[(\omega_m-\omega)\, \frac{\sin^2\theta}{[1-\omega(1-\cos\theta)]^2} -1\right]$$

$$F_A = 2(\omega_m-\omega)\left[-|G_A|^2\,\omega^2 \left\{ (\omega_m-\omega)\,\frac{\sin^2\theta}{[1-\omega(1-\cos\theta)]^2} -1 \right\}\right.$$

$$+ |G_A'|^2\,\frac{m^2}{2[(1-\omega_m)+\omega_m\cos\theta]}\left\{ \omega(1-\cos\theta) + \frac{\omega_m(\omega_m-\omega)\sin^2\theta}{[(1-\omega_m)+\omega_m\cos\theta]}\cdot\frac{1}{\omega^2}\right\}$$

$$\left. + 2R\ell(G_A' G_A^*)\,\frac{m^2}{2[(1-\omega_m+\omega_m\cos\theta]}\left\{\omega(1-\cos\theta) + \frac{(\omega_m-\omega)\sin^2\theta}{[1-\omega(1-\cos\theta)]}\right\}\right]$$

$$F_{A\cdot v} = 2(\omega_m-\omega)\left[2R\ell(G_v^*\, G_A)\,\frac{\omega^2[\omega(1-\cos\theta)+\cos\theta]}{[1-\omega(1-\cos\theta)]}\right.$$

$$\left. - 2R\ell(G_v^*\, G_A')\,\frac{m^2\omega(1-\cos\theta)}{2[(1-\omega_m)+\omega_m\cos\theta]}\right]$$

On integrating over the angle the following energy distributions for the photon are obtained.

For the direct decay we separate from $W(\omega)$ the infrared divergent part and write it as

$$W(\omega) = W_o(\omega) + W_1(\omega)$$

$$W_o(\omega) = 2\omega\left[\frac{2(\omega-\omega_m)}{(1-2\omega)} + \ln\left(\frac{1-2\omega}{m^2}\right)\right] + 4\omega_m\left[2-\ln\left(\frac{1-2\omega}{m^2}\right)-(\omega_m-1)\frac{\ln(1-2\omega)}{\omega}\right]$$

$$W_1(\omega) = 4\omega_m\left(1-\frac{\lambda^2}{\omega^2}\right)\frac{1}{\omega}\left[-(m^2-\lambda^2)\,\omega_m\frac{\omega^2}{[\omega^2(1-\omega_m)^2-|\underline{k}|^2\omega_m^2]} + \frac{(2\omega-1)\omega_m}{(1-2\omega+\lambda^2)}\right.$$

$$\left. +(\omega_m-1)\cdot\frac{\sqrt{\omega^2-\lambda^2}}{\omega}\cdot\ln\left\{\frac{(1-\omega+|\underline{k}|)\,[(1-\omega_m)\omega-|\underline{k}|\omega_m]}{\omega}\right\}\right]$$

Here we have put $\lambda = 0$ in the non-infrared divergent part W_o. W_1 reduces to

$$-(1-m^2)\left[2\,(1-m^2) + (1+m^2)\ln m^2\right]\left(\frac{1}{\omega}\right)$$

when λ is put equal to zero.

For the decay via Fermi interaction we get

$$W(\omega) = W_v + W_A + W_{A\cdot v}$$

where

$$W_v = |G_v|^2\, I_o$$

$$W_A = |G_A|^2\, I_o + |G_A'|^2\, I_2 + 2\, R\ell\,(G_A^* G_A')\, I_1$$

$$W_{A\cdot v} = -2\, R\ell\,(G_v^* G_A')\, |I_3|$$

The functions $I(\omega)$ are defined by

$$I_o = \frac{8}{3}\,\frac{\omega^3(\omega_m-\omega)^2[m^2+2(1-2\omega)]}{(1-2\omega)^2}$$

$$I_1 = 4m^2\,\omega(\omega_m-\omega)^2\left\{\frac{(1-\omega_m-2\omega)}{(\omega_m-\omega)(1-2\omega)} + \frac{(m^2-\omega)}{2(\omega_m-\omega)^2}\ln\left(\frac{m^2}{1-2\omega}\right)\right\}$$

$$I_2 = m^2\,(W_o(\omega) + W_1(\omega))$$

$$I_3 = 2m^2\omega^2 \left\{ \frac{2(\omega_m-\omega)}{(1-2\omega)} + \ln\left(\frac{m^2}{1-2\omega}\right)\right\}$$

In Fig. 1 we plot the energy functions I_o, I_1, I_2, W_o and W_1 ($\lambda = 0$). The correlation functions I_i (ω, θ) ($i = 0,1,2,3$) and I_3' (coefficient of 2Rl ($G_V^* G_A$) , defined likewise are plotted in Fig. 2 for an energy of photon about 14 Mev. It may be remarked that the infrared divergent term is an important term for the case of decay via Fermi interaction as well.

III. RADIATIVE CORRECTION TO DECAY OF PION

For simplicity, we shall discuss the radiative correction only for the case when the pion decay is elementary. Since the main contribution to the decay via Fermi interaction comes from a term similar to that involved in the direct decay, it may indicate the relative values of radiative corrections in the electron and the muon modes for the indirect decay as well.

The contributions to the second order radiative corrections arise from the vertex part and self energy diagrams. After mass and wave function renormalizations the contribution from self energy diagram of muon is (apart from common factors)[10]

$$-\left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi}\left[\ln\frac{\Lambda}{m} + 2\ln\frac{\lambda}{m} + \frac{9}{4}\right]$$

The meson self energy diagram gives

$$+\left(\frac{e^2}{4\pi}\right) \cdot \frac{1}{2\pi}\left[2\ln\Lambda - 2\ln\lambda - 1\right]$$

Λ is the high energy cut-off used to evaluate the divergent integrals[11]. The vertex part contribution is given by

$$\left(\frac{e^2}{4\pi}\right)\cdot\frac{1}{2\pi}\left[\frac{2}{(1-2m^2)}\left(\ln\Lambda - m^2\ln\frac{\Lambda}{m}\right) - 2\,\frac{(1+m^2)}{(1-m^2)}\left(\ln m^2\right)\left(\ln\lambda\right)\right.$$

$$\left. + \frac{1}{(1-m^2)}\left\{(1-m^2)-(2+m^2)(\ln m^2)+2(1+m^2)(\ln m)^2\right\}\right]$$

To these we must add the contribution of inner bremsstrahlung integrated from λ to a small energy Δ. This contribution is given by:

$$\left(\frac{e^2}{4\pi}\right)\frac{1}{2\pi}\,\frac{2}{(1-m^2)^2}\left\{(1-m^2)\left[2(1-m^2)+(1+m^2)\ln m^2\right](\ln\lambda)+C+\int_0^{\Delta}W_0(\omega)\,d\omega\right\}$$

Here C is the term arising due to the summing over the longitudinal polarization of neutral vector meson ($\lambda \neq 0$). It is given by[12]:

$$C = (1-m^2)\left[(1-m^2)(1-2\ln(2\Delta))+(1+m^2)\left\{-\frac{(3+m^2)}{4(1-m^2)} - \frac{1}{2}\ln 2\left(\ln 2+\left(\frac{1+m^2}{1-m^2}\right)^2\right)\right.\right.$$

$$+ (\ln m^2)\left(-1+\ln(1-m^2)-\frac{1}{2}\left(\frac{1+m^2}{1-m^2}\right)^2\right)$$

$$+\ln(1-m^2)\left(\ln 2-\ln(1-m^2)+\frac{1}{2}\left(\frac{1+m^2}{1-m^2}\right)^2\right)$$

$$\left.\left.+\left(\mathcal{L}\left(-\frac{1+m^2}{1-m^2}\right)-\mathcal{L}\left(\frac{-m^2}{1-m^2}\right)\right)+\frac{1}{2}\left(\mathcal{L}\left(-\frac{1}{m^2}\right)-\mathcal{L}(-1)\right)\right\}\right]$$

The infrared divergence is seen to disappear from the total radiative correction. The ultraviolet divergence (logarithmic), however, does not cancel out. This means that the direct interaction of pion muon and neutrino fields is not renormalizable. For a cut-off about a nucleon mass, the contribution of the cut-off dependent terms is 0.2 percent for electron decay while 10 percent for the muon case. The total radiative correction itself amounts ($\epsilon \sim 1$ Mev, $\Lambda = 6.7\ m_\pi$) to ~5 and ~6 percent for muon and electron modes respectively. The contribution of inner bremsstrahlung (which is mainly due to the term C arising due to the sum over all the three polarizations of photon with mass tending to zero) is ~85 percent for muon and ~27 percent (with negative sign) for electron decay.

The branching ratio for radiative to non-radiative decay is $\sim 2.7 \times 10^{-2}$ for the electron mode while $\sim 3.8 \times 10^{-2}$ for the muon mode. The contribution of inner bremsstrahlung is ~80 percent for muon case while it is negative and ~66 percent of that due to virtual photons for the electron case. The large negative contribution in the case of electron indicates that (in so far as direct decay is concerned) the radiative decay of electron may be difficult to observe. The radiative corrections when the pion decay is secondary will be published in a later report.

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APPENDIX

If $\underline{\epsilon}_x$, $\underline{\epsilon}_y$, and $\underline{\epsilon}_z = \underline{k}/|\underline{k}|$ for each $\underline{k}$ are three unit vectors forming a right-handed orthogonal system then we define

$$\sqrt{2}\ \underline{\epsilon}_\rho = \underline{\epsilon}_x - i(-1)^\rho\ \underline{\epsilon}_y$$

where $\rho = 1,2$ correspond to the right and left circular polarizations of the light quantum respectively. The contribution to $F(\omega,\theta)$, arising due to parity non-conservation is given in the center of mass frame of pion by ($m_\pi = 1$)

$$F' = F_V' + F_A' + F_{A\cdot V}'$$

where

$$F_V' = (-1)^\rho |G_V|^2 k_o \left[(1-k_o)\ (\underline{k}\cdot\underline{p}_\nu) - p_{\nu_o} k_o^{\ 2} \right]$$

$$F_A' = (-1)^\rho \Big[-|G_A|^2 k_o \left\{ p_{\nu_o} (\underline{k}\cdot\underline{p}) - p_o\ (\underline{k}\cdot\underline{p}_\nu) \right\}$$

$$- |G_A'|^2 \frac{m^2}{2(k\cdot p)^2} \left\{ (k\cdot p)^2 + \frac{1}{k_o} \left(k_o^2\ p_{\nu_o}^{\ 2} - (\underline{k}\cdot\underline{p}_\nu)^2 \right) \right\}$$

$$-2\, R\ell (G_A'^{*}\ G_A) \frac{m^2}{2(k\cdot p)}\ \ k_o (k\cdot p_\nu) \Big]$$

$$F_{A\cdot V}' = (-1)^\rho \left[2R\ell\ (G_A^{*}\ G_V) \left\{ k_o^2\ (k\cdot p_\nu) + (\underline{k}\cdot\underline{p}_\nu)^2 + k_o^2 (p_{\nu_o} - p_{\nu_o}^{\ 2}) \right\} \right.$$

$$\left. + 2R\ell\ (G_A'^{*} G_A) \frac{m^2}{k_o (k\cdot p)} \left\{ k_o^2 (k\cdot p_\nu) + (\underline{k}\cdot\underline{p}_\nu)^2 - k_o^2\ p_{\nu_o}^{\ 2} \right\} \right]$$

Here k, p and p_ν are four-momenta of photon, muon and neutrino respectively and the photon has vanishing mass. The terms corresponding to the direct interaction are contained in the coefficient of $|G_A'|^2$.

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7. $|\lambda| = 1.14$ A. Winther and O. Kofoed-Hansen, Kgl. Danske Vedenskab. Selskab, Mat.-fys. Medd. (to be published).

8. L is evaluated using a Feynman cut-off of the form $C(q^2) = \frac{\Lambda^2}{q^2+M^2+\Lambda^2}$. Its value is given by $(m_\pi = 1)$

$$L = 2\sqrt{4M^2-1}\ \tan^{-1}\frac{1}{\sqrt{4M^2-1}} - \frac{1+\Lambda^2}{2}\ln\left(\frac{M^2}{\Lambda^2+M^2}\right) - \frac{1}{2}\sqrt{\Delta}\ln\left(\frac{\Lambda^2+2M^2-1+\sqrt{\Delta}}{\Lambda^2+2M^2-1-\sqrt{\Delta}}\right)$$

For $\Lambda^2 > (2M+1)$

where,

$$\Delta = \Lambda^4 - 2(\Lambda^2+2M^2)+1$$

For $\Lambda^2 < (2M+1)$ replace the last term by $-\sqrt{-\Delta}\left\{\tan^{-1}\frac{1+\Lambda^2}{\sqrt{-\Delta}} - \tan^{-1}\frac{\Lambda^2-1}{\sqrt{-\Delta}}\right\}$,
Colber G. de Oliveira and J. Tiomno, Notas de Física 5, (1958).

9. The terms arising due to parity non-conservation cancel out when the sum over polarizations is made. The expressions for these terms are given in Appendix.

10. The common factor is $\frac{1}{2}\left(\frac{g'^2}{4\pi}\right)(1-m^2)^2$.

11. See R. P. Feynman, Phys. Rev. 76, 769, (1949) (Appendix); R. P. Feynman and L. M. Brown, Phys. Rev. 85, 231, (1952) (Appendix).

12. $\mathcal{L}(x)$ is Spence function defined by $\mathcal{L}(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt$
For a table and discussion of this function and related integrals, see K. Mitchell, Phil. Mag. 40, 351, (1949).

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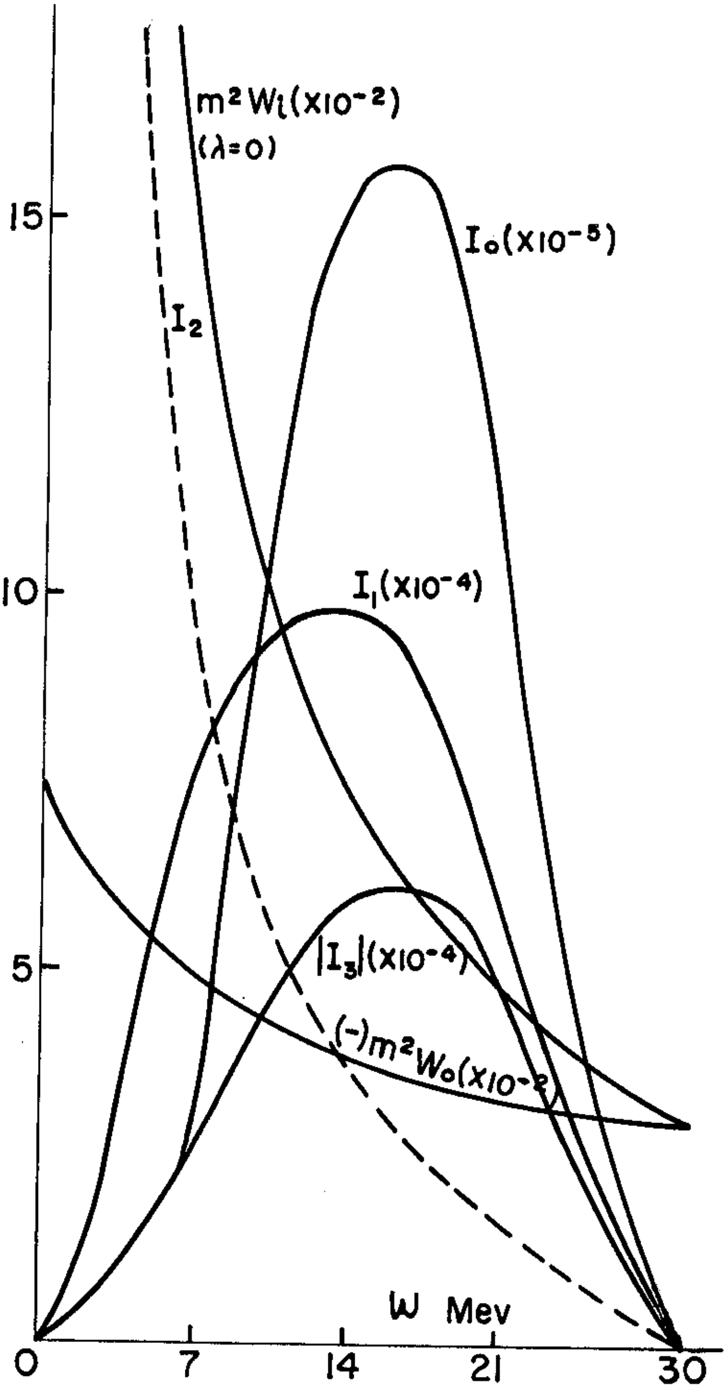

Figure 1

Energy distribution functions for photon. (The ordinates in the graphs should be multiplied by the appropriate factors (indicated in parenthesis), to to get the correct value.)

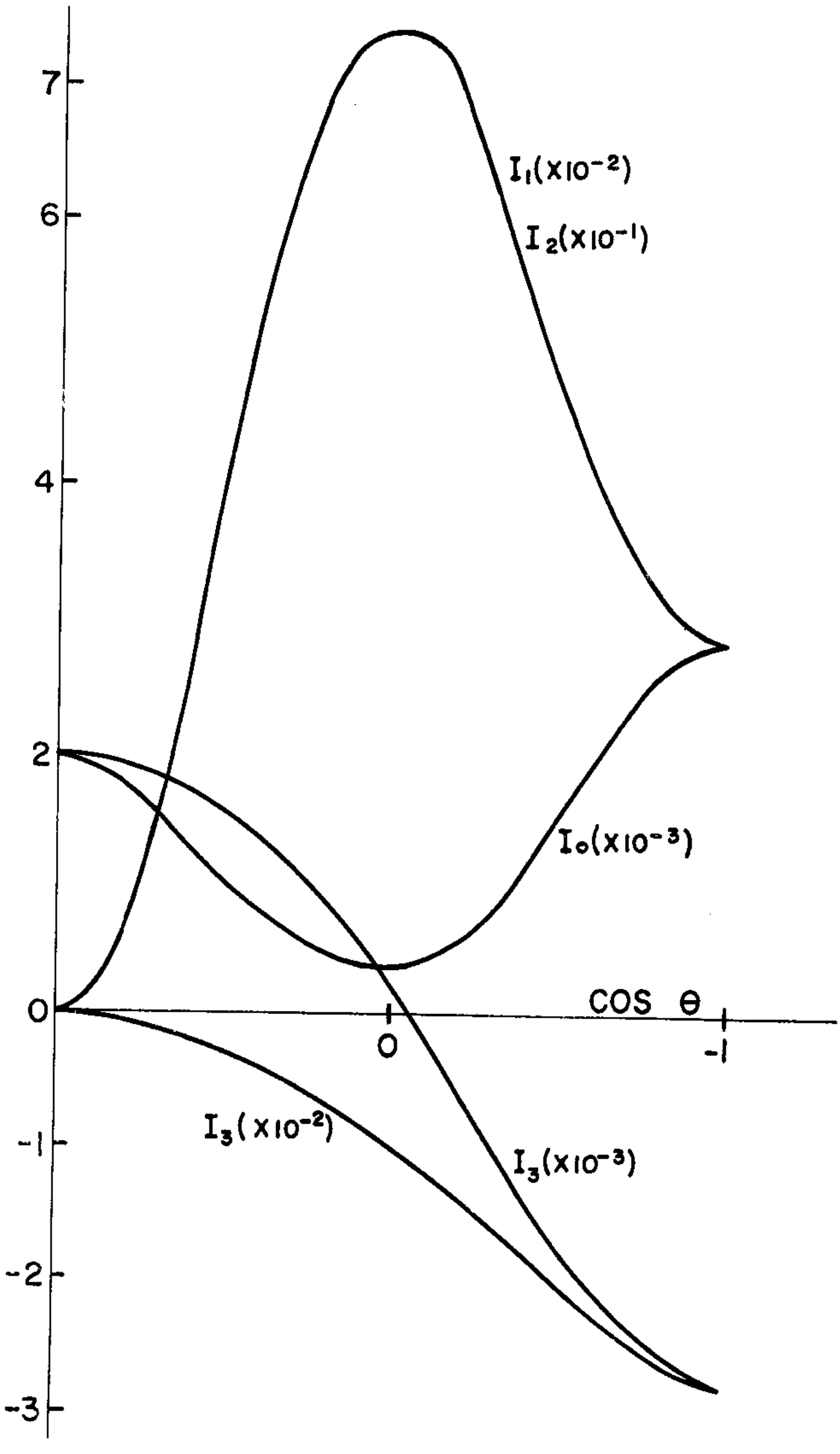

Figure 2.

Angular Correlation functions $I_i(\omega,\theta)$ for photon energy 14 Mev. (The ordinates in the graphs should be multiplied by the appropriate factors (indicated in parenthersis), to get the correct value).