(Split-)Octonions, Generalized Supersymmetries and M-Theory*

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Abstract. In this talk I discuss the results of a joint paper with Z. Kuznetsova, where the split-division algebras are introduced to construct generalized supersymmetries in different space-time signatures. In particular, in $D = 11$ dimensions, it is shown that split-octonions allow to introduce a split-octonionic M-algebra which extends to the $(6,5)$ signature the properties of the 11-dimensional octonionic M-algebras, only existing in $(10,1)$ and $(2,9)$ signatures. The three space-times above form a triality-related set of (split-)octonionic, eleven dimensional, spacetimes.

1 Introduction

The maximal division algebra of the octonions can be regarded as the responsible for the existence of various exceptional mathematical structures. It is well known, e.g., that four of the five exceptional simple Lie algebras arise through the so-called "Tits-Freudenthal magic squares construction", see [1], when octonions are involved. The remaining exceptional Lie algebra, G_2, corresponds to the group of automorphisms of the octonions.

Some arguments have been introduced linking the octonions to the possible existence of a "Theory Of Everything" based on exceptional mathematical structures. An interesting discussion of these topics can be found in [2]. On the other hand, it was proven in [3], that the eleven dimensional generalized supersymmetry algebra commonly known as "M-algebra" admits an octonionic reformulation, in terms of octonionic-valued spinors, with surprising new features. For instance, the bosonic sector is split into rank-1, 2 and 5 antisymmetric tensors which are no longer independent as in the standard M-algebra case. It has to be mentioned that the M-algebra is expected

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to be the fundamental building block underlying a still-to-be-constructed M-theory of unification of all interactions. The existence of an octonionic version of the M-algebra is therefore strictly linked to the cited "exceptional program" of ref. [2].

Higher dimensional (generalized) supersymmetries (i.e. formulated in space-time dimensions D, with $D \geq 8$) admit the peculiar feature that they come in several related copies of given signatures. The associated supersymmetric theories are all dually related ("the space-time dualities" of ref. [4]). The 10-dimensional superstrings only exist for the three $(9,1)$, $(1,9)$ and $(5,5)$ signatures (the latter with five time directions). The 11-dimensional supergravities are encountered, besides the Minkowskian $(10,1)$ signature, also in the exotic $(2,9)$ and $(6,5)$ signatures. It was proven in [5] that such dually-related versions are in consequence of the triality of the $D=8$ dimensions (the dually related theories are indeed triality related and close the S_3 group). We recall that it is eight-dimensional the transverse space of both the light-cone formulation of the 10-dimensional superstrings and of the supermembranes evolving in a flat 11-dimensional target spacetime. In $D=8$ the triality allowed signatures are $(8,0)$, $(0,8)$ and the exotic $(4,4)$.
Both the original Cartan's triality, see [6], and the space-time triality of ref. [5] are a consequence of the octonions. In this respect, it is quite puzzling that, while the standard M-algebra exists for the whole set of above signatures, the octonionic M-algebra only exists in $(10,1)$ and $(2,9)$ (the $(6,5)$ signature is missing). Essentially, the reason is due to the fact that the seven imaginary octonions have to be accommodated either in the time-like, or in the space-like directions. Obviously 7 cannot enter either 6 or 5. The puzzle is solved if we relax the condition of dealing with division-algebras. The exotic signature $(6,5)$ can be constructed in terms of the split-octonions. In [7] we reviewed the construction of the split-forms of the division-algebras and pointed out some of their applications. We used in the construction of graded algebras, Clifford algebras and spinors, of both unconstrained and constrained generalized supersymmetries and, finally, in the formulation of generalized Dirac equations of split-division algebra-valued spinors for the allowed space-times. Due to space constraint, only the application to generalized supersymmetry will be detailed here.

2 The Cailey-Dickson construction and the split-octonions

Following [8] we can construction the (split-)division algebras through repeated applications of the Cayley-Dickson doubling construction applied to the reals. A composition algebra possesses a unit, a non-degenerate quadratic form (norm) N and a conjugation denoted as "$*$". The Cayley-Dickson doubled algebra A^2 is obtained in terms of the operations of the original algebra A. Multiplication, conjugation and norm in A^2 are respectively given by

i)	multiplication in A^2:	$(x,y)\cdot(z,w) = (xz+\varepsilon w^* y, wx+yz^*)$,
ii)	conjugation in A^2:	$(x,y)^* = (x^*,-y)$,
iii)	norm in A^2:	$N(x,y) = N(x)-\varepsilon N(y)$.

The unit element $\mathbf{1}_{\mathbf{A}^2}$ of $\mathbf{A}^2$ is represented by $\mathbf{1}_{\mathbf{A}^2} = (\mathbf{1}_{\mathbf{A}},0)$.
In the above formulas ε is just a sign ($\varepsilon = \pm 1$).

It is convenient to denote the Cayley-Dickson's double of an algebra A by writing the ε sign on the right of the original algebra. For division algebras ε is always negative ($\varepsilon = -1$). We can therefore write

$\mathbf{C} = \mathbf{R}-$, $\mathbf{H} = \mathbf{C}- = \mathbf{R}--$, $\mathbf{O} = \mathbf{H}- = \mathbf{C}--- = \mathbf{R}---$.

The split division algebras are obtained by taking a positive ($\varepsilon = +1$) sign. We have $\widetilde{\mathbf{C}} = \mathbf{R}+$, $\widetilde{\mathbf{H}} = \mathbf{C}+ = \mathbf{R}-+$, $\widetilde{\mathbf{O}} = \mathbf{H}+ = \mathbf{C}-+ = \mathbf{R}--+$.

Other choices of the sign produce, at the end, isomorphic algebras.

Among the other properties, the seven imaginary split-octonions $\widetilde{E}_i$ satisfy the relations

$$\widetilde{E}_i\cdot\widetilde{E}_j = -\eta_{ij}\mathbf{1}+C_{ijk}\eta_{kr}\widetilde{E}_r, \qquad (1)$$

(the Einstein convention over repeated indices is understood) together with

$$\begin{aligned}\widetilde{E}_i^* &= -\widetilde{E}_i,\\ N(\widetilde{E}_i) &= \eta_{ii}.\end{aligned} \qquad (2)$$

In the above formulas η_{ij} denotes the diagonal matrix $(+++----)$ with three positive and four negative eigenvalues (normalized to ± 1), while C_{ijk} are the totally antisymmetric octonionic structure constants. The algebra of the split-octonions is, just like the ordinary octonions, an alternative algebra.

For our purposes here the most interesting feature is that the anticommutators $\{\widetilde{E}_i,\widetilde{E}_j\} = \widetilde{E}_i\widetilde{E}_j+\widetilde{E}_j\widetilde{E}_i$ between two imaginary split-octonions produce the basic relation of the generators of the Clifford algebra with signature $(4,3)$. Higher-dimensional Clifford algebra relations can be realized in terms of (split-)octonionic valued matrices, see [9], through repeated use of lifting algorithms. The 9-dimensional $(5,4)$ signature is realized in terms of 2×2 split-octonionic valued matrices, the 11-dimensional $(6,5)$ signature of interest here in terms of 4×4 split-octonionic valued matrices (4 purely real matrices while the remaining seven ones are given by the seven split-octonions each multiplying a unique, common, real 4×4 matrix). Following [9] it is possible to construct an octonionic (and split-octonionic) variant of the original Clifford algebra, by regarding it as the enveloping algebra produced by the (split-)octonionic valued matrix generators. The (split-)octonionic spinors can be constructed on similar lines. The "oxidized" forms (see [10] for a discussion) of the split-octonionic Clifford algebras are encountered for the $(4+k,3+8m+k)$ and $(5+8m+k,4+k)$ space-time signatures, where $k,m=0,1,2,\ldots$.

3 Split-octonionic generalized supersymmetries

The ordinary supersymmetry algebra is such that the anticommutator of two spinors produces a translation. Generalized supersymmetry is an extension, allowing the bosonic r.h.s. being decomposed into higher order antisymmetric tensors (in physical applications related to extended objects like branes). In the Minkowskian $D=11$ space-time, the M-algebra is given, e.g., by 32-component real spinors and a maximal saturated r.h.s. with 528 bosonic elements entering a 32×32 symmetric matrix. We have

$$\{Q_a,Q_b\} \quad = \quad Z_{ab} \tag{3}$$

with the bosonic symmetric matrix being decomposed into rank-1, rank-2 and rank-5 antisymmetric tensors, for a total number of $11+55+462=528$ elements.

The introduction of spinors valued in division or split-division algebras other than $\mathbf{R}$ (admitting a non-trivial conjugation) allows to split the supersymmetry algebra as follows

$$\{Q_a,Q_b\}=W_{ab},\quad \{{Q^*}_a,{Q^*}_b\}=W^*{}_{ab},\quad \{Q_a,{Q^*}_b\}=Z_{ab}, \tag{4}$$

with W_{ab} a symmetric matrix and Z_{ab} a hermitian one. Sets of constraints for the bosonic r.h.s. can be consistently imposed, see [11] and [10]. A consistent constraint sets $W_{ab}=W^*{}_{ab}=0$. In $D=11$, for the signatures supporting octonionic and split-octonionic spinors, the 4×4 (split-)octonionic valued bosonic hermitian matrix Z_{ab} admits 52 components. As in the real case, these components are accommodated into rank-1, rank-2 and rank-5 totally antisymmetric tensors. Unlike the real case, however, rank-5 tensors describe the same degrees of freedom as rank-1 and -2 tensors (whose total number is $52=11+41$, so that there is no further room to accommodate independent rank-5 tensors). The number of 41 rank-2 totally antisymmetric octonionic tensors is due to the relation $41 = 55 - 14$ (we recall that the octonions describe the $Spin(s,t)/G_2$ coset, see [9]). It can be proven [7] that the construction of [9], introduced for the octonions, can be extended also to split-octonions, leading to the split-octonionic M-algebra in $(6,5)$ signature.

The ordinary M-algebra admits an equivalent 12-dimensional presentation, named F-algebra, in terms of Maiorana-Weyl spinors for the non-Minkowskian $(10,2)$ space-time signature. This signature carries also the octonionic F-algebra whose bosonic r.h.s. is given by 12-dimensional rank-2 totally antisymmetric tensors with $52 = 66 - 14$ components. A split-octonionic version of the F-algebra is encountered for the $(6,6)$ signature of the space-time.

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