

Parametrization of the Kerr-NUT Solution

J. Gariel,¹ G. Marcihacy,² and N. O. Santos^{2,3,4}

Received January 30, 2004

The dragging of the Kerr-NUT solution does not tend to zero at infinity. To modify this solution in order to produce a good asymptotic behaviour we transform it by introducing two further parameters with the aid of a $SU(1,1)$ transformation followed by a unitary transformation. By imposing a certain relation between these parameters we obtain a new solution with a good asymptotic behaviour for any value of l , the NUT parameter. The new solution corresponds to a parametrized Kerr solution and we show that l is linked to the form of its ergosphere.

KEY WORDS: Black hole; Einstein equation; rotation.

1. INTRODUCTION

It is well known that the axisymmetric stationary Kerr-NUT (KN) solution [2] of Ernst equation has not a good asymptotic behaviour, the dragging ω does not tend to zero at infinity. This solution depends upon three parameters, one describing mass M of the source, another its angular momentum a and a third, the NUT parameter usually called l . In particular when $l = 0$ the solution reduces to Kerr.

¹ Université Pierre et Marie Curie – CNRS/FRE 2460, LERMA/ERGA, Tour 22-12, 4^{ème} étage, Boîte 142, 4 place Jussieu, 75252 Paris Cedex 05, France; e-mail: gariel@ccr.jussieu.fr

² Université Pierre et Marie Curie – CNRS/FRE 2460, LERMA/ERGA, Paris Cedex 05, France.

³ Laboratório Nacional de Computação Científica, 25651-070 Petrópolis RJ, Brazil; e-mail: santos@ccr.jussieu.fr and nos@cbpf.br

⁴ Centro Brasileiro de Pesquisas Físicas, 22290-180 Rio de Janeiro RJ, Brazil; e-mail: santos@ccr.jussieu.fr and nos@cbpf.br

The object of this paper is, starting from the KN solution, to obtain a new solution with a good asymptotic behaviour for any value of l and give to this parameter a physical interpretation which completes the one given in [3].

The method that we use to attain our aim has already been applied in a previous paper [4] to another solution of Ernst equation. The first part of this method lies in the introduction of two new parameters through a homographic transformation, belonging to the $SU(1, 1)$ group with one parameter, and a unitary transformation. The second part consists on imposing a relationship between these parameters such that they produce the required asymptotic behaviour for the new solution.

The Ernst equation is given by [5]

$$(\xi\bar{\xi} - 1)\nabla^2\xi = 2\bar{\xi}\nabla\xi \cdot \nabla\xi, \quad (1)$$

where ∇ and ∇^2 are the gradient and the three-dimensional Laplacian operators respectively, $\bar{\xi}$ is the conjugated complex potential of ξ , and in general its solution can be expressed as

$$\xi(\lambda, \mu) = P(\lambda, \mu) + iQ(\lambda, \mu), \quad (2)$$

where P and Q are real functions of the prolate spheroidal coordinates, λ a radial coordinate and μ an angular coordinate satisfying $-1 \leq \mu \leq 1$.

The KN solution ξ_{KN} of (1) is usually obtained from the Kerr solution ξ_K after a unitary transformation [6]

$$\xi_{KN} = \exp(i\theta_1)\xi_K(\lambda, \mu), \quad (3)$$

where θ_1 is a constant and

$$\xi_K = P_K + iQ_K, \quad P_K = p\lambda, \quad Q_K = q\mu, \quad (4)$$

with p and q being constants satisfying

$$p^2 + q^2 = 1. \quad (5)$$

The paper is organized as follows. In section 2 we present the $SU(1, 1)$ and unitary transformations and the new solution produced by them with two new parameters. In section 3 these parameters are determined by imposing a good asymptotic behaviour of the new solution. The two possible choices for these parameters are discussed in sections 4 and 5. The reduction to Kerr solution is presented in section 6 and the paper ends with a conclusion.

2. CONSTRUCTION OF A NEW SOLUTION OF ERNST EQUATION

To introduce two new parameters into ξ_{KN} in (3) we start with a homographic transformation $SU(1, 1)$ with two complex parameters c_1 and d_1

$$\xi_1 = \frac{c_1\xi_{KN} + d_1}{d_1\xi_{KN} + \bar{c}_1} = \frac{c_1 \exp(i\theta_1/2)\xi_K + d_1 \exp(-i\theta_1/2)}{d_1 \exp(i\theta_1/2)\xi_K + \bar{c}_1 \exp(-i\theta_1/2)}, \quad (6)$$

with

$$\begin{pmatrix} c_1 & d_1 \\ \bar{d}_1 & \bar{c}_1 \end{pmatrix} \in \text{SU}(1, 1) \subset \text{SL}(2\mathbb{C}) \quad (7)$$

and

$$|c_1|^2 - |d_1|^2 = 1. \quad (8)$$

We define

$$\exp(i\theta_1/2) \equiv p_1 + iq_1, \quad (9)$$

where p_1 and q_1 are two real constants, and to simplify the calculations we restrict ourselves to a one parameter transformation by introducing a real parameter α_1 such that

$$c_1 = 1 + i\alpha_1, \quad d_1 = \alpha_1. \quad (10)$$

With (4), (9) and (10) we can rewrite (6) like

$$\xi_1 = \frac{A_1 + iB_1}{C_1 + iD_1}, \quad (11)$$

where A, B, C and D are real quantities defined by

$$A_1 \equiv (p_1 - \alpha_1 q_1)P_K - (\alpha_1 p_1 + q_1)Q_K + \alpha_1 q_1, \quad (12)$$

$$B_1 \equiv (\alpha_1 p_1 + q_1)P_K + (p_1 - \alpha_1 q_1)Q_K + \alpha_1 p_1, \quad (13)$$

$$C_1 \equiv \alpha_1 q_1 P_K + \alpha_1 p_1 Q_K + p_1 - \alpha_1 q_1, \quad (14)$$

$$D_1 \equiv -\alpha_1 p_1 P_K + \alpha_1 q_1 Q_K - \alpha_1 p_1 + q_1. \quad (15)$$

Now we perform a unitary transformation on ξ_1 ,

$$\xi_2 = \exp(i\theta_0)\xi_1 \equiv (m + in)\xi_1, \quad (16)$$

where m and n are real constants. Then we obtain from (11) and (16)

$$\xi_2 = \frac{A + iB}{C_1 + iD_1} \equiv \frac{AC_1 + BD_1}{C_1^2 + D_1^2} + i \frac{BC_1 - AD_1}{C_1^2 + D_1^2} \equiv P_2 + iQ_2, \quad (17)$$

with

$$A \equiv mA_1 - nB_1, \quad B \equiv nA_1 + mB_1, \quad (18)$$

and

$$\begin{aligned} P_2 &\equiv -\alpha_1(\alpha_1 m + n)(P_K^2 + Q_K^2 + 1) \\ &+ \{[(1 - 2\alpha_1^2)m - 2\alpha_1 n](p_1^2 - q_1^2) - 2(n + 2\alpha_1 m)p_1 q_1\} P_K \\ &- \{(n + 2\alpha_1 m)(p_1^2 - q_1^2) + 2[(1 - 2\alpha_1^2)m - 2\alpha_1 n]p_1 q_1\} Q_K, \end{aligned} \quad (19)$$

$$\begin{aligned}
Q_2 = & -\alpha_1(\alpha_1 n - m)(P_K^2 + Q_K^2 + 1) \\
& + \{[(1 - 2\alpha_1^2)n + 2\alpha_1 m](p_1^2 - q_1^2) + 2(m - 2\alpha_1 n)p_1 q_1\} P_K \\
& - \{-(m - 2\alpha_1 n)(p_1^2 - q_1^2) + 2[(1 - 2\alpha_1^2)n + 2\alpha_1 m] p_1 q_1\} Q_K. \quad (20)
\end{aligned}$$

3. DETERMINATION OF TWO VALUES FOR α_1 OF THE SU(1, 1) TRANSFORMATION

Calculating the dragging ω from the Ernst equation (see (9) in [4]) we obtain an expression that is the quotient of two polynomials of the 6th degree each in λ (they are too long to be written here). The polynomials depend upon the parameters α_1 , p_1 , q_1 , p , q , m and n , which means that there are in fact 4 independent parameters since from (5), (9) and (16) we have

$$p^2 + q^2 = 1, \quad p_1^2 + q_1^2 = 1, \quad m^2 + n^2 = 1. \quad (21)$$

In order to have ω tending to zero at infinity we have to cancel the coefficient of highest order of λ in the numerator of ω , producing two independent values for α_1 , which we call α_{1a} and α_{1b} ,

$$2\alpha_{1a} = \cot\left(\frac{\theta_0}{2}\right) - \tan\left(\frac{\theta_1}{2}\right) = \frac{m+1}{n} - \frac{q_1}{p_1}, \quad (22)$$

$$2\alpha_{1b} = \cot\left(\frac{\theta_0}{2}\right) + \cot\left(\frac{\theta_1}{2}\right) = \frac{m+1}{n} + \frac{p_1}{q_1}. \quad (23)$$

We recall that θ_0 , or equivalently from (16) $n = \sin \theta_0$, is the parameter associated to the unitary transformation; that θ_1 , or equivalently from (9) $p_1 = \cos(\theta_1/2)$, is the parameter associated to the NUT parameter l by the relation $l = M \tan(\theta_1/2)$ as well as to another unitary transformation of the Kerr solution; and α_1 is associated to the SU(1, 1) transformation. Hence, it remains two independent parameters, θ_0 and l or θ_1 , plus Kerr parameters for the mass and angular momentum.

One can verify that for $q_1 = 0$, implying $p_1 = 1$, α_{1a} in (22) reduces to the expression (19) of the parameter α_1 of the SU(1, 1) transformation used in [4].

4. SOLUTION FOR α_{1a}

For this solution we obtain for the gravitational potentials f and ω (see (1) in [4])

$$f = \frac{(p^2\lambda^2 + q^2\mu^2 - 1)m_2}{(p\lambda - 1)^2 + q^2\mu^2}, \quad (24)$$

$$\omega = \frac{2kq(p\lambda - 1)(1 - \mu^2)}{p(p^2\lambda^2 + q^2\mu^2 - 1)m_2}. \quad (25)$$

where $k > 0$ is a constant and

$$m_2 \equiv \cos^2\left(\frac{\theta_1}{2}\right) \csc^2\left(\frac{\theta_0}{2}\right) = \frac{2p_1^2}{1-m}, \quad |m| < 1. \quad (26)$$

If $m_2 = 1$ then (24) and (25) reduce to the Kerr solution.

To interpret the parameters k , p and q we introduce the Boyer-Lindquist coordinates,

$$\lambda = \frac{r-M}{k}, \quad \mu = \cos\theta. \quad (27)$$

where r and θ are the radial and angular coordinates and M the Kerr mass. Then with (27) we obtain asymptotically for (24) and (25)

$$f \approx 1 + \frac{2k}{p} \frac{1}{r} + O\left(\frac{1}{r^2}\right), \quad (28)$$

$$\omega \approx \frac{k^2 q (1-m) \sin^2\theta}{p^2 p_1^2} \frac{1}{r} + O\left(\frac{1}{r^2}\right). \quad (29)$$

Hence, from (28) and (29) we have to choose

$$k = (M^2 - a^2 m_2^2)^{1/2}, \quad p = -\frac{k}{M}, \quad q = \frac{am_2}{M}. \quad (30)$$

where M is the mass and a is the angular momentum of the source. With these values for k , p and q we can finally write for (24) and (25)

$$f = 1 - \frac{2Mr}{r^2 + a^2 m_2^2 \cos^2\theta}, \quad (31)$$

$$\omega = \frac{2aMr \sin^2\theta}{r^2 - 2Mr + a^2 m_2^2 \cos^2\theta}. \quad (32)$$

5. SOLUTION FOR α_{1b}

The gravitational potentials f and ω now become

$$f = \frac{(p^2 \lambda^2 + q^2 \mu^2 - 1)m_3}{(p\lambda + 1)^2 + q^2 \mu^2}, \quad (33)$$

$$\omega = \frac{2kq(p\lambda + 1)(1 - \mu^2)}{p(p^2 \lambda^2 + q^2 \mu^2 - 1)m_3}, \quad (34)$$

where

$$m_3 \equiv \sin^2\left(\frac{\theta_1}{2}\right) \csc^2\left(\frac{\theta_0}{2}\right) = \frac{2q_1^2}{1-m}, \quad |m| < 1. \quad (35)$$

Here too if $m_3 = 1$ then (33) and (34) reduce to the Kerr solution.

To interpret k , p and q we use the same previous procedure and we obtain asymptotically for (33) and (34)

$$f \approx 1 - \frac{2k}{p} \frac{1}{r} + O\left(\frac{1}{r^2}\right), \quad (36)$$

$$\omega \approx -\frac{k^2 q (1-m) \sin^2 \theta}{p^2 q_1^2} \frac{1}{r} + O\left(\frac{1}{r^2}\right), \quad (37)$$

and we have to choose

$$k = (M^2 - a^2 m_3^2)^{1/2}, \quad p = \frac{k}{M}, \quad q = \frac{a m_3}{M}. \quad (38)$$

With (38) the solution (33) and (34) becomes

$$f = 1 - \frac{2Mr}{r^2 + a^2 m_3^2 \cos^2 \theta}, \quad (39)$$

$$\omega = \frac{2aMr \sin^2 \theta}{r^2 - 2Mr + a^2 m_3^2 \cos^2 \theta}. \quad (40)$$

6. REDUCTION TO KERR SOLUTION

We observe that the two solutions (31, 32) and (39, 40) are identical if the two parameters m_2 and m_3 are identified. Furthermore, these two solutions can be written in the form of Kerr if one introduces the transformation

$$M_1 = \frac{M}{m_2}, \quad r_1 = \frac{r}{m_2}, \quad (41)$$

without changing a , or a similar transformation for m_3 . These transformations introduce a parametrization of the Kerr solution, which has been studied in [4]. There it is shown that varying m_1 (see (30) in [4]), here m_2 or m_3 , produces a topological deformation of the Kerr ergosphere. Since m_2 and m_3 are linked, through (26) and (35), to θ_1 which constitutes the third parameter l in the KN metric, we can say that l is responsible for the form of the ergosphere of the new solution which has a suitable asymptotic behaviour, i.e. $\omega \rightarrow 0$ for $r \rightarrow \infty$, for any value of l .

7. CONCLUSION

Starting from KN solution, which is not asymptotically well behaved, we built a new one after using a one parameter group $SU(1, 1)$ transformation followed by a unitary transformation. The two new parameters thus introduced allowed us to

choose a relationship between them and M , a and l , given by (22) and (23), producing two new solutions with good asymptotic behaviour, $\omega \rightarrow 0$ when $r \rightarrow \infty$. The two transformations do not commute and only in the order here presented allowed to have conditions for the required asymptotic behaviour. Both new solutions have the limit of Kerr parametrized solution as studied in [4]. The two new parameters involved in the solutions have range $m_2 \in (0, \infty)$ and $m_3 \in (-\infty, 0)$. But since only the square of these parameters are involved in the two sets of solutions, (32, 33) and (40, 41), then both correspond in fact to the same solution. The convergence of these two solutions into a single one respects the unicity of Kerr solution. The parameters m_2 and m_3 describe the form of the ergosphere and are linked to the NUT parameter l . Hence we interpret l as being responsible of the form of the ergosphere.

REFERENCES

- [1] Newman, E. T., Tamburino, L. A., and Unti, T. (1963). *J. Math. Phys.* **4**, 915. A1
- [2] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., and Herlt, E. (2003). *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge, UK.
- [3] Aguilar-Sánchez, J. A., García, A. A., and Manko, V. S. (2001). *Grav. Cosmol.* **7**, 149.
- [4] Gariel, J., Marilhacy, G., and Santos, N. O. (2003). *Gen. Relat. Grav.* **35**, 1549.
- [5] Ernst, F. J. (1968). *Phys. Rev.* **167**, 1175.
- [6] Carmeli, M. (2001). *Classical Fields: General Theory and Gauge Theory*, World Scientific, Singapore.

Queries to Author:

- A1. Au: Reference is not cited Kindly cite it at an appropriate place in the text or delete it from the list.