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SIMPLE RENORMALIZATION GROUP METHOD FOR CALCULATING GEOMETRICAL AND OTHER EQUATIONS OF STATES

by

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Abstract

We describe a real space renormalization group procedure to calculate geometrical and thermal equations of states for the entire range of values of the external parameters. Its use is as simple as a Mean Field Approximation; however, it yields non trivial results and can be systematically improved. We illustrate it by calculating, for all bond concentrations, the site mass density for the complete and the backbone percolating infinite clusters in square lattice: the results are quite satisfactory.

Key-words: Equation of states; Renormalization group; Percolation; Backbone mass density.

PACS INDEX: 05.20.-y; 64.10.+h; 75.40.-s; 75.40.Bw.

Renormalization group (RG) techniques have been employed mainly to evaluate critical points and exponents. However a RG theory has been developped[1] to evaluate the free energy for the *whole range* of the thermodynamic parameters. From the free energy all other quantities of interest (such as magnetization, specific heat, susceptibility and so on) can be evaluated. This theory and similar ones[2] apply to systems which can be described by a Hamiltonian formalism.

In this paper we want to develop a *simpler* real space RG formalism which, besides covering Hamiltonian systems, may be useful in particular to calculate, for the *whole* range of variation of external parameters such as concentration, *geometrical* quantities which cannot be described by a Hamiltonian formalism. These include, for example, the density of sites in a full percolating cluster and in its backbone for different types of percolation problems (random, correlated, bootstrap, valence, chromatic), and the conductivity for random resistor and random superconducting networks, etc. An intensive effort has already been accomplished with great success in the use of *large cell* RG techniques[3] for calculating critical points and exponents. The present method illustrates *how* this effort can be easily adapted (and systematically improved) to extract, with precision, the interesting quantities for the *whole range of variation* of the external parameters. Last but not least, let us compare the present procedure with the standard Mean Field Approximation: (i) they share the fact that the entire equation of states is obtained; (ii) for small cells it is, operationally speaking, even simpler; (iii) it yields non trivial results which, as said

before, can be systematically improved.

To illustrate this method we consider first the particular case of random bond percolation. In this problem one supposes that each bond of a d-dimensional hypercubic lattice of linear size L has a probability p of being active. In the $L\to\infty$ limit, the order parameter P_∞ can be defined as $P_\infty = N_L(p)m_0/L^d$, where $N_L(p)$ is the average number of sites in the infinite cluster and m_0 is the adimensional "mass" associated with each site. Usually no such m_0 is introduced (i.e., $m_0=1$), but here we leave it as a variable since it will change under renormalization. Following the original idea of scaling as introduced by Kadanoff, we divide the system of L^d sites into a system of L'^d cells of linear size $b=L/L' > 1$. Then we associate with each cell new renormalized variables $p'=f_1(p,m_0)$ and $m_0'=f_2(p,m_0)$. The renormalized variables depend on the particular RG, but they all satisfy the condition that the total mass of the infinite cluster in the system of cells be identical to the mass of the infinite cluster in the original system, namely

$$N_{L'}(p')m_0' = N_L(p)m_0 \tag{1}$$

Dividing both terms by L^d we obtain

$$P_\infty(p')m_0' = b^d P_\infty(p)m_0 \tag{2}$$

After n iterations from (2) we have

$$P_\infty(p^{(n)})m_0^{(n)} = b^{nd}P_\infty(p)m_\infty \tag{3}$$

hence

$$P_\infty(p) = \lim_{n\to\infty} P_\infty(p^{(n)}) \frac{m_o^{(n)}}{b^{nd} m_o} \qquad (4)$$

For p less than the percolation threshold p_c, $\lim_{n\to\infty} p^{(n)}=0$. Since $P_\infty(0)$ and $\lim_{n\to\infty} m_o^{(n)}/b^{nd}=0$, we have the expected result $P_\infty(p)=0$, *for all* $p < p_c$. For $p > p_c$, $\lim_{n\to\infty} p^{(n)} = 1$, and since $P_\infty(1)=m_o$, we find

$$P_\infty(p) = \lim_{n\to\infty} \frac{m_o^{(n)}(p)}{b^{nd}} \qquad (p > p_c) \qquad (5)$$

This gives the desired expression of P_∞ *for all values of* p.

Let us illustrate the procedure on the square lattice by constructing a RG (noted RG_{12}) which maps the cell of Fig. 1(b) into that of Fig. 1(a) (hence b=2). The renormalized bond probability p' is given by the probability of connecting the bottom to the top of the cells, namely

$$p' = p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + 2p^2(1-p)^3 \qquad (6)$$

To find the equation for m_o we consider all possible cell configurations and the corresponding total mass of the spanning cluster. Then we impose that the average mass is preserved, and obtain

$$p'm_o' = [p^5 + 4p^5(1-p) + p^4(1-p) + \frac{21}{2} p^4(1-p)^2 + \frac{13}{2} p^3(1-p)^3 + p^2(1-p)^4]4m_o \qquad (7)$$

which, together with Eqs. (5) and (6), provides $P_\infty(p)$ for all values of p (see Fig. 2(a)). The resulting critical exponents ν (for the connectedness length $\xi(p)$) and β (for $P_\infty(p)$) are given

in Table 1. We have also performed the calculation renormalizing the cell of Fig. 1(c) into that of Fig. 1(a) (RG_{13};b=3), and into that of Fig. 1(b) (RG_{23};b=3/2). The results improve as shown in Fig. 2(a) and Table 1.

Backbone percolation. We have also calculated the mass density $P_\infty^B(p)$ in the backbone of the spanning cluster. The calculation is analogous to the percolation probability $P_\infty(p)$. The RG_{12} equation for p' still is Eq. (6), and Eq. (7) is replaced by

$$p'm'_{OB} = [p^5+4p^5(1-p)+p^4(1-p)+\frac{19}{2}p^4(1-p)^2+6p^3(1-p)^3+p^2(1-p)^4]4m_{OB} \qquad (8)$$

The full curve for $P_\infty^B(p)$ is given in Fig. 2(b); the corresponding critical exponent β_B equals $\ell n(64/49)/\ell n(13/8) \simeq 0.550$, to be compared with the Monte Carlo result[5] $\beta_B \simeq 0.53$.

Random resistor network. In this problem one supposes that each bond of a d-dimensional hypercubic lattice of linear size L curries a conductance g_o with probability p and O with probability (1-p). Following the same procedure as before we find for the average conductivity $\sum(p)$ the relation

$$\sum(p^{(n)})g_o^{(n)} = b^{n(d-2)}\sum(p)g_o \qquad (9)$$

where $p^{(n)}$ and $g_o^{(n)}$ respectively are the renormalized probability and conductance after n iterations. From (9) we have, for $p < p_c$, $\sum(p) = 0$, and, since $\sum(1) = g_o$, we have, for $p > p_c$,

$$\sum(p) = \lim_{n\to\infty} \frac{g_o^{(n)}(p)}{b^{n(d-2)}} \tag{10}$$

Calculations of $\sum(p)$ along similar lines are already available in the literature[6].

Connectedness length. The standard RG equation for the connectedness (correlation) length $\xi' = \xi/b$ can be rewritten as follows

$$\xi(p^{(n)}) = \frac{\xi(p)}{b^n} \tag{11}$$

In this case, ξ vanishes in both trivial fixed points p=0 and p=1. Therefore, in order to calculate $\xi(p)$, one needs to know the asymptotic behavior of ξ in the vicinity of these fixed points, which usualy can be done very easily within the RG framework itself. This problem has been investigated with very good results for the correlation length of the q-state Potts ferromagnet in d-dimensional hypercubic lattice[7].

Surface tension. Closely related to the correlation length, the (longitudinal) surface tension $\gamma(T)$ can be calculated as a function of the temperature T for all temperatures. The RG equation is given (see[8] and references therein) by

$$\frac{\gamma(T')}{T'} = b^{d-1}\,\frac{\gamma(T)}{T} \tag{12}$$

which, after n iterations, becomes

$$\frac{\gamma(T^{(n)})}{T^{(n)}} = b^{n(d-1)} \frac{\gamma(T)}{T} \tag{13}$$

Hence

$$\gamma(T) = T \lim_{n\to\infty} \frac{\gamma(T^{(n)})}{b^{n(d-1)} T^{(n)}} \tag{14}$$

By using the facts that $\gamma(\infty) = 0$ and that $\gamma(0)$ is expected to be finite, the entire function $\gamma(T)$ can be calculated. This has already been performed with very satisfactory results for the Ising model in square lattice[8].

In conclusion we have described a method based on real space renormalization group which allows to obtain the variation of various interesting quantities *for all values of the external parameters*. This method applies especially to geometrical quantities for which large cell renormalization groups have successfully been employed to get accurate critical points and exponents; it uses essentially the *same information*, provides the *full variation* of the quantities, and is normally expected to recover the *correct asymptotic behaviors* in the vicinity of the trivial fixed points, i.e. when $\xi \to 0$ ($p=0,1$ for percolation problems). For thermal problems, this method is, for small cells, operationally even simpler than the Mean Field Approximation; it gives, however, non trivial results, and can be systematically improved.

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CAPTION FOR FIGURES AND TABLE

Fig. 1 - RG cells used for the calculation of $P_\infty(p)$ and $P_\infty^B(p)$ (see Fig. 2) for square lattice. RG_{12}: cell (b) is renormalized into cell (a), hence b=2.RG_{13}: cell (c) is renormalized into cell (a), hence b=3.RG_{23}: cell (c) is renormalized into cell (b), hence b=3/2.

Fig. 2 - (a) RG percolation probability P_∞ and (b) RG backbone mass (reduced) density P_∞^B as functions of the bond concentration p for the square lattice. b is the RG linear expansion (see Fig. 1); p_c=1/2 for all values of b; the dots represent Monte Carlo data[4].

Table 1- RG critical exponents $\nu(\xi \propto |p-p_c|^{-\nu})$ and $\beta(P_\infty \propto (p-p_c)^{\beta})$ for several values of b (see Fig. 1).

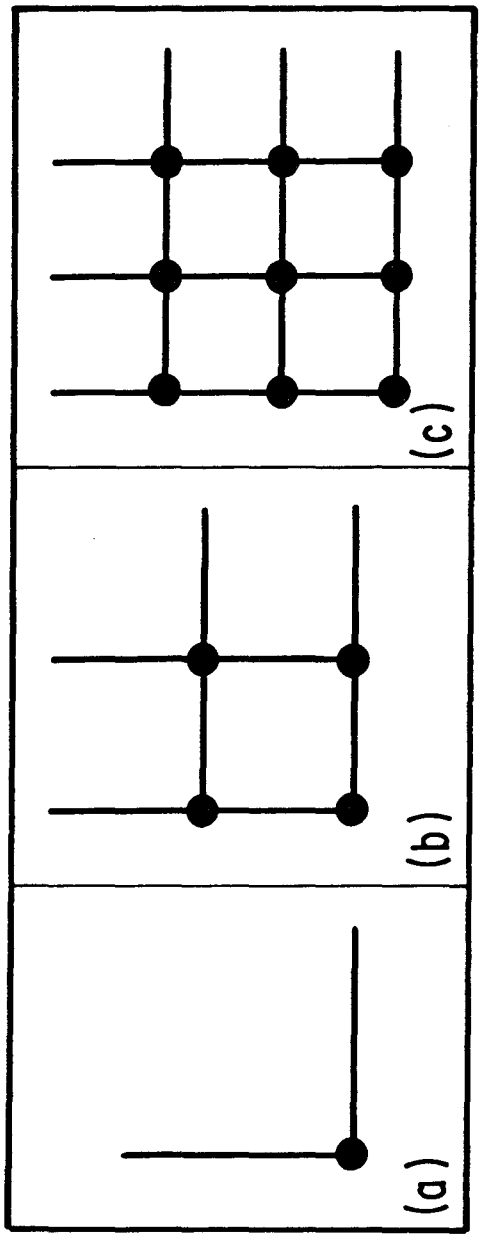

FIG.1

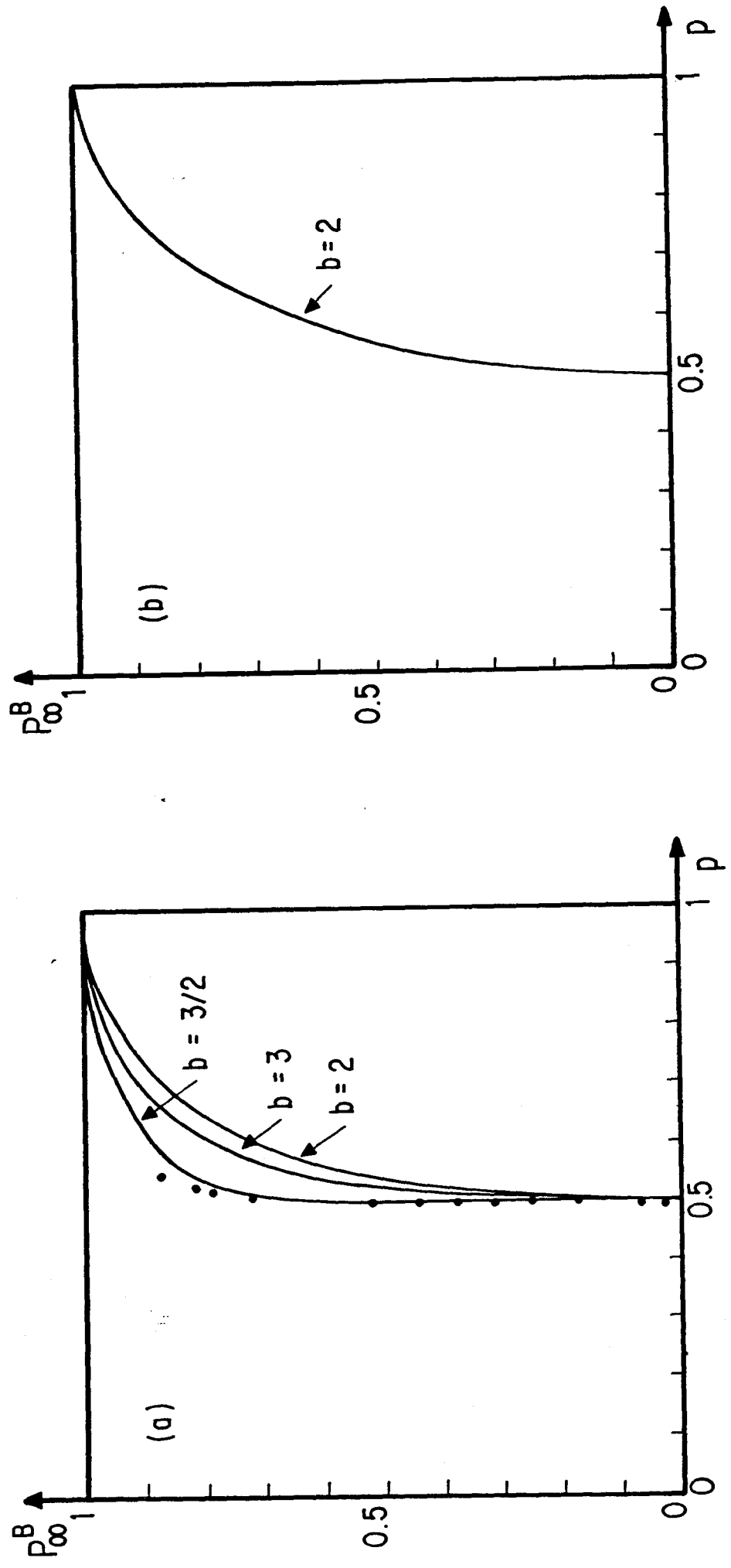

FIG.2

TABLE 1

	b=2	b=3	b=3/2	exact
ν	$\frac{\ell n 2}{\ell n \frac{13}{8}} \simeq 1.428$	$\frac{\ell n 3}{\ell n \frac{4541}{2048}} \simeq 1.380$	$\frac{\ell n \frac{3}{2}}{\ell n \frac{4541}{3328}} \simeq 1.305$	$\frac{4}{3} \simeq 1.333$
β	$\frac{\ell n \frac{16}{13}}{\ell n \frac{13}{8}} \simeq 0.428$	$\frac{\ell n \frac{49152}{37547}}{\ell n \frac{4541}{2048}} \simeq 0.338$	$\frac{\ell n \frac{39936}{37547}}{\ell n \frac{4541}{3328}} \simeq 0.198$	$\frac{5}{36} \simeq 0.139$