A0015/76
MAI, 1976

THE ELECTROMAGNETIC INTERACTIONS OF SCALAR PARTICLES IN RIEMANN SPACE - TIMES

Carlos A.P. Galvão

Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil

THE ELECTROMAGNETIC INTERACTIONS OF SCALAR PARTICLES IN RIEMANN SPACE - TIMES

Carlos A.P. Galvão*
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brasil

(Received April 1, 1976)

ABSTRACT

An analysis is made of the interaction between classical electromagnetic and gravitational fields and quantized complex scalar fields. In the case of homogeneous electromagnetic fields and a cosmological model of Kasner's type, we obtain explicit expressions for the parameters of the Bogoliubov transformation and for the number of created particles.

I - The Interaction Between Classical Gravitational and Scalar Fields

In this section we briefly review the interaction between classical gravitational and complex scalar fields in order to obtain some expressions which shall be used after. We also analyse the properties of the scalar fields under conformal

---

* On leave of absence from the Department of Physics, U.F.R.N., Natal, Brasil. Partially supported by C.A.P.E.S., Brasil.

transformations. In what follows, the sources of the gravitational and scalar fields are supposed to be specified independently of each other. The metric tensor has signature (+ - - -) and we use a system of units in which Planck's constant, the velocity of light in vacuum and tha gravitational constant are equal to one. In our conventions, for any arbitrary vector field V_α we set $V_{\mu||\lambda||\rho} - V_{\mu||\rho||\lambda} = R_{\mu\varepsilon\lambda\rho} V^\varepsilon$, where the double bar means covariant derivative.

It is well known[1,2] that conformal invariance of the equation of motion for the scalar fields has significant consequences on the mechanism of creation of particles by a non-stationary gravitational field. Due to this fact we shall consider the scalar field conformally coupled to the gravitational field and so the Lagrangean is

$$\mathcal{L} = \frac{1}{2}\left(g^{\alpha\beta}\phi_{|\alpha}\phi^{*}_{|\beta} - (m^2 + \frac{1}{6}R)\phi^{*}\phi\right) \qquad \text{(I-1)}$$

where $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar of curvature. From the Euler--Lagrange equations we obtain,

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\; g^{\alpha\beta}\phi_{|\alpha}\right)_{|\beta} + (m^2 + \frac{1}{6}R)\phi = 0 \qquad \text{(I-2)}$$

If m=0 this equation is invariant under the conformal transformation defined by $\phi \rightarrow \overline{\phi} = \Omega^{-1}(x)\phi$ and $g_{\mu\nu} \rightarrow \overline{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$.

Associated with the scalar field, the object of main interest in Cosmology and Astrophysics is the energy-momentum tensor $T_{\mu\nu}(x)$ which is defined as

$$\delta \int d^4 x \mathcal{L} = \int T_{\alpha\beta}\,\delta g^{\alpha\beta}\, d^4 x$$

The result of the calculation gives

$$T_{\alpha\beta} = \phi^{*}_{|\alpha}\phi_{|\beta} - g_{\alpha\beta}\mathcal{L} + \frac{1}{6}(R_{\alpha\beta} - g_{\alpha\beta} \quad + \nabla_\alpha\nabla_\beta)\phi^{*}\phi \qquad (I-3)$$

where $\Box \equiv g^{\lambda\rho}\nabla_\lambda\nabla_\rho$ and ∇_λ is the operator of covariant derivative. This tensor is symmetric, trace-free and divergenceless. With the help of the energy-momentum tensor as given above one obtains the conserved quantities S^α associated with the scalar field as

$$S^\alpha = T^\alpha{}_\beta\, \xi^\beta \qquad (I-4)$$

where ξ^β is the Killing vector field associated with the given metric tensor, defined as $\xi_{\mu||\lambda} - \xi_{\lambda||\mu} = 0$.

The momenta π, π^* conjugated to the fields ϕ and ϕ^* are defined as

$$\pi = \frac{\delta\mathcal{L}}{\delta\phi_{|o}} = \sqrt{-g}\; g^{o\mu}\; \phi^{*}_{|\mu} \quad , \quad \pi^{*} = \frac{\delta\mathcal{L}}{\delta\phi^{*}_{|o}} = \sqrt{-g}\; g^{o\mu}\phi_{|\mu} \qquad (I-5)$$

With these quantities we may calculate the Hamiltonian H defined in the usual way,

$$H = \int d^3x(\pi\phi_{|o} + \pi^{*}\phi^{*}_{|o} - \mathcal{L})$$

$$= \int d^3x \left[\frac{\pi\pi^{*}}{g^{oo}\sqrt{-g}} - \frac{\pi g^{oj}}{g^{oo}}\phi_{|j} - \frac{\pi^{*} g^{oj}}{g^{oo}}\phi^{*}{}_{|j} \right. \qquad (I-6)$$

$$\left. + \sqrt{-g}\left(\frac{g^{ok}}{g^{oo}}\phi_{|k}\frac{g^{oj}}{g^{oo}}\phi^{*}_{|j} - g^{ij}\phi^{*}_{|i}\phi_{|j} + (m^2 + \frac{1}{6}R)\phi^{*}\phi\right)\right]$$

Hamilton's equations of motion are

$$\frac{\partial\pi}{\partial x^o} = -\left(\frac{\pi g^{oj}}{g^{oo}}\right)_{|j} + \left(\frac{g^{ok}}{(g^{oo})^2}\phi^{*}_{|k}\, g^{oi}\right)_{|i} - \left(\sqrt{-g}\; g^{ij}\phi^{*}_{|j}\right)_{|i} - (m^2 + \frac{1}{6}R)\phi^{*}$$

$$\frac{\partial\phi}{\partial x^o} = \frac{1}{g^{oo}\sqrt{-g}}\left(\pi^{*} - \sqrt{-g}\; g^{oj}\phi_{|j}\right) \qquad (I-7)$$

For details see[3,4].

II - Field Quantization in a Non-Stationary Riemann Space-Time

To begin with, we shall consider a general linear free field ψ in a given Riemann space-time. The quantization procedure can be carried out in the same way as in the flat space-time case. Let

$$\phi\psi = 0 \qquad \text{(II-1)}$$

be the dynamical equation satisfied by the field ψ , obtained from a Lagrangian $\mathcal{L}$ through the Euler-Lagrange equations $\frac{\delta\mathcal{L}}{\delta\psi} = 0$. ϕ is a self-adjoint operator.

Let us assume that in the region Ω of the space-time of interest there are complete space-like Cauchy hypersurfaces σ for the equation (II-1). We introduce the equal time canonical commutation relations

$$\begin{aligned} [\psi(\vec{x},t),\psi(\vec{x}',t)] &= [\pi(\vec{x},t),\pi(\vec{x}',t)] = 0 \\ [\psi(\vec{x},t),\pi(\vec{x}',t)] &= i\delta^{(3)}(\vec{x}-\vec{x}') \end{aligned} \qquad \text{(II-2)}$$

where $\pi = \frac{\delta\mathcal{L}}{\delta\dot{\psi}}\Big|_0$ is the momentum conjugated to the field $\psi(x)$.

The next step should be a choice of the representation of the commutation relations (II-2). However, some difficulties appear at this point which are consequences of the fact that we are considering a non-stationary space-time[3]. As we shall see later, this choice is not unique due to the fact that there is not a conserved energy as a generator of the symmetry group. So, the vacuum state as defined in the usual way is not uniquely determined.

We shall further proceed to stablish the formalism. Let ψ_1 and ψ_2 be two complex solutions of equation

(II-1). Because ϕ is self-adjoint we may define an inner product for ψ_1 and ψ_2 in the usual way.

We may now introduce a complet set of conjugated pairs of solutions ϕ_j and ϕ_j^* of equation (II-1) which satisfy the orthornormality conditions

$$(\phi_i,\phi_j) = \delta_{ij} \quad , \quad (\phi_i^*,\phi_j) = 0 \tag{II-4}$$

and the following expansion for the field ψ :

$$\psi = \sum_k (A_k\phi_k + A_k^\dagger\phi_k^*) \tag{II-5}$$

In the above expansion A_k and $A_k^\dagger$ are operators which due to the commutation relations (II-2) must satisfy the following commutation relations

$$[A_k, A_{k'}^\dagger] = \delta(k,k') \quad , \quad [A_k, A_{k'}] = 0 \tag{II-6}$$

which define the algebra of these operators. Then, it is possible to define a Fock space of state vectors and a vacuum state by the relation

$$A_k|0\rangle = 0 \tag{II-7}$$

for any value of k.

Let us apply the scheme developed above for the case where ψ is a complex scalar field and the space-time is homogeneous with metric

$$dS^2 = dt^2 - g_{ij}(t)dx^i dx^j \tag{II-8}$$

The equation of motion for the scalar field is given by (I-2).

We may write the following expression for the scalar field:

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k \left[A_k\Phi_k(t)\psi_k(\vec{x}) + \underline{B}_k^\dagger\Phi_k^*(t)\psi_k^*(\vec{x})\right] \tag{II-9}$$

In (II-9), $\psi_k(\vec{x})$ are eigen-functions of the three dimensional Laplacian operator with eigen-values $-k^2$. Clearly, we have $\psi_k(x) \sim e^{\pm i\vec{k}.\vec{x}}$. The time dependent functions $\Phi_k(t)$ satisfy the equation

$$\frac{d^2}{dt^2}\Phi_k + \frac{1}{V}\left(\frac{dV}{dt}\right)\frac{d\Phi_k}{dt} + \Omega_k^2(t)\Phi_k = 0 \qquad \text{(II-10)}$$

where

$$\Omega_k^2(t) = m^2 + \frac{1}{6}R - g^{ij}(t)k_i k_j \quad , \quad \begin{matrix} V = g_{ij}g_{ij} \\ \text{(no sum)} \end{matrix} \qquad \text{(II-11)}$$

Due to the commutation relations (II-6) the functions $\Phi_k(t)$ must satisfy the relation

$$\Phi_k\dot{\Phi}_k^* - \Phi_k^*\dot{\Phi}_k = \frac{i}{V} \qquad \text{(II-12)}$$

It is clear that the functions Φ_k satisfying equation (II-10) are not uniquely determined by the above condition and so we may have different non equivalent representations for the canonical commutation relations (II-2) as we observed before. In other words, we may have alternative sets of operators $(\bar{A},\bar{B})$ in which the field may be expanded, and these operators will be related to the old ones by a time dependent Bogoliubov transformation(4,5):

$$\begin{aligned} \bar{A}_k &= \alpha_k^*(t)A_k + \beta_k(t)\underline{B}_k \\ \bar{B}_{-k} &= \beta_k^*(t)A_k + \alpha_k(t)\underline{B}_k \end{aligned} \qquad \text{(II-13)}$$

Due to the commutation relations (II-6) we have the following relation between the functions $\alpha_k(t)$ and $\beta_k(t)$:

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1 \qquad \text{(II-14)}$$

To see how this transformation comes into play, let us consider the case where the gravitational field is assymp totically flat in $t \to \pm\infty$. We shall write the scalar field in the Heisenberg representation as

$$\phi(x) \sim \int d^3k \, \frac{1}{\sqrt{2\Omega_k^{(-)}}} \, (\bar{A}_k(t)+\bar{B}_k(t))e^{+i\vec{k}.\vec{x}}$$
$$= \int d^3k \; G_k(t) \; e^{+i\vec{k}.\vec{x}} \qquad \text{(II-15)}$$

where $\Omega_k^{(-)} = \lim_{t\to-\infty} \Omega_k(t)$. Let us introduce the new variables defined by

$$dt = V^{1/3} \, d\eta \;, \quad \Phi_k = V^{-1/3}\psi_k \qquad \text{(II-16)}$$

Equation (II-10) then turns into

$$\frac{d^2}{d\eta^2} \psi_k + \Omega_k^2(\eta)\psi_k = 0 \qquad \text{(II-17)}$$

Note that the Wronskian $W(\psi_k,\psi_k^*)$ of any two solutions of (II-17) has the value $2i\Omega_k^{(-)}$.

The operator $G_k(\eta)$ in (II-15) satisfy the same equation as the function $\psi_k(\eta)$, namely

$$\frac{d^2}{d\eta} G_k + \Omega_k^2(\eta)G_k = 0 \qquad \text{(II-18)}$$

The solution to equation (II-18) may be written as

$$\bar{A}_k = \alpha_k(\eta)A_k + \beta_k(\eta)B^\dagger_{-k}$$
$$\bar{B}_k = \beta_k^*(\eta)A_k + \alpha_k(\eta)B^\dagger_{-k} \qquad \text{(II-19)}$$

where

$$\alpha_k^*(\eta) = \frac{1}{2\Omega_k^{(-)}} \, (\Omega_k^{(-)}\psi_k^* + i \, \frac{d\psi_k^*}{d\eta})$$
$$\beta_k(\eta) = \frac{1}{2\Omega_k^{(-)}} \, (\Omega_k^{(-)}\psi_k + i \, \frac{d\psi_k}{d\eta}) \qquad \text{(II-20)}$$

and A_k, B_{-k} are the operators $\bar{A}_k$ and $\bar{B}_{-k}$ in the limit $t\to\pm\infty$. Condition (II-14), which guarantee that the commutation relations if imposed at one time will hold for any subsequent time, can be easily verified to hold as $|\alpha_k(n)|^2-|\beta_k(n)|^2=$ $= \frac{1}{2i\Omega_k^{(-)}} W(\phi_k,\psi_k^*) = 1$, in virtue of (II-20).

It should be observed that charge conservation still holds, as it should be. From equations (II-19) we see that only operators of the type A, $B^\dagger$ and $A^\dagger$, B are mixed due to the process of evolution.

Let us now turn to equations (II-15). If the function $\beta_k(t)$ does not vanish then the vacuum as defined by (II-7) "changes with time". If we suppose that there were no particles present at the initial instant t_0 (which corresponds to the initial conditions $\beta_k(t_0) = 1$, and $\alpha_k(t_0)=0$ imposed on the functions β_k and α_k), so that ${}_{t_0}\langle 0|\bar{A}_k^\dagger \bar{A}_k|0\rangle_{t_0} = 0$ then at later times we shall have ${}_{t}\langle 0|\bar{A}_k^\dagger \bar{A}_k|0\rangle_{t} \neq 0$ and so the old vacuum contains particles. This is a consequence of the metric being non stationary, which means that these new particles are created due to the expansion (contraction) of the Universe. Note that if m=0 and the space-time is conformally flat, there is no particle creation due to the expansion of the Universe[6,7]. A simple calculation shows that the total momentum associated with the created particles as given by $P_j = \int d^3x\sqrt{-g}\, T_j^0$ vanishes and so, the new particles are created in pairs.

One of the main difficulties of this theory is that we cannot always distinguish between positive and negative frequence solutions[4,7]. These concepts may not have a meaning depending on what kind of geometry we are working with. The only situation in which we can distinguish between the two kinds of frequency is that with a stationary geometry or in the case where the Universe is assymptotically flat.

We mention also that some problems arise when we calculate the mean value of the energy-momentum tensor $T_{\mu\nu}$ associated with the ϕ-field as we find it expressed in terms of divergent integrals. We shall not be concerned with this problem and refer the interested reader to the existent literature[7,8].

III - The Electromagnetic Interaction of Scalar Particles

We shall consider now the interaction between a quantized complex scalar field and classical electromagnetic and gravitational fields. It is supposed that the scalar field is conformally coupled with the gravitational field and minimally coupled with the electromagnetic field. The Lagrangian for the ϕ-field is now

$$\mathcal{L} = (\phi^{\dagger}_{|\mu} + ieW_{\mu}\phi^{\dagger})(\phi^{|\mu} - ieW^{\mu}\phi)-(m^2+\tfrac{1}{6}R)\phi^{\dagger}\phi \qquad \text{(III-1)}$$

where W_{μ} is the electromagnetic potential. The part of the Lagrangian which describes the interaction between the electromagnetic and scalar fields may be written as

$$\mathcal{L}_I = - J^{\mu}W_{\mu} \qquad \text{(III-2)}$$

where J^{μ} is the current defined as

$$J^{\mu} = ig^{\mu\nu}\left[(\phi^{\dagger}_{|\nu} + ie\phi^{\dagger}W_{\nu})\phi-\phi^{\dagger}(\phi_{|\nu}-ieW_{\nu})\right] \qquad \text{(III-3)}$$

The equation of motion of the scalar field becomes

$$\frac{1}{\sqrt{-g}}\,(g^{\alpha\beta}\sqrt{-g}\,\phi_{|\alpha})_{|\beta} - 2ieg^{\alpha\beta}W_\alpha\phi_{|\beta} - e^2 g^{\alpha\beta}W_\alpha W_\beta\phi + (m^2+\tfrac{1}{6}R)\phi = 0 \qquad \text{(III-4)}$$

We observe that the presence as the electromagnetic field breaks the conformal invariance of equation (III-4) for massless ϕ-field, the conformal transformation being defined as $\phi\to\bar{\phi} = \Omega^{-1}(x)\phi$, $g_{\mu\nu}\to\bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$, and $W^\mu\to\bar{W}^\mu = \Omega^{-2}(x)W^\mu$. Conformal invariance is mantained if the space-time is conformally flat and the potential vector is of the form $W_\mu = (0,\vec{W}(\vec{x}))$ (9).

The energy-momentum tensor is now given by

$$\begin{aligned} T_{\alpha\beta}(x) = {} & \phi^\dagger_{|\alpha}\phi_{|\beta} - \tfrac{1}{2}\,g_{\alpha\beta}(g^{\mu\nu}\phi^\dagger_{|\mu}\phi_{|\nu} - (m^2+\tfrac{1}{6}R)\phi^\dagger\phi) \\ & + \tfrac{1}{6}(R_{\alpha\beta}-g_{\alpha\beta}\Box+\nabla_\alpha\nabla_\beta)\phi^\dagger\phi \\ & - ie(\phi^\dagger_{|\alpha}\phi - \phi^\dagger\phi_{|\alpha})W_\beta + e^2W_\alpha W_\beta\phi^\dagger\phi \end{aligned} \qquad \text{(III-5)}$$

and the Hamiltonian takes the form

$$\begin{aligned} H = \int d^3x\sqrt{-g}\Big[& g^{oo}\phi^\dagger_{|o}\phi_{|o} - g^{ij}\phi^\dagger_{|i}\phi_{|j} - ieg^{ij}\phi^\dagger_{|i}W_j\phi \\ & - ieg^{ij}\phi^\dagger\phi_{|i}W_j - e^2g^{\mu\nu}W_\mu W_\nu\phi^\dagger\phi + (m^2+\tfrac{1}{6}R)\phi^\dagger\phi\Big] \end{aligned} \qquad \text{(III-6)}$$

Let us now consider the case where the electromagnetic field is homogeneous

$$W_\mu = (0,\vec{W}(t)) \qquad \text{(III-7)}$$

and the space-time metric is homogeneous and anisotropic (Kasner's type),

$$dS^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dZ^2 \qquad \text{(III-8)}$$

We shall suppose the electromagnetic and gravitational fields satisfy the following assymptotic conditions

$$\lim_{t\to\pm\infty} (A,B,C) = \text{Const.}$$
$$\lim_{t\to\pm\infty} W_i(t) = \text{Const.} \qquad \text{(III-9)}$$

For the scalar field we write

$$\Phi_\pm(x) \sim e^{i\vec{k}.\vec{x}}\, \Phi_\pm(t) \qquad \text{(III.10)}$$

From equation (III-4) it follows that the amplitudes $\Phi_\pm(t)$ satisfy the equation

$$\ddot{\Phi}_\pm + \frac{\dot{V}}{V}\, \dot{\Phi}_\pm + \Omega_k^2(t)\, \Phi_\pm = 0 \qquad \text{(III-11)}$$

where

$$\Omega_k^2(t) = m^2 + \frac{1}{6}R - g^{ij}(k_i - eW_i)(k_j - eW_j) \qquad \text{(III-12)}$$

and the dot means the derivative with respect to t. Introducing new variables defined by

$$\tau = \int \frac{dt}{V^{1/3}} \quad , \quad \Phi_\pm = V^{-1/3}\, \psi_\pm \qquad \text{(III-13)}$$

where V = A.B.C , it follows that

$$\psi''_\pm + \Delta_k^2(\tau)\psi_\pm = 0 \qquad \text{(III-14)}$$

where

$$\Delta_k^2(\tau) = V^{2/3}\Omega_k^2(\tau) + \frac{1}{18}\left\{\left(\frac{A'}{A}-\frac{B'}{B}\right)^2 + \left(\frac{A'}{A}-\frac{C'}{C}\right)^2 + \left(\frac{B'}{B}-\frac{C'}{C}\right)\right\} \qquad \text{(III-15)}$$

and the prime means derivative with respect to τ.

Due to (III-9), the amplitudes $\psi_\pm(\tau)$ satisfy the assymptotic conditions

$$\lim_{\tau\to\pm\infty} \psi_\pm(\tau) = e^{\pm i\Delta_k^{(\pm)}\tau} \qquad \text{(III-16)}$$

where

$$\Delta_k^{(\pm)} = \lim_{\tau\to\pm\infty} \Delta_k(\tau) \qquad \text{(III-17)}$$

With these conditions one can prove that the amplitudes $\psi_\pm$ satisfy the following relations[10]:

$$\begin{aligned}
&\psi_+^*\psi_+' - \psi_+\psi_+^{*'} = 2i\Delta_k^{(-)} \\
&\psi_-^*\psi_-' - \psi_-\psi_-^{*'} = 2i\Delta_k^{(-)} \\
&\psi_+\psi_-' - \psi_+'\psi_- = -2i\Delta_k^{(-)} \\
&\psi_+^*\psi_-' = \psi_+^{*'}\psi_- \ , \ |\psi_+|^2 = |\psi_-|^2 \ , \ |\psi_+'|^2 = |\psi_-'|^2
\end{aligned} \qquad \text{(III-18)}$$

We may decompose the fields ϕ and $\phi^\dagger$ as follows

$$\phi(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d^3k}{\sqrt{\Delta_k^{(-)}}} \left[A_k e^{i\vec{k}.\vec{x}}\psi_-(\vec{k},\tau) + B_k^\dagger e^{-i\vec{k}.\vec{x}}\psi_+(-\vec{k},\tau)\right]$$

$$\text{(III-19)}$$

$$\phi^\dagger(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d^3k}{\sqrt{\Delta_k^{(-)}}} \left[B_k e^{i\vec{k}.\vec{x}}\psi_+^*(-\vec{k},\tau) + A_k^\dagger e^{-i\vec{k}.\vec{x}}\psi_-^*(\vec{k},\tau)\right]$$

$(A_k, A_k^\dagger)$ and $(B_k, B_k^\dagger)$ are creation and anihilation operators that satisfy the comutation relations (II-6).

Introducing (III-19) in the Hamiltonian (III-6) and using (III-18), we obtain

$$\begin{aligned}
H(\tau) = \frac{1}{2}\int \frac{d^3k}{\Delta_k^{(-)}} \Big\{ & U(\vec{k},\tau)(A_k^\dagger A_k + B_{-k}^\dagger B_{-k}) \\
& + V(k,\tau)A_k B_{-k}^\dagger + V^*(\vec{k},\tau)B_{-k}A_k^\dagger \Big\}
\end{aligned} \qquad \text{(III-20)}$$

where we defined

$$\begin{aligned}
U(\vec{k},\tau) &\equiv |\psi_+'(\vec{k},\tau)| + \Delta_k^2(\tau)|\psi_+(\vec{k},\tau)|^2 \\
V(\vec{k},\tau) &\equiv \psi_+'(\vec{k},\tau)\psi_-^*(\vec{k},\tau) + \Delta_k^2(\tau)\psi_+(\vec{k},\tau)\psi_-^*(\vec{k},\tau)
\end{aligned} \qquad \text{(III-21)}$$

It is clear that the above Hamiltonian does not define a self-adjoint operator in a Hilbert space due to the presence of the non diagonal terms. However, it can be diagonalized if we perform the following Bogoliubov transformation on the operators $(A_k, A_k^\dagger)$ and $(B_k, B_k^\dagger)$

$$
\begin{aligned}
A_k^\dagger &= \frac{1}{\sqrt{1-|\lambda(\vec{k},\tau)|^2}}\left[a_k^\dagger(\tau_1)+\lambda^*(\vec{k},\tau)b_{-k}(\tau_1)\right] \\
A_k &= \frac{1}{\sqrt{1-|\lambda(\vec{k},\tau)|^2}}\left[a_k(\tau_1)+\lambda(\vec{k},\tau)b_{-k}^\dagger(\tau_1)\right] \\
B_k^\dagger &= \frac{1}{\sqrt{1-|\lambda(-\vec{k},\tau)|^2}}\left[b_k^\dagger(\tau_1)+\lambda(-\vec{k},\tau)a_{-k}(\tau_1)\right] \\
B_k &= \frac{1}{\sqrt{1-|\lambda(-\vec{k},\tau)|^2}}\left[b_k(\tau_1)+\lambda(-\vec{k},\tau)a_{-k}(\tau_1)\right]
\end{aligned}
\qquad \text{(III-22)}
$$

In the above expressions τ_1 is a given initial instant and $\lambda(\vec{k},\tau)$ is a function which satisfies the condition

$$\lim_{\tau\to\pm\infty} \lambda(\vec{k},\tau) = 0 \qquad \text{(III-23)}$$

In the limits $t\to\pm\infty$ the operators $(A_k^\dagger, B_k^\dagger)$ and (A_k, B_k) coincide with the creation and anihilation operators for the free field. Substituting (III-22) into (III-20) we obtained

$$
\begin{aligned}
H(\tau) = \frac{1}{2}\int\frac{d^3k}{\Delta_k^{(-)}}\,\frac{1}{1-|\lambda(\vec{k},\tau)|^2}\Big\{\Big[&U(\vec{k},\tau)(1+|\lambda(\vec{k},\tau)|^2)+ \\
&V(\vec{k},\tau)\lambda^*(\vec{k},\tau)+V^*(\vec{k},\tau)\lambda(\vec{k},\tau)\Big]\,(a_k a_k^\dagger+b_{-k}b_{-k}^\dagger) \\
&+(V^*(\vec{k},\tau)\lambda^2(\vec{k},\tau)+2U(\vec{k},\tau)\lambda(\vec{k},\tau)+V(\vec{k},\tau))a_k^\dagger a_{-k}^\dagger \\
&+(V(\vec{k},\tau)\lambda^{*2}(\vec{k},\tau)+2U(\vec{k},\tau)\lambda^*(\vec{k},\tau)+V^*(\vec{k},\tau))a_k b_{-k}\Big\}
\end{aligned}
$$

(III-24)

Then, the above Hamiltonian will be diagonalized if we require that

$$\lambda(\vec{k},\tau) = \frac{-U(\vec{k},\tau) + \sqrt{U^2(\vec{k},\tau) - |V(\vec{k},\tau)|^2}}{V^*(\vec{k},\tau)} \qquad \text{(III-25)}$$

where the positive sign of the square root has been chosen so that condition (III-23) is fulfilled.

Then, at the instant τ to which a given value of the function $\lambda(\vec{k},\tau)$ is associated, and consequently a representation of the canonical commutation relations is defined, the Hamiltonian operator is diagonal and has the meaning of a well defined operator for the energy associated with the particles, which number is also well defined in this representation. We note that this does not mean that the Hamiltonian will be diagonal in other instant $\tau_1 \neq \tau$ and so the number of particles in the new representation is different from that in the old representation. The function $\lambda(\vec{k},\tau)$ may be viewed as a parameter which caracterize a continuous set of representations of the canonical commutation relations.

Now let us introduce the vacuum state which is anihilated by the operators a_k and b_k. The mean number of particles in the mode k, created by both electromagnetic and gravitational fields is given by

$$\begin{aligned} \langle N_k(\tau)\rangle_{\tau_1} &= {}_{\tau_1}\langle 0|A_k^{\dagger}A_k|0\rangle_{\tau_1} = {}_{\tau_1}\langle 0|B_{-k}^{\dagger}B_k|0\rangle_{\tau_1} \\ &= \frac{|\lambda(\vec{k},\tau)|^2}{1-|\lambda(\vec{k},\tau)|^2} = \frac{U(\vec{k},\tau)-2\Delta_k(\tau)\Delta_k^{(-)}}{4\Delta_k(\tau)\Delta_k^{(-)}} \end{aligned} \qquad \text{(III-26)}$$

From the above expression we see that the function $|\lambda(\vec{k},\tau)|^2$ may be interpreted as the relative probability of creation of one pair of particles.

The total momentum of the created particle is

$$ {}_{\tau_1}\langle 0|P^i|0\rangle_{\tau_1} = {}_{\tau_1}\langle 0|:T^i_o:|0\rangle_{\tau_1} = 0 \qquad \text{(III-27)} $$

where $: \, :$ indicates normal ordering of the operators $(A_k, A_k^\dagger)$ and $(B_k, B_k^\dagger)$. The total momentum is then conserved which means that particles are created in pairs. This result is true in any representation and is a consequence of the homogeneity of the fields responsible by the process of particle creation. We also have

$$ {}_{\tau_1}\langle 0|:J^o:|0\rangle_{\tau_1} = 0 \qquad \text{(III-29)} $$

$$ {}_{\tau_1}\langle 0|:J^i:|0\rangle_{\tau_1} = -\frac{1}{A_i^2(\tau)} \int \frac{d^3k}{\Delta_k^{(-)}\Delta_k(\tau)} (k_i + eW_i)(\Delta_k^{(-)} - \Delta_k(\tau)|\psi_+(\vec{k},\tau)|^2; \qquad \text{(III-30)} $$

where A_i are the corresponding components of the metric tensor.

One could argue about how the electromagnetic fields influence the process of particle creation. Some aspects of this problem have been analyzed in[9]. There it is shown that homogeneous electromagnetic fields always increase the particle creation rate in directions along the field and decrease this rate in directions transverse to the field. As a consequence if one believes that the process of particle creation acts as an isotropization mechanism of the Universe, then the existence of primordial electromagnetic fields would have slowed down the isotropization process due to particle creation.

REFERENCES

(1) Penrose, R.; in "Les Houches Lectures 1963", Ed. by B. Dewitt;

(2) Fulton, T. and Rohrlich, R.; Rev. Mod. Phys.,34,442 , 1962.

(3) Gribb, A.A. and Mamaev, S.G.; Sov. J. of Nucl. Phys. , 10,6, 722, 1970.

(4) Fulling, S.A.; Thesis, Princeton University, 1973.

(5) Berezin, F.A.; "The Method of Second Quantization",Academic Press, New York, 1966.

(6) Parker, L.; Phys. Rev. 183, 562, 1968.

(7) Zel'dovich, Ya.B., and Starobinskii, A.A.; Sov. Phys. JETP, 34, 6, 1159 (1972).

(8) Parker,L. and Fulling, S.A.; Phys. Rev. D9, 2,341,1974.

(9) Novello, M. and Galvão, Carlos A.P.; Notas de Física XXIII,5, 1974.

(10) Galvão, Carlos A.P.; Thesis, Centro Brasileiro de Pesquisas Físicas, 1974.