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ON THE POLES OF THE S-MATRIX FOR LONG RANGE POTENTIALS * **

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ABSTRACT: The analytic properties in the complex k-plane of the S-matrix for scattering by a screened Coulomb potential are studied. Particular attention is given to the limit as the screening radius tends to infinity, so as to show in an explicit example the effect of the tail of the potential on the properties of the analytically extended S-matrix. It is shown that the pole configuration obtained in this way is different from that obtained in the usual description of the analytic properties of the Coulomb S-matrix.

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1. INTRODUCTION

It is well known that the analytic behaviour of the non-relativistic S-matrix for infinite range potentials depends fundamentally on the way the potentials behave at large distances. In particular, this analytic behaviour is strongly affected by the existence of an infinite tail, however thin it may be.

It seems to us, however, that this dependence has never been shown explicity in a convenient way in a meaningful example. A rectangular barrier or well whose range tends to infinity [1], thus establishing a constant potential in all space, does not provide us with an interesting situation. A soluble and convenient problem for this purpose is that of a screened Coulomb potential. By varying the radius of the screening layer, we tend continuously to the Coulomb potential case. However, as we shall see, the properties of the S-matrix in the complex momentum plane, as obtained by this limiting process, are not the same as those obtained by the usual analytic extension of the Coulomb S-matrix to the complex momentum plane [2]. We believe that the procedure of analytic extension we use here is a more natural one, and that at any rate we can learn from this example something about the treatment of this sort of infinite range potentials.

We take a potential of the form

$$V(r) = \mu C/r, \qquad r < b$$
$$= 0 , \qquad r > b \qquad (1)$$

where C is a positive quantity and $\mu = +1$ for repulsive and $\mu =$

= -1 for attractive potentials. We shall choose as unity the quantity $mC/\hbar^2 = 1/a_o$, where a_o is the first Bohr-radius for an attractive potential of strength C, and m is the mass of the particle. Thus we measure lengths (the range b for example) in units of a_o and the momentum k in units of $1/a_o$. The dimensionless product kb is independent of the potential strength and is a convenient variable for many purposes. We introduce also the quantity $\lambda = i\mu/k$, where $i = \sqrt{-1}$. We call $\mathrm{Re}(k) = x$, $\mathrm{Im}(k) = y$, so that $k = x + iy$.

The S-function for the ℓ-th partial wave can be written down directly. It is

$$S_\ell(k,b) = -Y_\ell^{(2)}(k,b)/Y_\ell^{(1)}(k,b) \tag{2}$$

where

$$\begin{aligned} Y_\ell^{(j)}(k,b) &= (\ell+1/2)\Big\{{}_1F_1(\ell+1+\lambda;\ 2\ell+1;-2ikb)\Big[H^{(j)}_{\ell+\frac{1}{2}}(kb) - iH^{(j)}_{\ell-\frac{1}{2}}(kb)\Big] + \\ &+ {}_1F_1(\ell+\lambda;2\ell+1;-2ikb)\Big[H^{(j)}_{\ell+\frac{1}{2}}(kb) + iH^{(j)}_{\ell-\frac{1}{2}}(kb)\Big]\Big\} = \\ &= -ikb\,{}_1F_1(\ell+1+\lambda;\ 2\ell+2;-2ikb)\Big[H^{(j)}_{\ell+\frac{1}{2}}(kb) - iH^{(j)}_{\ell-\frac{1}{2}}(kb)\Big] + \\ &+ (2\ell+1)\,{}_1F_1(\ell+\lambda;2\ell+1;-2ikb)\,H^{(j)}_{\ell+\frac{1}{2}}(kb) \end{aligned} \tag{3}$$

with j = 1, 2. The last expression is obtained from the second one by well known relations among the confluent hypergeometric functions.

From the fact that ${}_1F_1(a;\ c;\ x)$ is a real function of its arguments, that the complex conjugate to $H_p^{(2)}(z)$ is $H_p^{(1)}(z^*)$, that $H^{(2)}_{\ell+\frac{1}{2}}(-z) = \exp[i\pi(\ell+\frac{1}{2})]\ H^{(1)}_{\ell+\frac{1}{2}}(z)$ and the use of the Kummer transformation of the confluent hypergeometric functions, we see immediately that (2) satisfies the well known [3] symmetry properties of the S_ℓ-function such as $S_\ell(k, b)\ S_\ell(-k, b) = 1$, $S_\ell^*(k, b) = S_\ell(-k^*, b)$, and $S_\ell(k, b)\ S_\ell^*(k^*, b) = 1$. These relations imply that the zeros and poles of $S_\ell(k, b)$ in the complex k plane are symmetric with respect to the imaginary axis, and that if k is a zero of $S_\ell(k, b)$, k^* is a pole of the same function. Thus we need only studying the half-plane defined by $\text{Re}\ k \geqslant 0$.

We are now considering finite values of b. If $\text{Im}(k) \longrightarrow +\infty$, we have that $S_\ell(k, b)$ behaves like $\exp[2b\ \text{Im}(k)]$ thus presenting the well-known essential singularities for $\text{Im}(k) \longrightarrow \infty$. For $\text{Im}(k) \longrightarrow -\infty$, $S_\ell(k, b)$ tends to zero exponentially. For $k \longrightarrow 0$, b finite, and also for $b \longrightarrow 0$, k finite, we have $S_\ell(k, b) \longrightarrow 1$, as will be shown later. There can be no poles of $S_\ell(k, b)$ in the upper half plane, except on the imaginary axis. All these are well known general properties of finite range potentials [3].

In section 2 of this paper we discuss the asymptotic behaviour of the wave-functions when $b \longrightarrow \infty$. In sections 3-6 we shall discuss the distribution and displacements in the complex k-plane of the poles $S_\ell(k,b)$ as a function of the range b. Since $Y_\ell^{(j)}(k, b)$ are regular functions of k (except at the origin), the singularities of $S_\ell(k, b)$ will be poles due to zeros in the denominator $Y_\ell^{(1)}(k, b)$. Our general equation for the poles will then be $Y_\ell^{(1)}(k, b) = 0$.

2. ASYMPTOTIC COULOMB WAVE-FUNCTIONS FOR COMPLEX k

For $|kb| \gg 1$ the asymptotic (large r) ℓ-wave-function for the Schroedinger equation with potential (1) is

$$R_\ell(r) \sim \frac{1}{kr}\left\{ e^{ik(r-b)}\left[\frac{e^{ikb}(2ikb)^{-\ell-\lambda}}{\Gamma(\ell+1-\lambda)} + \frac{e^{-ikb}\,\lambda(-2ikb)^{-\ell-1+\lambda}}{\Gamma(\ell+1+\lambda)}\right] + \right.$$

$$\left. + e^{-ik(r-b)}\left[\frac{e^{ikb}\,\lambda(2ikb)^{-\ell-1-\lambda}}{\Gamma(\ell+1-\lambda)} - \frac{e^{-ikb}\,(-2ikb)^{-\ell+\lambda}}{\Gamma(\ell+1+\lambda)}\right]\right\}. \qquad (4)$$

If in (4) we just substitute $b = r$ and keep only the dominating terms we obtain the asymptotic Coulomb wave-function

$$R_\ell^{Coul}(r) \sim \frac{1}{kr}\left[e^{ikr}\frac{(2ikr)^{-\ell-\lambda}}{\Gamma(\ell+1-\lambda)} - e^{-ikr}\frac{(-2ikr)^{-\ell+\lambda}}{\Gamma(\ell+1+\lambda)}\right] \qquad (5)$$

and consequently the Coulomb S_ℓ-function

$$S_\ell^{Coul}(k) = \frac{\Gamma(\ell+1+\lambda)}{\Gamma(\ell+1-\lambda)}(-2kr)^{-2\lambda}.$$

However this substitution is quite arbitrary, since for $|kb| \gg 1$ (4) is valid for all $r > b$ and the definition of S_ℓ is independent of r. Putting $r = b$ in (4) implies in that we have made r and b tend together to infinity, with $r - b = 0$. This is not the case in our problem, where we have the solution for a finite range potential whose range b we let then increase. We might as well for instance keep $r - b$ finite, non-zero. The two ways of taking the limit may lead to different properties of the resulting S_ℓ-function.

So, let us keep $r - b$ finite while $r, b \to \infty$.

If k is real the exponentials appearing is (4) are just oscillating functions and the limit b $\to \infty$ will give, selecting the dominating terms

$$R_l(r) \sim \frac{1}{kr}\left[e^{ikr}\frac{(2ikb)^{-l-\lambda}}{\Gamma(l+1-\lambda)} - e^{-ikr}\frac{(-2ikb)^{-l+\lambda}}{\Gamma(l+1+\lambda)}\right], \text{ real } k \quad (5)$$

which is again the asymptotic Coulomb wave-function (put b = r inside the brackets), and gives the usual Coulomb S_l-function. So, nothing is different in case k is real.

Let us now assume that Im(k) $\neq$ 0 in (4). Now there will be real exponentials, and at least for values of k which are not in the neighbourhood of poles of $\Gamma(l+1+\lambda)$ or $\Gamma(l+1-\lambda)$ we have that when taking the limits r, b $\to \infty$ the dominating terms will be those containing positive exponents. Thus we will have

$$R_l(r) \sim \frac{e^{-ikb}}{kr}\left[e^{ikr}\frac{\lambda(-2ikb)^{-l-1+\lambda}\, e^{-2ikb}}{\Gamma(l+1+\lambda)} - e^{-ikr}\frac{(-2ikb)^{-l+\lambda}}{\Gamma(l+1+\lambda)}\right] \quad (6)$$

for Im(k) > 0 and

$$R_l(r) \sim \frac{1}{kr}\left[e^{ikr}\frac{(2ikb)^{-l-\lambda}}{\Gamma(l+1-\lambda)} + e^{-ikr}\frac{\lambda(2ikb)^{-l-1-\lambda}e^{2ikb}}{\Gamma(l+1-\lambda)}\right] \quad (7)$$

for Im(k) < 0 with the possible exception of isolated points for which $\Gamma(l+1+\lambda)$ and $\Gamma(l+1-\lambda)$ increase without limit.

None of these two formulas (6) and (7) for Im(k) $\neq$ 0 reproduces the Coulomb wave-function. One of them (6) contains the Coulomb incoming wave, combined with a different outgoing part, the other (7) contains the Coulomb outgoing wave combined with a different incoming part. These wave-functions (6) and (7) or the corresponding S_l-functions show the behaviour we obtain if we solve the problem for finite range b, then taking larger and larger values of b. Perhaps this would correspond more appropriately to a physical situation, where a trully infinite Coulomb potential cannot really exist.

We can easily see that the expressions (6) and (7) for the asymptotic wave-functions do not allow for poles in the corresponding S_ℓ-function. Thus, as $b \to \infty$ poles can only be located in points where these expressions are not valid.

Let us return to (4). The position of the poles of S_ℓ will be determined by equating to zero the coefficient of $\exp(-ikr)$, that is by

$$e^{2ikb}\,\lambda(2ikb)^{-2\lambda-1}\,\Gamma(\ell+1+\lambda) = (-1)^{-\ell+\lambda}\,\Gamma(\ell+1-\lambda)\,. \qquad (8)$$

Using this relation to evaluate the coefficient of $\exp(+ikr)$ in (4), we then obtain that near a pole of S_ℓ the asymptotic wave-function in the limit $b \to \infty$ behaves like

$$R_\ell^{pole}(r) \sim \frac{1}{kr}\left\{e^{ikr}\,\frac{(2ikb)^{-\ell-\lambda}}{\Gamma(\ell+1-\lambda)} + e^{-ikr}\left[\frac{e^{2ikb}\,\lambda(2ikb)^{-\ell-1-\lambda}}{\Gamma(\ell+1-\lambda)} - \frac{(-2ikb)^{-\ell+\lambda}}{\Gamma(\ell+1+\lambda)}\right]\right\}$$

If $\mathrm{Im}(k) > 0$ equation (8) has solutions only for points in the k-plane which tend to the poles of $\Gamma(\ell+1+\lambda)$ as $b \to \infty$. For $\mathrm{Im}(k) < 0$ the poles of S_ℓ tend to the poles of $\Gamma(\ell+1-\lambda)$. Thus the structure of the pole distribution for the S_ℓ-function obtained this way is different from that obtained with formula (5). We now have that for an attractive potential, there will be poles in both the positive and negative imaginary axis, for $\mathrm{Im}(k) = \pm(\ell+1+n)^{-1}$, where $n = 0, 1, 2, \ldots$. For a repulsive potential the poles can only move to the origin in the k-plane, where the above formulas are not valid in general, as we shall see later. We remind here that in the usual discussion of the Coulomb potential problem [5] there appear poles in the positive imaginary axis for the attractive potential and in the negative imaginary axis for the repulsive potential. We must remark that there is no different behaviour concerning the bound state poles (those on the positive imaginary axis).

3. THE LIMIT $b \to 0$.

From (2) it is easy to see that when the range b tends to zero with k non-infinite, $S_\ell(k,b)$ tends to one. This follows immediately from the fact that ${}_1F_1(a^{-1};c;ax) \to 1$ as $x \to 0$ and that $H_p^{(1)}(z)/H_p^{(2)}(z)$ goes to 1 when $z \to 0$. Thus, when the range of the potential tends to zero, there can be poles of the S_ℓ-function only for $|k| \to \infty$.

We can now ask what happens in the variable kb. To keep kb finite when $b \to 0$ we must have $|k| \to \infty$, but taking this limit $|k| \to \infty$ in (3) is equivalent to make $\lambda \to 0$, or in other words, to reduce to zero the strength of the potential. We expect that S_ℓ goes to zero in this limit. In the Appendix we show that this is in fact true.

Thus we have that there can be no poles in the finite kb plane when $b \to 0$, the poles being pushed to the infinity when this limit is taken. To find how the poles displace themselves, we can use well known asymptotic expressions, valid for large kb, for the Hankel and hypergeometric functions ocurring in $Y_\ell^{(1)}(k,b)$. The asymptotic formula for the confluent hypergeometric function is

$$ {}_1F_1(a;c;z) \xrightarrow[|z| \to \infty]{} \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}\left[1+\mathcal{O}(z^{-1})\right] + \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c}\left[1+\mathcal{O}(z^{-1})\right]. \tag{10} $$

We obtain after a lengthy but straightforward calculation

$$ Y_\ell^{(1)}(k,b) \sim \sqrt{2/\pi kb}\; e^{-ikb} \frac{(2\ell+1)!}{\ell!} i^{-\ell} k^{-1} (2ikb)^{-\ell-1} \left[(-)^\ell \cdot 2k^2 b - \mu e^{2ikb}\right]. \tag{11} $$

The equation for the poles is then

$$ \mu \exp(2ikb) = (-1)^\ell \cdot 2k^2 b \tag{12} $$

which has solutions

$$ xb = \left[2n + \ell + (\mu+1)/2\right]\pi/2 \tag{13} $$

$$ yb \longrightarrow -\infty, \quad \text{with} \quad \exp(-2yb) = 2\, y^2 b \tag{13'} $$

where n is a positive or negative integer, or zero.

This result is similar to the one obtained bu Nussenzveig[1] in the case of a rectangular well or barrier and by Humblet [4] in a more general case. It says that in the limit of very short range or very weak screened Coulomb potential the poles tend to $yb \rightarrow -\infty$ approaching the asymptotic lines $xb = \pm N(\pi/2)$ where N is odd for ℓ odd (even) and N is even for ℓ even (odd) in the case of attractive (repulsive) potentials. Thus the odd ℓ-wave poles in the attractive and the even ℓ-wave poles in the repulsive potentials both tend to the lines $xb = \pm \pi/2$, $\pm 3\pi/2$, ..., while in other cases the poles tend to $0, \pm \pi, \ldots$ This is in fact a particular example of a quite general property of weak potentials [5]. This behaviour is shown in the curves at sections 5 and 6, where trajectories described by the poles when b varies are drawn.

The asymptotic behaviour of the numerator $Y_\ell^{(2)}(k,b)$ when $b \rightarrow 0$ is given by

$$Y_\ell^{(2)}(k,b) \sim (2/\pi kb)^{\frac{1}{2}} \exp(-ikb)\left[(2\ell+1)!/\ell!\right] i^\ell \, k^{-1}(-2ikb)^{-\ell-1} \left[(-1)^\ell \, 2k^2 b - \mu \exp(-2ikb)\right] \tag{14}$$

and then

$$S_\ell(k,b) \sim -i/\left[2b(k-k_o)\right] \tag{15}$$

in the neighbourhood of a pole k_o. Then in the plane kb the residues of the poles tend to $-i/2$, and in the k plane they increase as $-i/2b$ when $b \rightarrow 0$. It is interesting to note that these results for the residues do not depend on ℓ. Also Equation (13') which determines how fast the poles are pushed to the infinity, does not depend on the particular pole (that is, on N) neither on ℓ. This means that the centrifugal barrier at the origin has no effect in the strength of the poles for a limiting short range potential nor in the speed with which they are sent away to infinity. Since the centrifugal barriers are just equivalent to a potential behaviour like C/r^2 at the origin, we can predict that the potentials of this sort and whose range is made approach zero, the S_ℓ-function will have the same pole structure as given by (13), (13'), (15), except that the asymptotic lines will be given by (13) with ℓ modified so as to include the effect of the potential, that is $\ell(\ell+1)$ is substituted by $\ell(\ell+1) + C$. We must notice that in this case of potentials behaving like C/r^2 at the origin we obtain different behaviour of the poles in the two cases where we reduce the strength and where we reduce the range to zero.

4. THE LIMIT $b \to \infty$

When taking the limit $b \to \infty$ we have several cases to consider. We may have k finite (non-zero) and thus $|kb| \to \infty$, or we may have $k \to 0$. In this last case we may have kb finite or kb increasing to infinity. Let us consider each of these possibilities separately. This is essential, as the asymptotic behaviour of the functions involved are different under these different conditions.

a) We keep k finite, non-zero, and let b increase without limit. Since $|kb| \to \infty$ and all other parameters are limited, we may use well known asymptotic expressions for the Hankel and hypergeometric functions. We now show that the S_ℓ-function tends to limits which are in general different from the usual Coulomb S_ℓ-function. The results here indicated are in agreement with those of Section 2.

Writing $Y_\ell^{(j)}(k,b)$ in the form

$$Y_\ell^{(j)}(k,b) = \left\{-ikb\left[H_{\ell+\frac{1}{2}}^{(j)}(kb) - iH_{\ell-\frac{1}{2}}^{(j)}(kb)\right] + (\ell+\lambda)H_{\ell+\frac{1}{2}}^{(j)}(kb)\right\} \times$$

$${}_1F_1(\ell+1+\lambda;2\ell+2;-2ikb) + (\ell+1-\lambda)H_{\ell+\frac{1}{2}}^{(j)}(kb)\,{}_1F_1(\ell+\lambda;2\ell+2;-2ikb) \qquad (16)$$

we obtain, for $|kb|\gg 1$,

$$Y_\ell^{(2)}(k,b)\sim(2/\pi kb)^{\frac{1}{2}}\exp(-ikb)i^{\ell+1}\Big\{\lambda\,{}_1F_1(\ell+1+\lambda;2\ell+2;-2ikb) + (\ell+1-\lambda)\,{}_1F_1(\ell+\lambda;2\ell+2;-2ikb)\Big\} \qquad (17)$$

and

$$Y_\ell^{(1)}(k,b)\sim(2/\pi kb)^{\frac{1}{2}}\exp(-ikb)i^{-\ell-1}\Big\{(\ell+1+\lambda)\,{}_1F_1(\ell-\lambda;2\ell+2;2ikb) - \lambda\,{}_1F_1(\ell+1-\lambda;2\ell+2;2ikb)\Big\}. \qquad (18)$$

We can prove that the asymptotic forms of the ${}_1F_1$'s which occur in both expressions (17) and (18) are such that the first one predominates over the second when $\mathrm{Im}(k)>0$ and the second ${}_1F_1$ dominates when $\mathrm{Im}(k)<0$. This dominance is not true for the isolated points where the dominating ${}_1F_1$ happens to be zero (this will happen in the poles and in the zeros of S_ℓ). In these cases we have to keep the second dominating terms in the expressions for $Y_\ell^{(1)}$ and $Y_\ell^{(2)}$. The expression for the S_ℓ-function which is valid in all cases is

$$S_\ell(k) = -\frac{\left[(2ikb)^{-\ell-\lambda}/\Gamma(\ell+1-\lambda)\right]+\left[\lambda(-2ikb)^{-\ell-1+\lambda}\exp(-2ikb)/\Gamma(\ell+1+\lambda)\right]}{\left[(-2ikb)^{-\ell+\lambda}/\Gamma(\ell+1+\lambda)\right]-\left[\lambda(2ikb)^{-\ell-1-\lambda}\exp(2ikb)/\Gamma(\ell+1-\lambda)\right]}. \qquad (19)$$

For $\mathrm{Im}(k)>0$ we have that, for $|kb|\gg 1$, $S_\ell(k,b)$ behaves like

$$S_\ell(k,b) \underset{b\to\infty}{\sim} -\lambda(-2ikb)^{-1}\exp(-2ikb), \qquad \mathrm{Im}(k)>0 \qquad (19')$$

in all points not in the neighbourhood of the zeros of the denominator in (19). Thus, $S_\ell(k,b)$ explodes exponentially as $b \to \infty$ in the whole upper k-plane. On the other hand, for $Im(k)<0$, except in the neighbourhood of zeros of the dominator in (19), $S_\ell(k,b)$ bahaves like

$$S_\ell(k,b) \underset{b \to \infty}{\simeq} \lambda^{-1}(2ikb)\exp(-2ikb) , \qquad Im(k)<0 . \qquad (19'')$$

Thus, when $b \to \infty$, S_ℓ tends to zero for every finite k in the lower k-plane, with the possible exception of isolated points.

Let us discuss the behaviour of the poles of $S_\ell(k,b)$ as b increases to infinity. For $Im(k)>0$ the exponential in the denominator of (19) will tend to zero as b gets large, and the poles will then move to the points where $\Gamma(\ell+1+\lambda)$ also gets large, that is, to the points such that $\ell+1+\lambda = -n$, with $n = 0, 1, \ldots$ For $Im(k)<0$ the exponential in the denominator in (19) increases with b, and the poles must tend to the points for which $\Gamma(\ell+1-\lambda)$ is also large. Thus the poles in the lower k-plane will move to the points given by $\ell+1-\lambda = -n$. Since $\lambda = i\mu/k$, μ being the sign of the potential, we see that in the attractive case ($\mu = -1$) there will be poles in both positive and negative imaginary axis, while for a repulsive ($\mu = 1$) potential in the limit $b \to \infty$ there will be no poles with finite non-zero k. This is in agreement with what has been said in Section 2. We should remark that (19) still obeys the symmetry properties $S_\ell(k,b) = S_\ell^{-1}(-k,b)$ and $S_\ell^*(k,b) = S_\ell(-k^*,b)$.

The residues of the bound-state poles are given by

$$\frac{iy^2(2yb)^{2y^{-1}}}{(y^{-1}-\ell-1)!\;(y^{-1}+\ell)!} , \qquad y = Im(k) > 0$$

and they increase as b increases. The residues of the poles in the negative imaginary axis for large b values are given by

$$\frac{-iy^4(2yb)^{2-2y^{-1}}\exp(4yb)}{(-y^{-1}-\ell-1)!\,(-y^{-1}+\ell)!} , \quad y = Im(k) < 0$$

and thus they tend to zero as b increases.

b) Now let us consider the case in which k goes to zero while $b \to \infty$, with kb being kept finite. In these conditions the first parameter in ${}_1F_1(a; c; z)$ is large, while the second parameter and the variable are limited. We must then use the asymptotic expressions valid under these conditions [6]. We obtain for $|k| \ll 1$, $|kb|$ finite,

$$Y_\ell^{(j)}(k,b) \approx (2\ell+1)!\ \pi^{-\frac{1}{2}} \exp(-ikb)\ (2b)^{-\ell-1/4} \exp[-i(\ell+1/4)(3+\mu)\pi/2]$$
$$\Big\{ kb\left[H^{(j)}_{\ell+\frac{1}{2}}(kb) + iH^{(j)}_{\ell-\frac{1}{2}}(kb)\right]\left[\exp(2i\sqrt{-2\mu b}) + \exp(i3\pi/2 - 2i\sqrt{-2\mu b}\right] +$$
$$H^{(j)}_{\ell+\frac{1}{2}}(kb)\exp[i(1+\mu)\pi/4]\ \sqrt{2b}\ [\exp(2i\sqrt{-2\mu b}) + \exp(i\pi/2 - 2i\sqrt{-2\mu b})]\Big\} \quad (20)$$

It is now easy to see that in both cases of $\mu = \pm 1$, the second part in the above expression dominates over the first. This means that in the conditions here studied we may simplify $Y_\ell^{(j)}(k,b)$ to

$$Y_\ell^{(j)}(k,b) \approx (2\ell+1)\exp(-2ikb)\,{}_1F_1(\ell-\lambda;\ 2\ell+1;\ 2ikb)\ H^{(j)}_{\ell+\frac{1}{2}}(kb),\quad |k| \ll 1. \quad (21)$$

The S_ℓ-function will then be

$$S_\ell(k,b) \approx -\ H^{(2)}_{\ell+\frac{1}{2}}(kb)/H^{(1)}_{\ell+\frac{1}{2}}(kb) \quad (22)$$

The result is that when $b \longrightarrow \infty$ there will be poles in the plane kb at the point given by the roots $H^{(1)}_{\ell+\frac{1}{2}}(kb) = 0$. The residues of these poles are not zero, since the roots of $H^{(1)}_{\ell+\frac{1}{2}}$ and of $H^{(2)}_{\ell+\frac{1}{2}}(kb)$ never coincide (they are symmetric with respect to the origin; $H^{(1)}_{\ell+\frac{1}{2}}(kb)$ has roots only for $\mathrm{Im}(kb) < 0$ and $H^{(2)}_{\ell+\frac{1}{2}}(kb)$ only for $\mathrm{Im}(kb) > 0$). For $\ell = 0$ there will be no such poles. For $\ell = 1$ we have a pole at $kb = -i$, for $\ell = 2$ there are poles at $kb = \mp 3/2 - i\sqrt{3/2}$, and so on. There will be ℓ poles of this kind for a given ℓ-wave. For odd ℓ-values one pole will be on the imaginary axis and the $(\ell-1)$ remaining ones will be distributed symmetrically with respect to this axis. For even ℓ-values there will be no poles on the imaginary axis, since $H^{(1)}_{\ell+\frac{1}{2}}(kb)$ does not admit double roots. In the k-plane all those poles tend to the origin.

This pole structure does not depend on whether the potential is attractive or repulsive. These conclusions are confirmed by the numerical calculations and are shown clearly in figures of the next sections. These poles are responsible for the essential singularities of S in the Coulomb case.

c) Only one possibility remains to be discussed in this limit $b \to \infty$, that in which $k \to 0$ and $|kb| \to \infty$. We now have that both the first parameter a and the variable z in the functions ${}_1F_1(a; c; z)$ increase without limit. The asymptotic expressions for the attractive and repulsive cases are different, and also different expressions have to be used for different regions of the complex plane [7]. We shall then go directly to the points we wish to demonstrate, avoiding more general calculations which are not absolutely necessary.

As we shall see in the next Section, the behaviour of the poles in the attractive potential is already completely described in terms of the cases previously analysed. We then specialize to the repulsive case. We shall show that there exists an infinite number of poles such that $|k^2b|$ is kept finite while $k \to 0$.

We want to study the behaviour of the S_ℓ-function in the right hand side of the lower half-plane of the variable k, that is, we have

$$-\pi/2 < \arg(k) < 0 \,, \qquad 0 < \arg\,(2ikb) < \pi/2 \,.$$

In the denominator of the S_ℓ-function two hypergeometric functions enter,

$F_1 = {}_1F_1(\ell+1-\lambda;\ 2\ell+2;\ 2ikb)$ and $F_2 = {}_1F_1(\ell-\lambda;\ 2\ell+1;\ 2ikb)$.

By using the appropriate asymptotic expressions [7] we can prove, after a rather long but straightforward calculation that kbF_1 and F_2 are of the same order of magnitude in the limits considered, as long as $k^2 b$ is a finite quantity, that is we have $\underline{O}(kbF_1/F_2) = (k^2b)^{\frac{1}{2}}$. Since F_1 and F_2 appear in $Y_\ell^{(1)}(k,b)$ in the combination

$$Y_\ell^{(1)}(k,b) = \exp(-2ikb)\left\{ikb\left[H^{(1)}_{\ell+\frac{1}{2}}(kb) + iH^{(1)}_{\ell-\frac{1}{2}}(kb)\right]F_1 + (2\ell+1)H^{(1)}_{\ell+\frac{1}{2}}(kb)F_2\right\} \tag{23}$$

and since

$$H^{(1)}_{\ell+\frac{1}{2}}(kb)\Big/\left[H^{(1)}_{\ell+\frac{1}{2}}(kb) + i\,H^{(1)}_{\ell-\frac{1}{2}}(kb)\right] \approx ikb/\ell$$

when $|kb| \longrightarrow \infty$, we have that the term containing F_2 will dominate over that containing F_1.

A similar evaluation can be performed with the terms contributing to $Y_\ell^{(2)}(k,b)$. We now take the form

$$Y_\ell^{(2)}(k,b) = \exp(-2ikb)\left\{-ikb\left[H^{(2)}_{\ell+\frac{1}{2}}(kb) - iH^{(2)}_{\ell-\frac{1}{2}}(kb)\right]F_1 + {} + (2\ell+1)\,H^{(2)}_{\ell+\frac{1}{2}}(kb)\,F_3\right\} \tag{24}$$

where $F_3 = {}_1F_1(\ell+1-\lambda;\ 2\ell+1;\ 2i\,kb)$

and prove that $\underline{O}(kbF_1/F_3) = (k^2b)^{\frac{1}{2}}$.

Now since $H^{(2)}_{\ell+\frac{1}{2}}(kb)$ is larger than $H^{(2)}_{\ell+\frac{1}{2}}(kb) - iH^{(2)}_{\ell-\frac{1}{2}}(kb)$

when $|kb| \to \infty$, the term containing F_3 dominates over the other, and the S_ℓ-function becomes

$$S_\ell(k,b) \xrightarrow[\substack{b\to\infty \\ k^2 b \text{ finite}}]{} H^{(2)}_{\ell+\frac{1}{2}}(kb)\,{}_1F_1(\ell+\lambda;\ 2\ell+1;\ -2ikb)/$$

$$\left[\exp(-2ikb)\ H^{(1)}_{\ell+\frac{1}{2}}(kb)\ {}_1F_1(\ell-\lambda;\ 2\ell+1;\ 2ikb)\right] \tag{25}$$

This satisfies the symmetry properties known for the S-matrix, mentioned in Section 1.

With this result our problem reduces to the search for the existence of zeros of ${}_1F_1(\ell-\lambda;\ 2\ell+1;\ 2ikb)$ in the conditions considered, that is, with $k \to 0$, $b \to \infty$, $t = 2ikb/[4(\lambda+1/2)]$ finite, $0 \leqslant \arg\,(i/k+1/2) \leqslant \pi/2$. We can then use the appropriate asymptotic formula for ${}_1F_1$, and equate it to zero.

We have [7]

$$F_2 \underset{\substack{k\to 0 \\ k^2 b \text{ finite}}}{\approx} \exp(2ikb)(2ikb)^{-\ell-\frac{1}{2}}\,\Gamma(2\ell+1)\left[1-(2\lambda+1)/ikb\right]^{-1/4} A_\ell(k,b) \tag{26}$$

where

$$A_\ell(k,b) = (\lambda+1/2)^{-\lambda-\frac{1}{2}}\exp(E+\lambda+1/2)/\Gamma(\ell-\lambda) + $$
$$+ (\lambda+1/2)^{\lambda+\frac{1}{2}}\exp(-E-\lambda-1/2)\ \exp[i\pi(\ell-\lambda)]/\Gamma(\ell+1+\lambda) \tag{27}$$

and

$$E = (2\lambda+1)\left\{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}} + \log\left[t^{\frac{1}{2}} - (t-1)^{\frac{1}{2}}\right]\right\}. \tag{28}$$

Using the asymptotic limit

$$\Gamma(z) \approx (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} e^{-z} \qquad \text{(large } |z|)$$

for the Γ functions and noticing that λ is large (so that $(\lambda + 1/2)^{\lambda+\frac{1}{2}} \approx \lambda^{\lambda+\frac{1}{2}} e^{\frac{1}{2}}$) we obtain that

$$A_{\ell}(k,b) \approx (2\pi)^{-\frac{1}{2}} \lambda^{-\ell}(-1)^{\ell} \exp(-i\pi\lambda - E)[1 + \exp(2E - i\pi/2)]. \tag{29}$$

Thus the necessary condition for the existence of poles in the S_{ℓ}-function is then that

$$\text{Re }(E) = 0$$

$$\text{Im }(E) = 3\pi/4 + m\pi, \quad m \text{ integer.}$$

We first notice that these equations determining the poles are independent of ℓ. We now look for solutions of these equations such that $y/x \longrightarrow 0$ while $x, y \longrightarrow 0$. In these conditions we have that the system of equations is

$$(1/2 + y/x^2)\left[(bx^2/2)^{\frac{1}{2}}(bx^2/2-1)^{\frac{1}{2}} + \log|(bx^2/2)^{\frac{1}{2}} - (bx^2/2-1)^{\frac{1}{2}}|\right] -$$

$$- (1 + 4y/x^2)(bx^2/2)^{\frac{1}{2}}(bx^2/2 - 1)^{\frac{1}{2}} = 0 \tag{30}$$

$$(2/x)\left[(bx^2/2)^{\frac{1}{2}}(bx^2/2-1)^{\frac{1}{2}} + \log|(bx^2/2)^{\frac{1}{2}} - (bx^2/2-1)^{\frac{1}{2}}|\right] +$$

$$+ (x/2)(1/2 + y/x^2)(1 + 4y/x^2)(bx^2/2)^{\frac{1}{2}}(bx^2/2-1)^{\frac{1}{2}} = 3\pi/4 + m\pi \tag{31}$$

As $x \longrightarrow 0$ the first term in (31) increases without limit, unless $b\,x^2/2$ tends fast to one. In this case (31) becomes simply

$$(bx^2/2-1)/x = 3\pi/4 + m\pi \tag{32}$$

which gives an explicit solution

$$xb = (2b)^{\frac{1}{2}} + 3\pi/4 + m\pi \tag{33}$$

for one of the coordinates of the poles. Taking (33) into (30) we obtain that

$$yb \longrightarrow - 1/2 \ . \qquad (34)$$

Thus the result is that for repulsive potentials, we have an infinite number of poles which, as b increases, tend to infinity in the kb-plane, approaching asymptotically the line $yb = - 1/2$. The existence and position of these poles are independent of the value of the angular momentum. All this can be seen in the curves of Section 6.

5. THE POLES OF THE S_ℓ-FUNCTION FOR ATTRACTIVE POTENTIALS

We have seen that (13) and (13') show that for increasing b the poles approach lines parallel to the imaginary axis in the kb plane. For curves with even ℓ-values the poles approach the lines $xb = N(\pi/2)$, with $N = 0, \pm 2, \pm 4$, and so on. For the odd waves the asymptotes are given by the same formula, but with $N = \pm 1, \pm 3, \pm 5$, and so on. We shall label a pole by the number N that defines the asymptote of its trajectory when $b \rightarrow 0$.

From (13') it can be shown that as $b \longrightarrow 0$ we have $d(yb)/d(\log b) = 1/2$. This means that dividing b by a given factor implies that all the poles go down the same vertical distance in the plane kb. This can be easily observed in figures 1, 3 and 5, which show the trajectories of the poles for the attractive s, p and d-waves respectively. Since equation (13') is also valid for repulsive potentials, this behaviour can also be observed in the corresponding curves for the repulsive case (Fig. 7).

As b increases the poles move upwards in the kb plane, and in the attractive case they turn so as to run towards the imaginary axis.

a) <u>The s-wave poles</u>

We first describe in detail the behaviour of the s-wave poles. There is a pole ($N = 0$) which moves along the imaginary axis, from $kb = - i\infty$ to $kb = + i\infty$. The pair of poles ($N = \pm 2$) coming from the asymptotes $xb = \pm \pi$ reach the imaginary axis in the point $yb = - 1.5774$ for $b = 3.4115$. All pairs of symmetric poles meet at the imaginary axis; as b is further increased one pole goes up along the axis towards $yb = \infty$, while the other moves downwards to $yb = - \infty$. The poles $N = \pm 4$ reach the imaginary axis in the point $yb = 1.5227$ for $b = 8.9781$. For $N=\pm 6$ this happens for $yb = - 1.5101$ and $b = 17.001$.

The most distant (large $|N|$) poles reach the imaginary axis for larger and larger values of b. We now prove that $|N| \to \infty$, the poles reach the axis in points closer and closer to $yb = -1.5$. The denominator of the S_ℓ-function for $\ell = 0$ is $Y_0^{(1)}(k,b) = {}_1F_1(i/k;\ 1;\ 2\ ikb)$. Poles in the imaginary axis are determined by ${}_1F_1(1/y;\ 1;\ -2\ yb) = 0$. This gives y as a function of b. For the points in which the poles leave the imaginary axis we must have $\frac{db}{dy} \equiv - (\partial F/\partial y)/(\partial F/\partial b) = 0$, and thus these points are determined by the simultaneous solution of these two equations. We want to study the distant poles, i.e., these which enter the axis for large b. We can try a solution with $y \longrightarrow 0$, yb finite.

In these conditions the hypergeometric functions can be expanded in terms of a series of Bessel functions,

$$ {}_1F_1(1/y;\ 1;\ -2yb) \approx \exp(-yb)\Big\{J_0(2\sqrt{(2-y)b}) + $$
$$ + (b^{\frac{1}{2}}\ 2^{\frac{1}{2}}/6)\ y^2\ J_1(2\sqrt{(2-y)b})\Big\}. \qquad (35) $$

Thus for small y the "distant" poles must satisfy

$$ J_0(2\sqrt{(2-y)b}) = 0\,. \qquad (36) $$

From (35) we obtain

$$ \partial\,{}_1F_1/\partial y = \exp(-yb)(b/2)^{\frac{1}{2}}\ (1+2by/3)\ J_1(2\sqrt{(2-y)b})\ . \qquad (37) $$

Since J_0 and J_1 cannot be simultaneously zero, we must then have $by = -3/2$ as the only possibility to satisfy simultaneously the two equations. This is what we want to proof. All the poles (except the pole $N=0$) enter the imaginary axis for values of yb between -1.5774 and -1.5. From (36) we then obtain that the values of b for which the distant poles enter the imaginary axis are approximately given by the larger roots of $J_0(2\sqrt{2b+3/2})=0$. The poles $N=0$ can be said to enter the imaginary axis for $yb = -\infty$.

When a pole crosses the origin and enters the positive imaginary axis a new bound state is formed. The values of b for which this happens can be determined in the following way. For $y \to 0$, b finite we have

$$ {}_1F_1(1/y;\ 1;\ -2yb) \approx J_0(2\sqrt{2b}) + y\ (b/2)^{\frac{1}{2}}\ J_1(2\sqrt{2b}) \qquad (38) $$

Thus, new bound states arise whenever $2\sqrt{2b}$ reaches a root of the Bessel function of order zero. b is measured in units of the

Bohr - radius $a_o = \hbar^2/mZe^2$.

The displacement of the poles along the imaginary axis can be better observed in Fig. 2, where Im(k) is plotted against b. Only the purely imaginary poles are represented.

For values of b corresponding to the vertices A, B, C, D new poles reach the imaginary axis. Every time one of these values is reached, two new purely imaginary poles arise. This always happens for a value of Im(k) such that $-2/|N| < \mathrm{Im}(k) < 0$. One of these poles moves downwards along the imaginary axis and tends asymptotically to the point $\mathrm{Im}(k) = -2/|N|$ as $b \to \infty$. The "twin" pole moves upwards, and as $b \to \infty$ it approaches asymptotically the point $\mathrm{Im}(k) = +2/(|N|+2)$. Thus in the attractive Coulomb potential limit the pole configuration in the k-plane in the s-wave case is the following: there are poles in the points of the positive imaginary axis, corresponding to the usual bound states ($\mathrm{Im}(k) = 2/(|N|+2)$, $|N| = 0, 2, 4, \ldots$) and in points of the negative imaginary axis (given by $\mathrm{Im}(k) = -2/|N|$, $|N| = 0, 2, 4, \ldots$). We again remark that this is not the same on the usual pole description of the S_o-function for Coulomb potential, where the poles in the negative imaginary axis are not present.

It is interesting to note how the bound state poles tend to the points determined by the Rydborg formula as the range b increases. According to (38) new bound states (with zero binding energy) appear for $b = 0{,}74\ a_o$, $3.74\ a_o$, $9.33\ a_o$, $17.35\ a_o$, that is, for values of the range close to the values of the Bohr radius n^2a_o. As b increases from this value, the binding energy tends to the maximum values, given by Rydberg formula. In the first level (the ground state, N = 0) the maximum is reached rapidly: for b equal to 2 a_o the binding energy is less that 0.5 per cont different from the limit value. Thus this binding energy is not much affected by the existence of a tail in the potential. The same is true of the other bound states: if the range of the potential is twice as large as the range necessary to create the bound state, the binding energy and the position of the two "twin" poles are almost the same as if the tail were complete.

From (38), which is valid for finite b and small y, we can see that $(\partial F/\partial y)_{y=0} \neq 0$. This means that the s-wave poles for attractive potential do not enter the imaginary axis in the origin k = 0. In other words, the vertices A', B', C', D' of the curves in Fig. 2 do not coincide with the points A, B, C, D where bound states are formed. Comparing the equation $J_o(2\sqrt{2b+3/2})=$ $=0$ which determines the vertices for distant poles and the equation $J_o(2\sqrt{2b})=0$ which determines the values of b for which the poles cross the origin, we see that the two values of b tend to differ by 0.75 for the very distant poles.

b) <u>The p-wave poles</u>

According to (13), for small values of b the poles are close to the lines $\mathrm{Re}(kb)=N\pi/2$ ($N = \pm 1, \pm 3, \pm 5, \ldots$). As b increases the poles move towards the origin. This is shown in

Fig. 3, where the trajectories in the kb-plane are drawn. For certain values of b two symmetric poles reach the imaginary axis at the origin.

With increasing b one pole moves upwards along the positive imaginary axis to $yb = +\infty$, which other moves downwards to $yb = -\infty$. That the entry point of all the poles is at the origin can be proved in the following way. For small y we have that the equation $Y_\ell^{(1)}(k,b) = 0$ defining pole becomes if $\ell \neq 0$,

$$J_{2\ell}(2\sqrt{2b}) + y^2\, p(\ell,b)\, J_{2\ell+1}(2\sqrt{2b}) = 0 \tag{39}$$

where

$$p(\ell,b) = (\ell+1)(4\ell^2-1+4b)(2b)^{\frac{1}{2}}/[12(2\ell-1)]. \tag{40}$$

Thus the values of b for which the poles pass the origin are given by the roots of

$$J_{2\ell}(2\sqrt{2b}) = 0\,. \tag{41}$$

Due to the square dependence on y, we have that in that points also $(\partial Y_\ell^{(1)}(k,b)/\partial y)_{y=0} = 0$. This means that the origin is also the point in which the trajectory enter the imaginary axis. Clearly these results are valid for all $\ell \neq 0$.

Call b_n a solution of $J_{2\ell}(2\sqrt{2b}) = 0$. Since $p(\ell,b)$ is a positive definite quantity and since

$$J_{2\ell}(2\sqrt{2b}) \xrightarrow[b \to b_n]{} -\left[2\ell/(2b_n)^{\frac{1}{2}}\right](b-b_n)\, J_{2\ell+1}(2\sqrt{2b_n})$$

we can conclude that the curves $y = y(b)$ defined by (39) have for small y, concavities directed towards the large values of b. Clearly these results are valid for all $\ell \neq 0$. All this can be better seen in Fig. 4, where Im(k) is plotted against b, and the purely imaginary poles are indicated. Near the points where the curves $y = y(b)$ cross the b axis, they are symmetric with respect to y (due to the square dependence mentioned above). Thus the vertices coincide with the axis.

The lower parts of the curves in Fig. 4 present plateaux which all occur in the neighbourhood of $kb = -i$. This point has the character of a "sink", attracting the poles: la e variation in the value of b is necessary to remove a pole from the neighbourhood of this point of the kb-plane. All the poles are "attracted" by this point. For b large enough, we can say that almost always there will be a pole around $kb = -i$. This is the pole corresponding to the solution of $H^{(1)}_{3/2}(kb) = 0$, as found in Sec. 4.b.

The bound state poles again tend rapidly to the Rydberg values $Im(k) = 2/(|N| + 3)$ as b increases. Unlike the s-wave case, the "twin" poles tendo to symmetric points $Im(k) = -2/(|N| + 3)$.

c) The d-wave poles

We have here a few complications as compared to the previous cases.

According to (13), for small b the poles are close to lines $xb = N\pi/2$, with $N = 0, \pm 2, \pm 4$, etc. For increasing b the complex poles ($N \neq 0$) move in the complex plane (see Fig. 5) and reach the origin in pairs, for values of b given by (41).

While one of the two poles $|N| = 2$ moves along the positive imaginary axis to $yb = \infty$, the other moves downwards along the negative imaginary axis. In the meantime the pole $N = 0$ is climbing up the negative imaginary axis. For $b = 7.54$ the two poles meet each other in the point $yb = -1.08$. As b is further increased they leave the imaginary axis and pass to the complex plane, one for each side, symmetrically. The symmetry of the S_ℓ-function is thus not destroyed. The two poles describe two "semi-circles", and join again in the imaginary axis in the point $yb = 2.35$ for $b = 13.6$. Now, one of them chooses to go down the imaginary axis, running to $yb = -\infty$, $y = -1/3$, as b increases. The other one (we cannot tell which one) goes up the imaginary axis, travelling towards the origin. In the meantime, the pair of poles with $|N| = 4$ has arrived at the origin, and decided that one of them would go to $yb = +\infty (y = +1/4)$ and the other would travel down the imaginary axis. When $b = 15.6$ this pole reachs the point $yb = -1.04$, and there it meets the pole which was just going up the imaginary axis after having described the semi-circles already described. Since these two poles meet there for the some value of b, they can pass symmetrically to the complex plane without destroying the right-left symmetry of the S_ℓ-function. They do it, describing "semicircles" in the complex plane. The process is thus repeated continuously.

In can be remarked that the poles, when describing the "semicircles", are very much slowed down when near the points $kb = \mp \sqrt{3/2} - i\ 3/2$. This effect is more and more important as

$|N|$ becomes larger. This is analogous to what happened with the point $kb = -i$ in the p-wave case and we can say that as $b \to \infty$, there will always be found a pole in those points. This is in agreement with the results obtained in Sec. 4-b.

In Fig. 6 this rather complicated situation can be observed in a different way. We there plot Im(k) against b. The poles on the imaginary axis are represented by full lines, the complex poles by dotted lines. We have an infinite number of valid labelling of certain lines, since after two poles join each other we cannot tell which is which. Two of the simplest descriptions are attempted in Fig. 6. In one of them the poles $N = 0$ never goes to infinity: it is always describing semicircles, taking charge of meeting other poles to keep symmetry of the S_2-funtion. As $b \to \infty$ it tends to be retained by one of the two points which are the roots of $H^{(1)}_{5/2}(kb) = 0$. In the other interpretation each pole $N \neq 0$ describes semicircles twice: one time symmetrically to a pole of smaller $|N|$, another time to the next larger one. In thin case the $N = 0$ describes only one "semi circle".

6. THE POLES FOR REPULSIVE POTENTIALS

For repulsive potentials the behaviour of the poles is rather simple. In Fig. 7 are drawn the trajectories of the s, p and d-wave poles in the kb plane.

For small values of b the trajectories are close to the vertical lines $xb = N\pi/2$, in agreement with (15). As b increases an infinite number of poles turn towards large values of Re(kb), approaching asymptotically the line yb = - 0.5. In all of them $k^2b \longrightarrow 2$ as $b \longrightarrow \infty$. They form groups, consisting of one trajectory for each ℓ-value which go together to infinity. Each group tending to keep a distance π (in the kb plane) from the next one. All this is in agreement with the results of item in Sec. 4.

For each value of ℓ there are ℓ trajectories (counting those in both left and right hand side planes) with an special behaviour: they remain in the finite kb-plane as $b \longrightarrow \infty$, tending asymptotically to the points which are the roots of $H^{(1)}_{\ell+\frac{1}{2}}(kb) = 0$. These solutions have been mentioned in Sec. 4-b, and are clearly shown in Fig. 7. Only in these trajectories do the several waves differ from each other when b is large.

In the k-plane all the poles go asymptotically to the origin as $b \longrightarrow \infty$. Those which correspond to the trajectories which go to infinity in the k-plane, approach the origin taking the x axis as a tangent, since for them $y/x \longrightarrow 0$. The ℓ trajectories which end in points with finite |kb| are the only ones which do not take the x-axis as a tangent direction.

* * *

APPENDIX I - Proof that $S_\ell(k,b) \longrightarrow 1$ when $b \longrightarrow 0$ with kb finite.

Taking the functional re.at..on among confluent hypergeomatic functions

$$ {}_1F_1(a+1;\ 2a+1;\ z) + {}_1F_1(a;\ 2a+1;\ z) \equiv 2\ {}_1F_1(a;\ 2a;\ z) $$

multiplying by

$$ (z/2)\,{}_1F_1(a+1;\ 2a+2;z) \equiv (a+1/2)\left[{}_1F_1(a+1;\ 2a+1;\ z) - {}_1F_1(a;\ 2a+1;\ z)\right] $$

and rearranging terms we obtain

$$ -\frac{{}_1F_1(a+1;2a+1;z)}{{}_1F_1(a;\ 2a+1;\ z)} = \frac{(z/2)\,{}_1F_1(a+1;2a+2;z) + (2a+1)\,{}_1F_1(a;2a;z)}{(z/2)\,{}_1F_1(a+1;2a+2;z) - (2a+1)\,{}_1F_1(a;2a;z)} $$

The convenience of this is that we obtained a relation between the two F's that appear in $S_\ell(k,\ b;\ \lambda = 0)$ (put $a = \ell$, $z = -2ikb$) and functions of the type ${}_1F_1(p;\ 2p;\ z)$ which can be expressed in terms of Bessel functions. We obtain

$$ -\frac{{}_1F_1(a+1;\ 2a+1;\ z)}{{}_1F_1(a;\ 2a+1;\ z)} = \frac{J_{a+\frac{1}{2}}(iz/2) + i\ J_{a-\frac{1}{2}}(iz/2)}{J_{a+\frac{1}{2}}(iz/2) - i\ J_{a-\frac{1}{2}}(iz/2)} \qquad \text{(A-1)} $$

which can also be written

$$ -\frac{{}_1F_1(a+1;\ 2a+1;\ z)}{{}_1F_1(a;\ 2a+1;\ z)} = \frac{\left[H^{(1)}_{a+\frac{1}{2}}(iz/2) + i\,H^{(1)}_{a-\frac{1}{2}}(iz/2)\right] + \left[H^{(2)}_{a+\frac{1}{2}}(iz/2) + iH^{(2)}_{a-\frac{1}{2}}(iz/2)\right]}{\left[H^{(1)}_{a+\frac{1}{2}}(iz/2) - iH^{(1)}_{a-\frac{1}{2}}(iz/2)\right] + \left[H^{(2)}_{a+\frac{1}{2}}(iz/2) - iH^{(2)}_{a-\frac{1}{2}}(iz/2)\right]} $$

which is enough to proof that $S_\ell(k,b) \longrightarrow 1$ when $\lambda \longrightarrow 0$. Since this corresponds to $|k| \longrightarrow \infty$, it also implies in that $b \rightarrow 0$, since $|kb|$ is supposed to be kept finite.

* * *

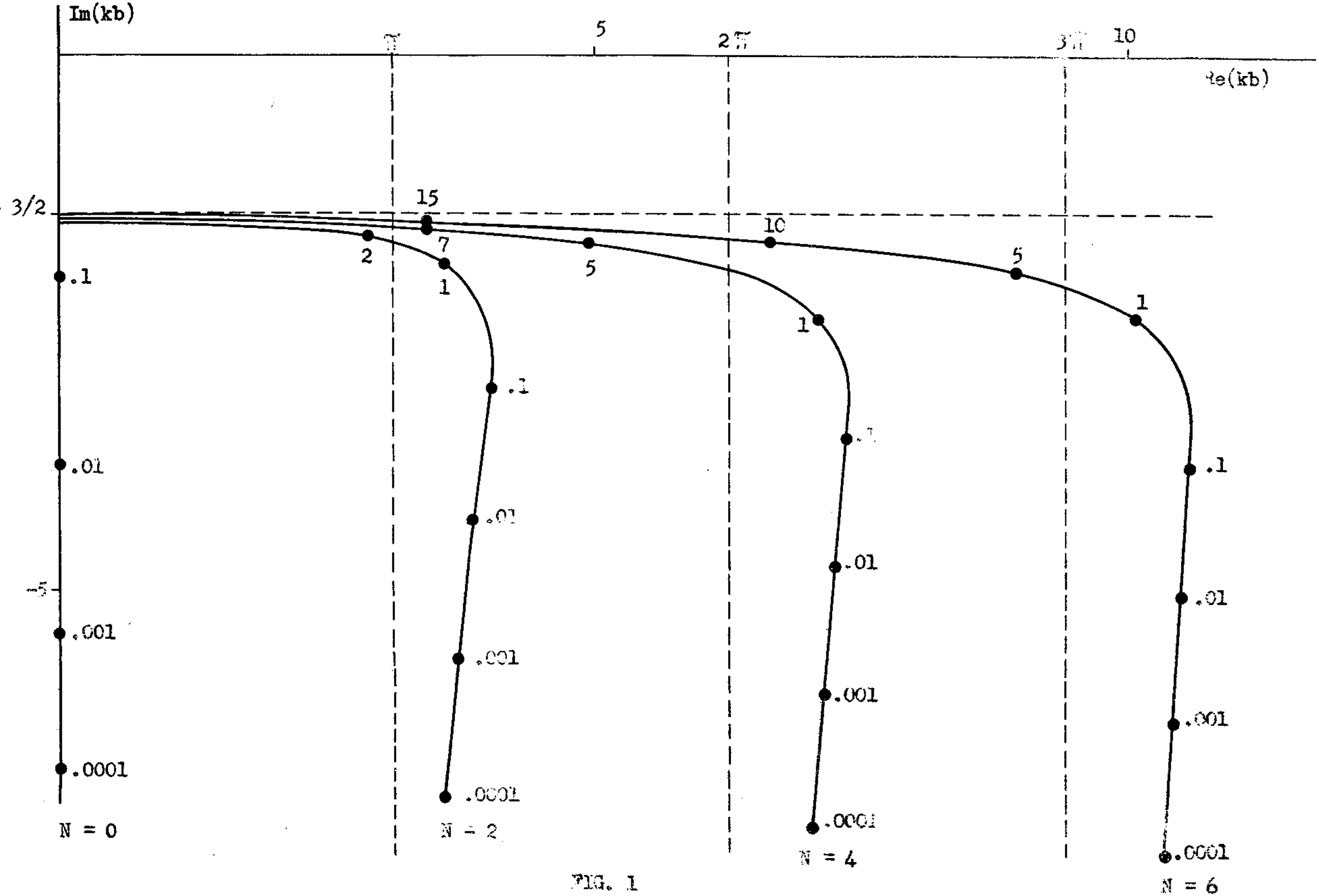

FIG. 1

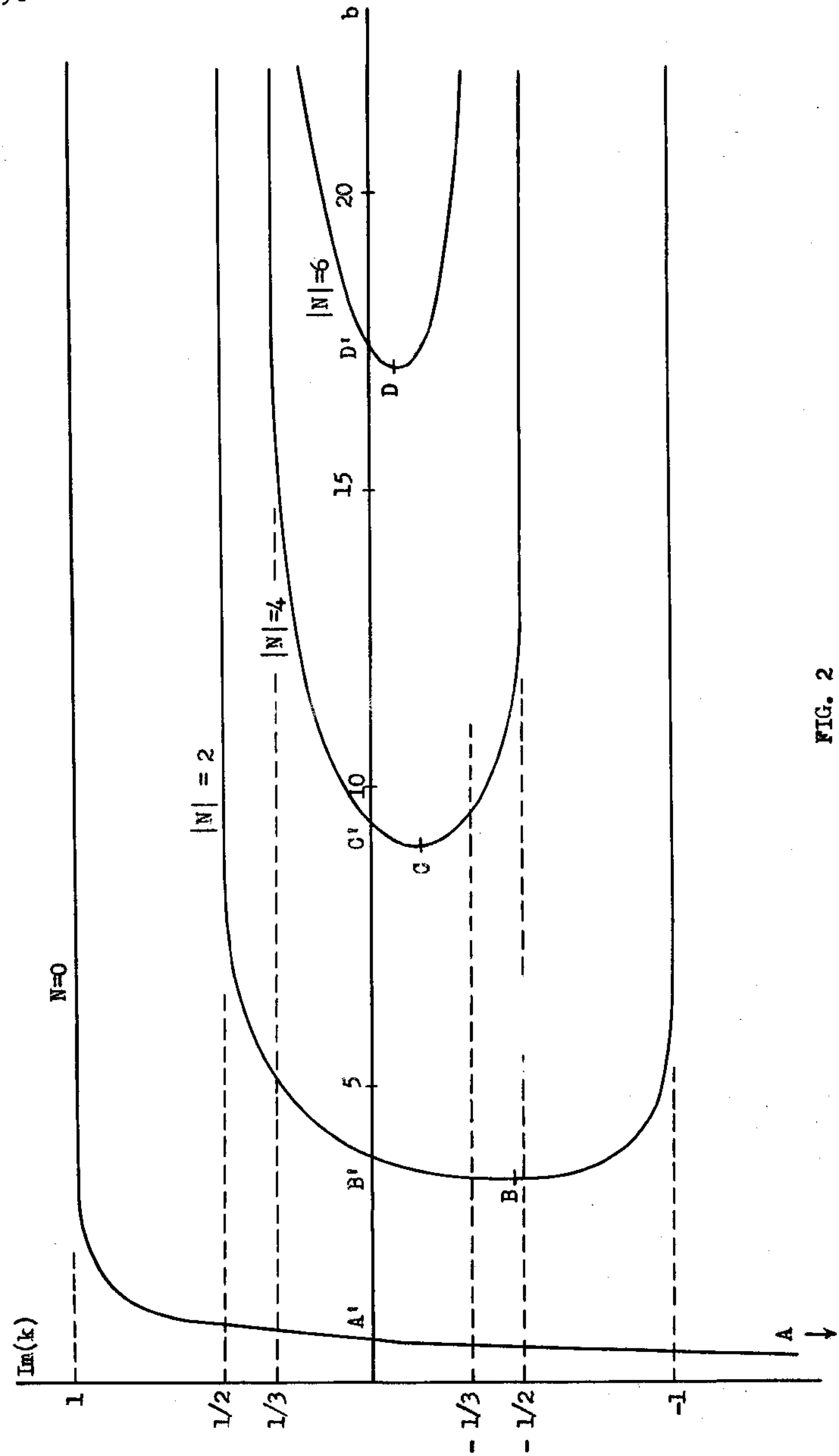

FIG. 2

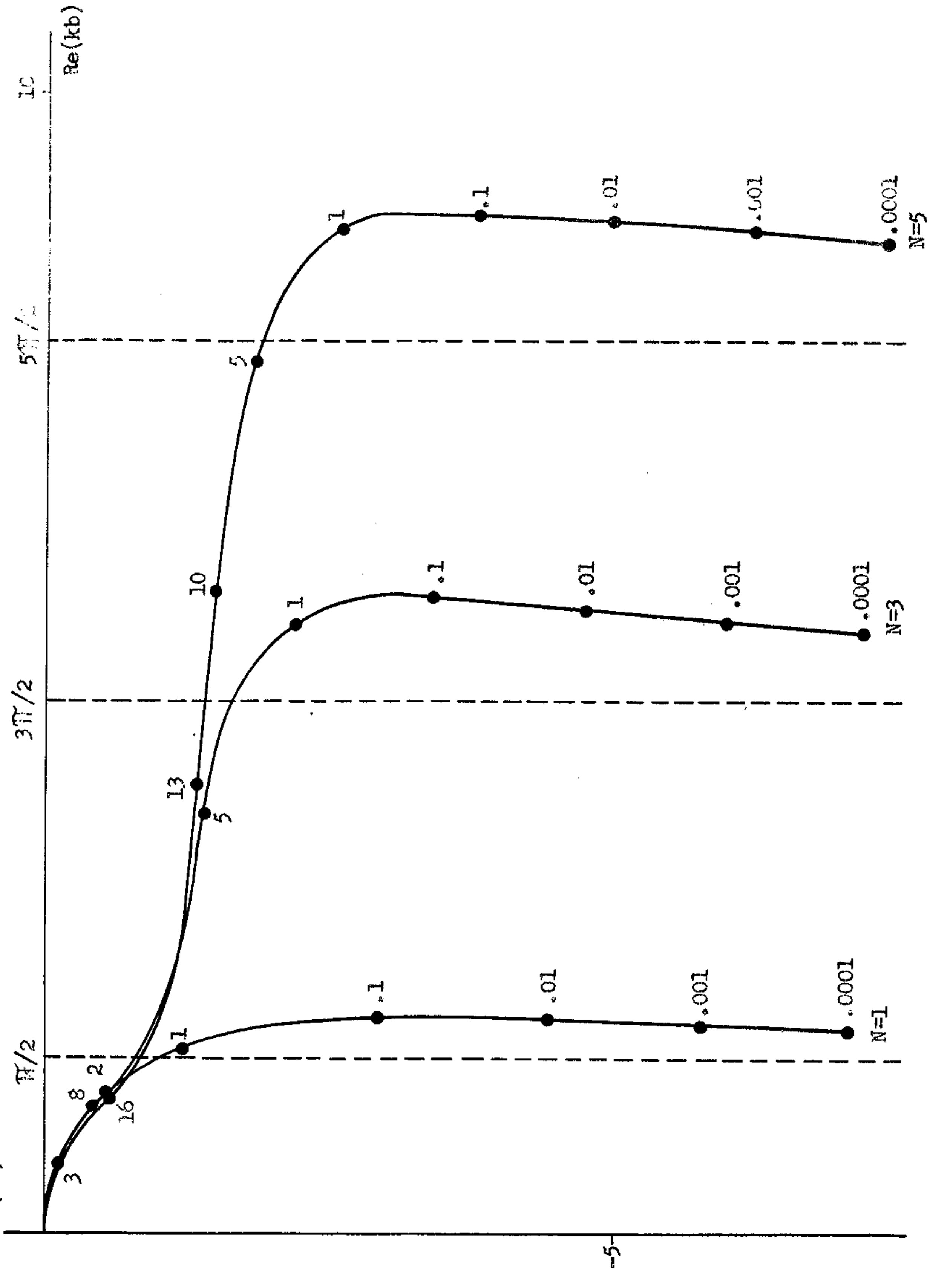

FIG. 3

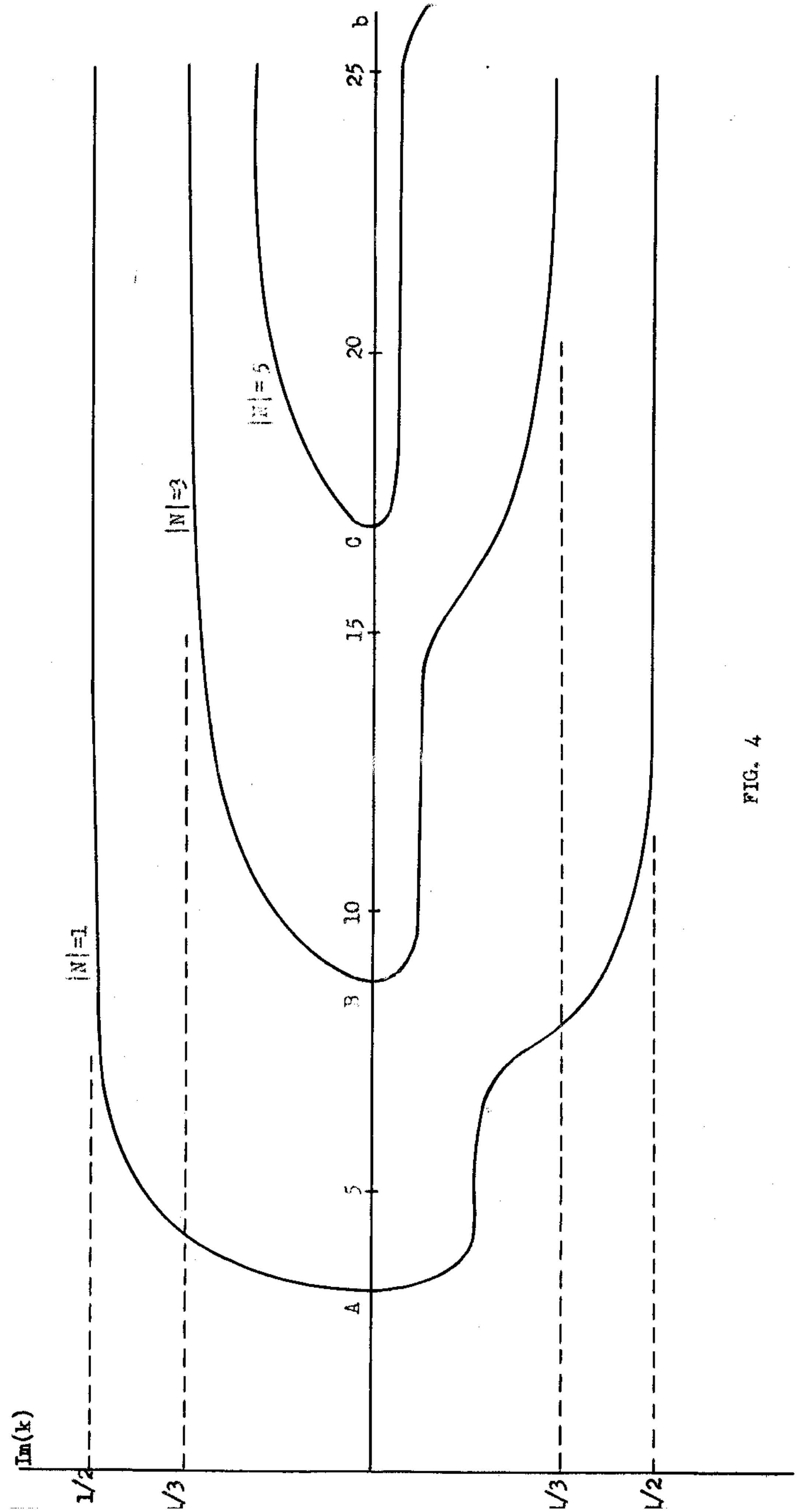

FIG. 4

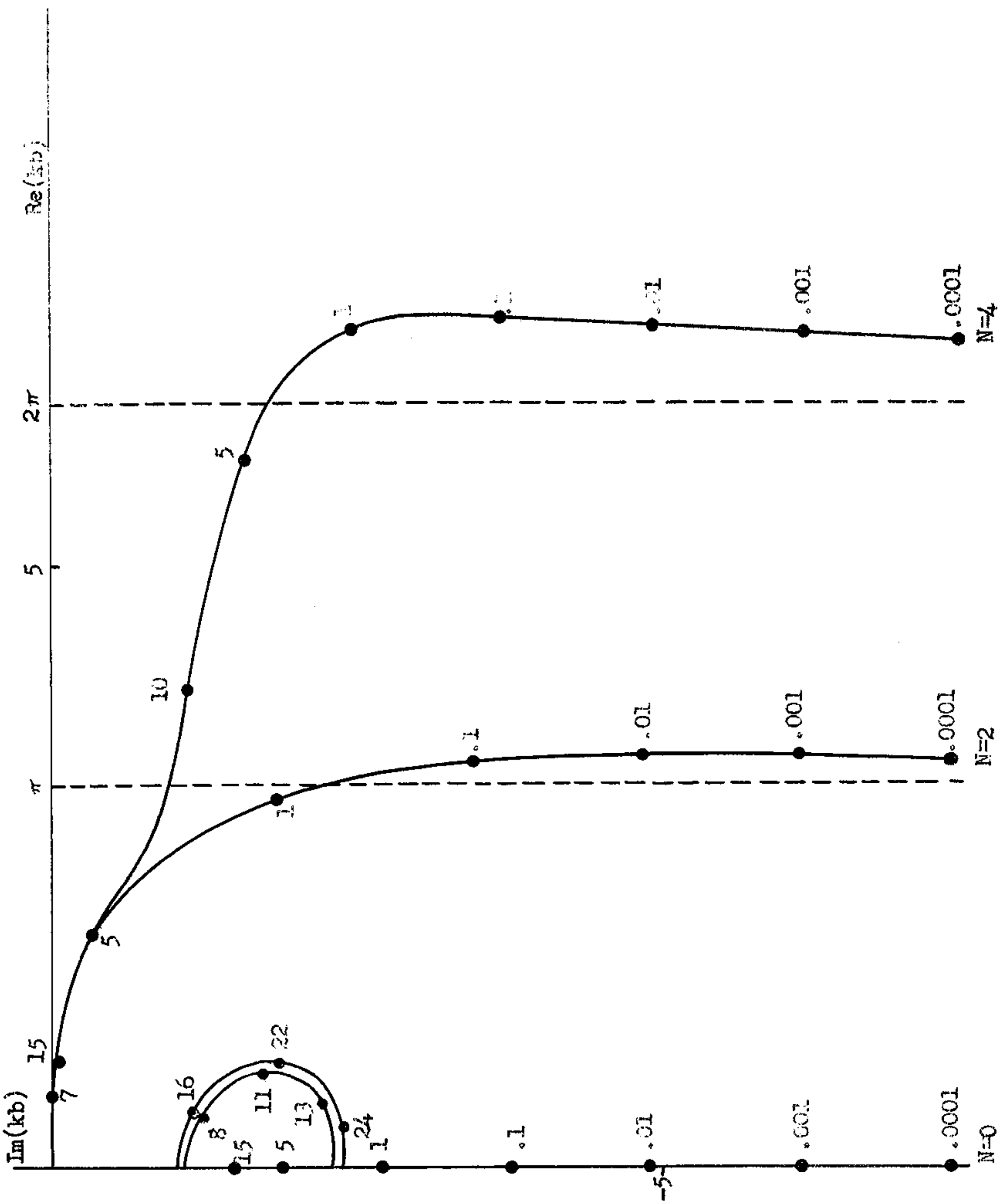

FIG. 5

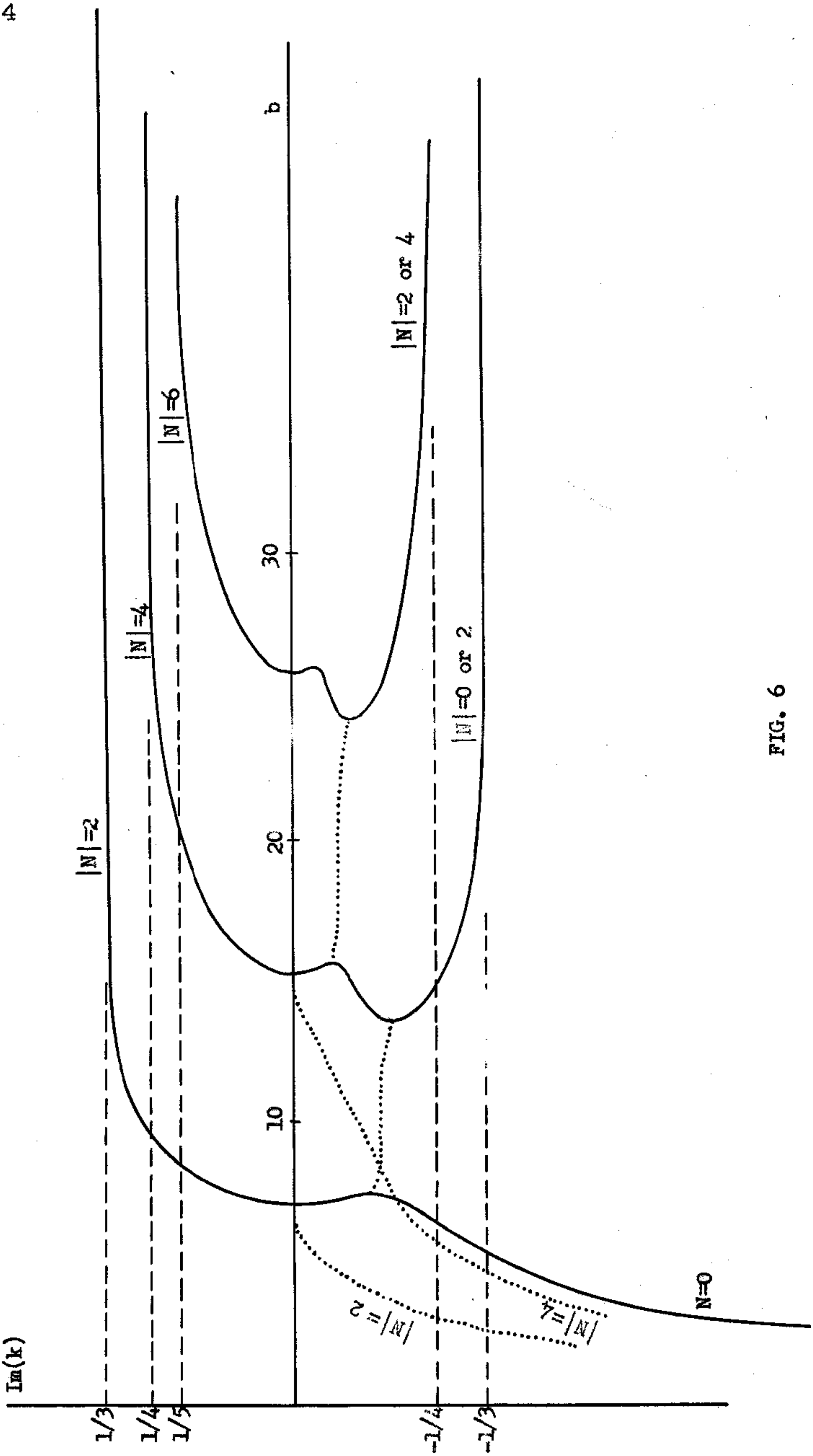

FIG. 6

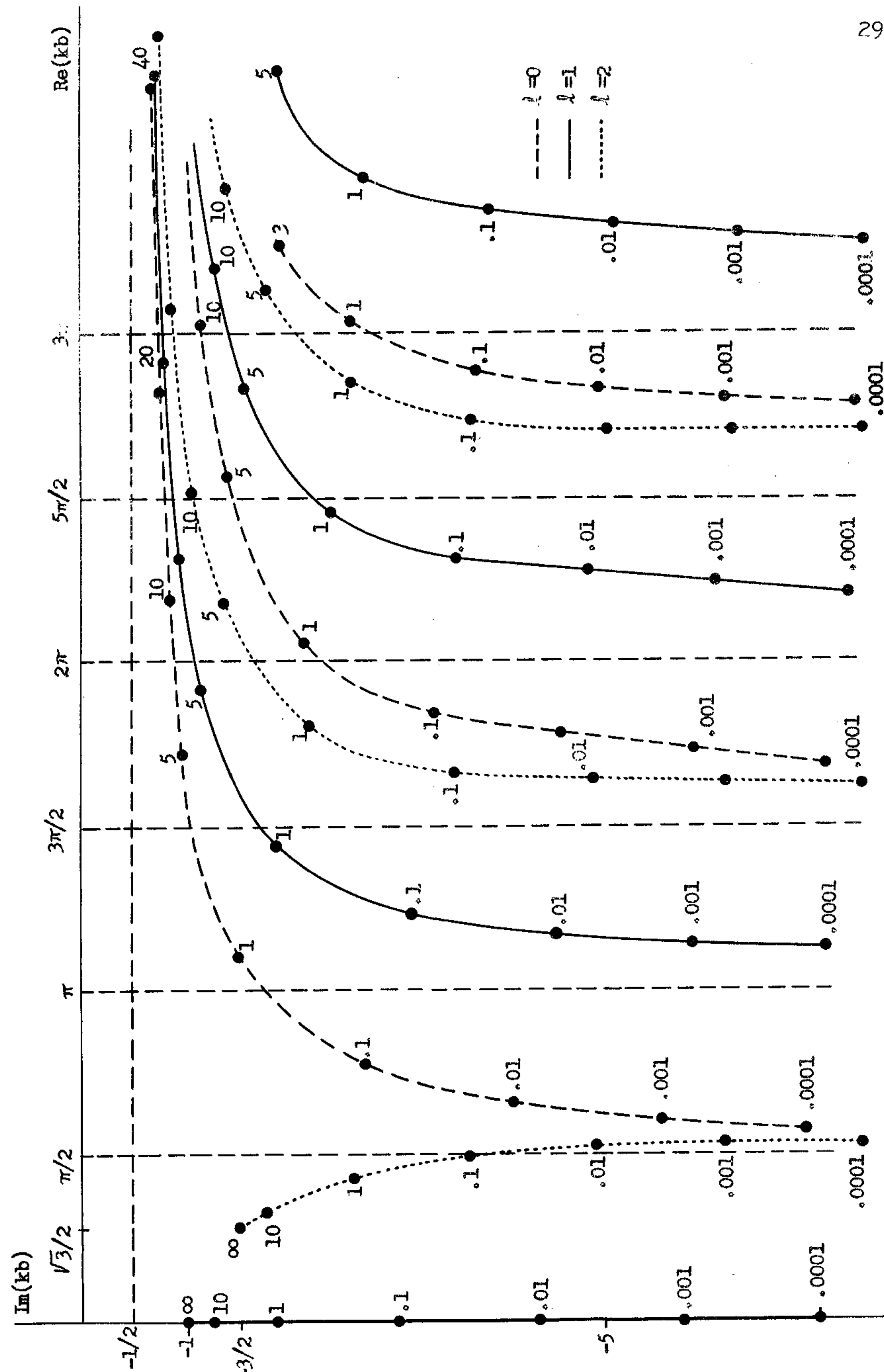

FIG. 7

CAPTIONS FOR THE FIGURES

FIG. 1 - Poles for the S-matrix fo[illegible] s-[illegible]ve scattering by a screened attractive Coulomb potential of [illegible]ge b. The values of b are shown on the curves. The trajectory $N = 0$ is always on the imaginary axis. Each trajectory has a symmetric one for negative values of Re(kb). After two symmetric poles reach the imaginary axis as b increases, one moves upwards and the other moves downwards along the imaginary axis.

FIG. 2 - Purely imaginary poles for s-wave scattering by an attractive screened Coulomb potential. The points A, B, C, D show the values for which the poles reach the imaginary axis. A', B', C', D' correspond to bound states of zero binding energy. The asymptotes $\mathrm{Im}(k) = 2/(|N| + 2)$ give the binding energies of the Rydberg formula. The poles that tend to $\mathrm{Im}(k) = -2/|N|$ are not present in the usual analytic extension of the Coulomb s-matrix.

FIG. 3 - Trajector[illegible] scattering by attractive potentials. The values of the range b are indicated on the curves. All trajectories enter the imaginary axis at the origin, and then run upwards and downwards along the imaginary axis.

FIG. 4 - Purely imaginary poles for p-wave scattering by attractive potentials. The points A, B, C [illegible]he values of b for which the poles reach the origin and ne[illegible] states are formed. As $b \to \infty$ the bound state pole[illegible] to the values of Im(k) corresponding to the binding ener[illegible] given by Rydberg formula. In the Coulomb limit there are [illegible]ric poles in the negative imaginary axis.

FIG. 5 - d-wave poles for attractive potential. As b increases the complex poles move towards the origin, and then follow the imaginary axis. Those which run along the negative imaginary axis pass again to the complex plane, describing "semicircles", and turning back to the imaginary axis and at the points $\mp\sqrt{3}/2 - i\,3/2$. See text for detailed description.

FIG. 6 - Poles of the S-matrix for d-wave scattering by an attractive potential. In full lines are represented the poles on the imaginary axis, in dotted lines the imaginary part of the complex poles.

FIG. 7 - s, p and d-wave poles for repulsive screened Coulomb potentials. All poles which move to infinity tend asymptotically to the line $Im(kb) = -0.5$, with $k^2b \longrightarrow 2$ and $Re(kb) = (2b)^{1/2} + 3\pi/4 + m\pi\ (m = 1, 2, \ldots)$. There are poles which as $b \longrightarrow \infty$ tend to the points defined by $H^{(1)}_{\ell+1/2}(kb) = 0$.

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