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Confluent Heun Equations: Convergence of Solutions in Series of Coulomb Wavefunctions

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Abstract

The Leaver solutions in series of Coulomb wave functions for the confluent Heun equation are given by two-sided infinite series, that is, by series where the summation index n runs from minus to plus infinity [E. W. Leaver, J. Math. Phys. **27**, 1238 (1986)]. First we show that, in contrast to the D'Alembert test, under certain conditions the Raabe test assures that the domains of convergence of these solutions include an additional singular point. We also consider solutions for a limit of the confluent Heun equation. For both equations, new solutions are generated by transformations of variables. Finally we discuss the time dependence of the Klein-Gordon equation in two cosmological models and the spatial dependence of the Schrödinger equation to a family of quasi-exactly solvable potentials. For a subfamily of these potentials we obtain infinite-series solutions which converge and are bounded for all values of the independent variable, in opposition to a common belief.

1 Introduction

In 1986 Leaver [1] found two types of solutions in series of confluent hypergeometric functions for the confluent Heun equation (CHE) and presented a limit procedure to generate solutions for the double-confluent Heun equation (DCHE) out of solutions for the CHE. Later on we have found that there are two other physically relevant equations whose solutions can also be derived from the Leaver solutions for the CHE and DCHE by means of a procedure called Whittaker-Ince limit [2, 3, 4]. Further, from solutions of the CHE and/or DCHE, we can find solutions for the Mathieu, Whittaker-Hill and spheroidal equations [1, 5].

In view of the above connections, from the inception we establish the convergence properties of Leaver's solutions. We consider only the expansions in series of Coulomb wave functions which are given by a set of three solutions, one in series of regular confluent hypergeometric functions and two in series of irregular functions. By redefining the Coulomb functions, we avoid difficulties arising from the Leaver definitions and find that the convergence of the solutions for the CHE and its Whittaker-Ince limit follows from the Raabe test. Furthermore, we investigate the transformations of these solutions.

First we write the aforementioned equations, present the connections among them and call attention for the fact that there are three types of series expansions whose convergence require different treatments. After that we introduce the D'Alembert and Raabe tests for convergence and outline the structure of the article.

The Leaver form for the CHE is [1]

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3-2\eta\omega(z-z_0)+\omega^2z(z-z_0)\right]U=0, \tag{1}$$

where B_i, η and ω are constants. The points $z=0$ and $z=z_0$ (if $z_0\neq 0$) are regular singular points, whereas $z=\infty$ is an irregular point. Since z_0 is free, by taking $z_0=0$

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Leaver obtained the DCHE

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left(B_3-2\eta\omega z+\omega^2z^2\right)U=0,\qquad [B_1\neq 0,\ \omega\neq 0] \tag{2}$$

in which case $z=0$ and $z=\infty$ are both irregular singularities.

In addition, the CHE and the DCHE admit a (Whittaker-Ince) limit which changes the nature of the singularity at $z=\infty$, keeping unaltered the other singular points. This limit is given by [2, 3]

$$\omega\rightarrow 0,\quad \eta\rightarrow\infty,\ \text{such that}\quad 2\eta\omega=-q,\quad \text{(Whittaker-Ince limit)} \tag{3}$$

where q is a nonvanishing constant. The Whittaker-Ince limit of the CHE is

$$z(z-z_0)\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+\left[B_3+q(z-z_0)\right]U=0,\qquad (q\neq 0) \tag{4}$$

(if $q=0$ this equation can be transformed into a hypergeometric equation), while the limit of the DCHE is

$$z^2\frac{d^2U}{dz^2}+(B_1+B_2z)\frac{dU}{dz}+(B_3+qz)\,U=0,\qquad (q\neq 0,\ B_1\neq 0) \tag{5}$$

(if $q=0$ and/or $B_1=0$ the equation degenerates into a confluent hypergeometric equation or simpler equations). This last equation also follows from Eq. (4) when $z_0=0$ (Leaver's limit).

The Mathieu, Whittaker-Hill and spheroidal equations have been studied by themselves, but they are particular cases of the above equations. The Mathieu equation reads [6]

$$\frac{d^2w}{du^2}+\sigma^2[\mathrm{a}-2k^2\cos(2\sigma u)]w=0,\quad \text{(Mathieu equation)} \tag{6}$$

where a and k are constants, while $\sigma=1$ or $\sigma=i$ for the Mathieu or modified Mathieu equation, respectively. This equation is transformed into particular instances of Eqs. (1), (2) and (4) by the substitutions of variables [3, 5]. The Whittaker-Hill equation (WHE) can be written in the form [7, 8]

$$\frac{d^2W}{du^2}+\varsigma^2\left[\vartheta-\frac{1}{8}\xi^2-(p+1)\xi\cos(2\varsigma u)+\frac{1}{8}\xi^2\cos(4\varsigma u)\right]W=0,\quad \text{(WHE)}. \tag{7}$$

where ϑ, ξ and p are parameters; if u is a real variable, this represents the WHE when $\varsigma=1$ and the modified WHE when $\varsigma=i$. The WHE reduces to the CHE (1) and DCHE (2) by the substitutions [3, 5]. Finally, the spheroidal equation reads [9]

$$\frac{d}{dy}\left[\left(1-y^2\right)\frac{dS(y)}{dy}\right]+\left[\lambda+\gamma^2(1-y^2)-\frac{\mu^2}{1-\ y^2}\right]S(y)=0, \tag{8}$$

where γ, λ and μ are constants. The substitutions (58a) transform this into a special case of the CHE (1).

On the other side, in general there are solutions given by three different types of series, called two-sided infinite series, one-sided infinite series and finite series. These take, respectively, the forms

$$\sum_n a_n\ f_n^{\nu}(z):=\sum_{n=-\infty}^{\infty}a_n\ f_n^{\nu}(z),\qquad \sum_{n=0}^{\infty}b_n\ g_n(z),\qquad \sum_{n=0}^{N}b_n\ g_n(z), \tag{9}$$

where $f_n^{\nu}(z)$ and $g_n(z)$ are functions of the independent variable z, N is a non-negative integer and ν is a parameter which does not appear in the differential equations. In the present case, the series coefficients a_n and b_n satisfy three-term recurrence relations. No finite-series solutions are known for the Mathieu equation (6) nor for the Whittaker-Ince limit (5) of the DCHE.

Two-sided infinite series, the only considered by Leaver, are necessary to assure the convergence of solutions of equations in which there is no free parameter, as in scattering problems [2] or in some wave equations in curved spacetimes [10]. In such cases the parameter ν must be determined as solutions of a transcendental (characteristic) equation. However, when truncated on the left ($n \geq 0$), the two-sided infinite series give one-sided infinite series which are useful for equations having a free parameter; in turn, these lead to solutions given by finite series if the parameters of the equation satisfy certain constraints.

Finite-series solutions are suitable for quasi-exactly solvable (QES) problems, that is, for quantum-mechanical problems where one part of energy spectrum and the respective eigenfunctions can be computed explicitly [11, 12]. For QES problems obeying equations of the Heun family [3], that part of the spectrum may be derived from finite-series solutions if these are known. Indeed, a problem is also said to be QES if it admits solutions given by finite series whose coefficients necessarily satisfy three-term or higher order recurrence relations [13], and is said to be exactly solvable if its solutions can be expressed by hypergeometric functions.

The convergence of two-sided infinity series is obtained from the limits

$$L_1(z) = \lim_{n\to\infty}\left|\frac{a_{n+1}^{\nu} f_{n+1}^{\nu}(z)}{a_n^{\nu} f_n^{\nu}(z)}\right|, \qquad L_2(z) = \lim_{n\to-\infty}\left|\frac{a_{n-1}^{\nu} f_{n-1}^{\nu}(z)}{a_n^{\nu} f_n^{\nu}(z)}\right|. \tag{10}$$

By the D'Alembert test the series converges in the intersection of the regions for which $L_1 < 1$ and $L_2 < 1$, and diverges otherwise (if $L_1 = L_2 = 1$, the test is inconclusive). Leaver's definitions for the Coulomb wave functions lead to ratios between terms presenting square roots (except if $\eta = 0$) which make difficult to deal with the convergence tests. To avoid this problem, we use alternative definitions that, in addition, permit to apply the Raabe test for the solutions of the CHEs. By the Raabe test [14, 15], if

$$L_1(z) = 1 + (A/n), \qquad L_2(z) = 1 + (B/|n|), \tag{11}$$

where A and B are constants, then the series converges in the region where $A < -1$ and $B < -1$, and diverges otherwise (if $A = B = -1$, the test is inconclusive). For one-sided series the convergence may be enhanced since we use only the limit L_1, while for finite series the convergence must be decided from the behaviour of each term of the series.

Furthermore, by using transformations of variables we find four sets of two-sided solutions instead of one set as in Leaver. By the Raabe test, under certain conditions these solutions converge absolutely for $|z| \geq |z_0|$ or $|z - z_0| \geq |z_0|$ rather than for $|z| > |z_0|$ or $|z - z_0| > |z_0|$; the one-sided solutions given by series of regular confluent hypergeometric functions converge for $|z| \geq 0$. Nevertheless, the behaviour of each solution for $z \to \infty$ must be analysed carefully because, in computing $L_1(z)$ and $L_2(z)$, we assume that z is bounded. We have also to examine the behaviour of the solutions at the finite singular points because the series appear multiplied by factors which may become unbounded at such points.

For brevity, here we do not consider all the above points. Indeed, we deal only with the two-sided solutions for the CHE and its Whittaker-Ince limit. However, for later reference, elsewhere [16] we provide the one-sided solutions as well as the solutions for the DCHE (2) and its limit (5).

In Section 2.1 we discuss the two-sided infinite expansions for CHE (1), and in Sec. 2.2 we consider the corresponding Whittaker-Hill limit. For the spheroidal equation, we get the

Meixner solutions in series of Bessel functions [9] instead of the Chu and Stratton solutions [17] mentioned by Leaver. For the Mathieu equation we recover known solutions, but now the convergence is improved by the Raabe test.

In section 3, we consider examples which illustrate the consequences of Raabe test. In particular, we show that the Schrödinger equation for some quasi-exactly solvable potentials admits infinite-series solutions which are convergent and bounded for all values of the independent variable. Thus, in addition to the energy levels resulting from finite series, in principle it is possible to get additional energy levels as solutions of characteristic equations corresponding to the infinite series.

In section 4 there are some final considerations. Appendix A gives the normalization used for the Coulomb functions and takes the case $\eta = 0$ as a criterion to decide in favour of one of two possibilities for the ratio between successive Coulomb functions. The derivation of the recurrence relations for the series coefficients is given in Appendix B.

2 The two-sided Series Expansions

In this section we examine separately the two-sided expansions for the CHE and for its Whittaker-Ince limit. In this limit, the expansions in series of Coulomb functions give solutions in series of Bessel functions. The solutions of the CHE with $\eta = 0$ are also expressible by series of Bessel functions. In all cases, given an initial set of solutions, new sets are generated by transformations of variables which preserve the form of the differential equations. Notice that the linear independence of the functions used as basis for the series expansions will impose restrictions on the characteristic parameter ν and/or on some parameters of the differential equations.

In Eqs. (60-62) we recover the Meixner solutions for the spheroidal equation as particular cases of the solutions for the CHE, while in Eqs. (81a-81c) we recover the usual solutions in series of Bessel functions for the Mathieu equation as particular cases of the solutions for the Whittaker-Ince limit of the CHE.

2.1 Solutions for the CHE

The initial set of solutions, $\mathbb{U}_1(z)$, is reconstructed in Appendix B. It reads

$$\mathbb{U}_1(z) = z^{-\frac{B_2}{2}} \sum_n b_n^1 \mathscr{U}_{n+\nu}(\eta, \omega z), \quad \mathscr{U}_{n+\nu}(\eta, \omega z) = \left(\phi_{n+\nu}, \psi^+_{n+\nu}, \psi^-_{n+\nu}\right)(\eta, \omega z), \tag{12}$$

where $\sum_n$ denotes two-sided series, $\phi_{n+\nu}$ and $\psi^{\pm}_{n+\nu}$ represent the Coulomb wave functions defined in Eqs. (A.12) and (A.13), and the coefficients b_n^1 satisfy three-term recurrence relations. So, we have a set of three expansions, one in series of regular confluent hypergeometric functions and two in series of irregular functions. This set corresponds to Leaver's solutions who have used the definitions (A.14) and (A.15) for the Coulomb functions. In addition, if $U(z) = U(B_1, B_2, B_3; z_0, \omega, \eta; z)$ denotes an arbitrary solution of the CHE, we can find other solutions by means of the transformations T_1, T_2, T_3 and T_4 which operate as [3, 5]

$$\begin{aligned}
&T_1 U(z) = z^{1+B_1/z_0} U(C_1, C_2, C_3; z_0, \omega, \eta; z), \\
&T_2 U(z) = (z - z_0)^{1-B_2-B_1/z_0} U(B_1, D_2, D_3; z_0, \omega, \eta; z), \\
&T_3 U(z) = U(B_1, B_2, B_3; z_0, -\omega, -\eta; z), \\
&T_4 U(z) = U(-B_1 - B_2 z_0, B_2, B_3 + 2\eta\omega z_0; z_0, -\omega, \eta; z_0 - z),
\end{aligned} \tag{13}$$

where

$$\begin{aligned} &C_1=-B_1-2z_0, \qquad C_2=2+B_2+\tfrac{2B_1}{z_0}, \qquad C_3=B_3+\left(1+\tfrac{B_1}{z_0}\right)\left(B_2+\tfrac{B_1}{z_0}\right), \\ &D_2=2-B_2-\tfrac{2B_1}{z_0}, \qquad D_3=B_3+\tfrac{B_1}{z_0}\left(\tfrac{B_1}{z_0}+B_2-1\right). \end{aligned} \tag{14}$$

These transformations allow constructing a group with 4 sets of two-sided series $\mathbb{U}_i$ ($i=1,\cdots,4$) where coefficients b_n^i satisfy recurrence relations having the form

$$\alpha_n^i b_{n+1}^i+\beta_n^i b_n^i+\gamma_n^i b_{n-1}^i=0, \qquad [-\infty<n<\infty] \tag{15}$$

where α_n^i, β_n^i and γ_n^i depend on the parameters of the differential equation as well as on ν and n. These relations lead to transcendental (characteristic) equations given as a sum of two infinite continued fractions. By omitting the superscripts of α_n^i, β_n^i and γ_n^i, the characteristic equations read

$$\beta_0=\frac{\alpha_{-1}\gamma_0}{\beta_{-1}-}\;\frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2}-}\;\frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3}-}\cdots+\;\frac{\alpha_0\gamma_1}{\beta_1-}\;\frac{\alpha_1\gamma_2}{\beta_2-}\;\frac{\alpha_2\gamma_3}{\beta_3-}\cdots \tag{16}$$

which are equivalent to the vanishing of the determinants of infinite tridiagonal matrices, as in Eq. (32). If the CHE has no free parameter, Eq. (16) may be used to find the possible values of ν (characteristic parameter); if the CHE has an arbitrary parameter, Eq. (16) permits to find the values of that parameter corresponding to suitable values of ν.

To analyse the properties of the solutions we write explicitly each of the three solutions, instead of using the abbreviated form (12). Thus, we denote by $\mathbb{U}_1=\left(U_1,U_1^+,U_1^-\right)$ the solutions associated respectively with the functions $\left(\phi_{n+\nu},\ \psi_{n+\nu}^+,\ \psi_{n+\nu}^-\right)$. This gives the solutions (19) which, by the transformations (13), generate the three sets of solutions that have not been considered by Leaver. The four sets of two-sided solutions are denoted by

$$\mathbb{U}_i(z)=\left[U_i(z),U_i^+(z),U_i^-(z)\right], \qquad i=1,\cdots,4, \tag{17}$$

if $\eta\neq0$; if $\eta=0$ the notation is given in Eq. (47). These solutions correspond to eight sets of one-sided solutions [16] which are denoted by

$$\mathring{\mathbb{U}}_i(z)=\left[\mathring{U}_i(z),\mathring{U}_i^+(z),\mathring{U}_i^-(z)\right], \qquad i=1,\cdots,8, \tag{18}$$

and do not depend on ν. In fact they are generated by expressing the parameter ν of each $\mathbb{U}_i$ as two different functions of the parameters of the CHE. The convergence of the solutions $\mathring{\mathbb{U}}_i$ is obtained by considering only the limits $n\to\infty$ in the computations given in the present article.

2.1.1 The four sets of the solutions

Explicitly the first set $\mathbb{U}_1$, given in Eq. (12), reads

$$\begin{aligned} U_1(z)&=e^{i\omega z}\sum_n\frac{b_n^1\,[2i\omega z]^{n+\nu+1-\frac{B_2}{2}}}{\Gamma[2n+2\nu+2]}\Phi\left[n+\nu+1+i\eta,2n+2\nu+2;-2i\omega z\right] \\ U_1^{\pm}(z)&=e^{\pm i\omega z}\sum_n\frac{b_n^1\,[-2i\omega z]^{n+\nu+1-\frac{B_2}{2}}}{\Gamma[n+\nu+1\mp i\eta]}\Psi\left[n+\nu+1\pm i\eta,2n+2\nu+2;\mp2i\omega z\right], \end{aligned} \tag{19}$$

where, in the recurrence relations (15) for b_n^1,

$$\begin{aligned}
\alpha_n^1 &= \tfrac{2i\omega z_0\left[n+\nu+2-\frac{B_2}{2}\right]\left[n+\nu+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)},\\
\beta_n^1 &= -B_3-\eta\omega z_0-\left(n+\nu+1-\tfrac{B_2}{2}\right)\left(n+\nu+\tfrac{B_2}{2}\right)-\tfrac{\eta\omega z_0[B_2-2]\left[B_2+\frac{2B_1}{z_0}\right]}{(2n+2\nu)(2n+2\nu+2)},\\
\gamma_n^1 &= -\tfrac{2i\omega z_0\left[n+\nu+\frac{B_2}{2}-1\right]\left[n+\nu+\frac{B_2}{2}+\frac{B_1}{z_0}\right](n+\nu+i\eta)(n+\nu-i\eta)}{(2n+2\nu-1)(2n+2\nu)}.
\end{aligned} \tag{20}$$

By applying the transformation T_3 on $\mathbb{U}_1$, we find the equivalence

$$T_3\left[U_1(z),U_1^+(z),U_1^-(z)\right] \quad \Leftrightarrow \quad \left[U_1(z),U_1^-(z),U_1^+(z)\right]. \tag{21}$$

Precisely, we find $T_3\left[U_1,U_1^+,U_1^-\right]=\left[\bar{U}_1,\bar{U}_1^-,\bar{U}_1^+\right]$ with

$$\bar{U}_1(z)=e^{i\omega z}\sum_n \frac{\bar{b}_n^1\ [-2i\omega z]^{n+\nu+1-B_2/2}}{\Gamma[2n+2\nu+2]}\Phi\left(n+\nu+1+i\eta,2n+2\nu+2;-2i\omega z\right),$$

$$\bar{U}_1^{\pm}(z)=e^{\pm i\omega z}\sum_n \frac{\bar{b}_n^1\ [2i\omega z]^{n+\nu+1-B_2/2}}{\Gamma[n+\nu+1\mp i\eta]}\Psi\left(n+\nu+1\pm i\eta,2n+2\nu+2;\mp 2i\omega z\right),$$

where the recurrence relations for $\bar{b}_n^1$ are

$$-\alpha_n^1\ \bar{b}_{n+1}^1+\beta_n^1\ \bar{b}_n^1-\gamma_n^1\ \bar{b}_{n-1}^1=0.$$

Up to a multiplicative constant independent of n, we can set $\bar{b}_n^1=(-1)^n b_n^1$ in order to establish relation (21). Thus, the transformation T_3 is ineffective in the present case. The remaining transformations allow to form a group constituted by four sets of solutions, namely,

$$\mathbb{U}_1(z),\quad \mathbb{U}_2(z)=T_2\mathbb{U}_1(z);\quad \mathbb{U}_3(z)=T_4\mathbb{U}_1(z),\quad \mathbb{U}_4(z)=T_4\mathbb{U}_2(z)=T_1\mathbb{U}_3(z). \tag{22}$$

In fact, from the explicit forms of these sets, one verifies that T_1 does not generate new solutions when applied on $\mathbb{U}_1$ and $\mathbb{U}_2$; similarly, T_2 has no effect on $\mathbb{U}_3$ and $\mathbb{U}_4$.

The three sets of solutions generated by the preceding transformations are written below.

$$\begin{aligned}
U_2(z)&=f_2(z)e^{i\omega z}\sum_n \frac{b_n^2[i\omega z]^n}{\Gamma[+2\nu+2]}\Phi\left[n+\nu+1+i\eta,2n+2\nu+2;-2i\omega z\right]\\
U_2^{\pm}(z)&=f_2(z)e^{\pm i\omega z}\sum_n \frac{b_n^2[-2i\omega z]^n}{\Gamma[n+\nu+1\mp i\eta]}\Psi\left[+\nu+1\pm i\eta,2n+2\nu+2;\mp 2i\omega z\right]
\end{aligned} \tag{23}$$

where $f_2=f_2(z)=z^{\nu+(B_1/z_0)+(B_2/2)}\left(z-z_0\right)^{1-B_2-(B_1/z_0)}$. The coefficients for the recurrence relations are given by

$$\begin{aligned}
\alpha_n^2&=\tfrac{2i\omega z_0\left[n+\nu+\frac{B_2}{2}\right]\left[n+\nu+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)},\qquad \beta_n^2=\beta_n^1,\\
\gamma_n^2&=-\tfrac{2i\omega z_0\left[n+\nu+1-\frac{B_2}{2}\right]\left[n+\nu-\frac{B_2}{2}-\frac{B_1}{z_0}\right](n+\nu+i\eta)(n+\nu-i\eta)}{(2n+2\nu-1)(2n+2\nu)}.
\end{aligned} \tag{24}$$

The transformation T_4 acting on $\mathbb{U}_1$ gives the set $\mathbb{U}_3=\mathbb{U}_3(z)$, namely,

$$\begin{aligned}
U_3&=f_3e^{i\omega[z-z_0]}\sum_n \frac{b_n^3\left[2i\omega(z-z_0)\right]^n}{\Gamma[2n+2\nu+2]}\Phi\left[n+\nu+1+i\eta,2n+2\nu+2;-2i\omega(z-z_0)\right],\\
U_3^{\pm}&=f_3e^{\pm i\omega[z-z_0]}\sum_n \frac{b_n^3\left[2i\omega(z_0-z)\right]^n}{\Gamma[n+\nu+1\mp i\eta]}\times\\
&\qquad\Psi\left[n+\nu+1\pm i\eta,2n+2\nu+2;\mp 2i\omega(z-z_0)\right],
\end{aligned} \tag{25}$$

where $f_3=f_3(z)=(z-z_0)^{\nu+1-(B_2/2)}$ and, in the recurrence relations,

$$\begin{aligned}
&\alpha_n^3=-\frac{2i\omega z_0\left[n+\nu+2-\frac{B_2}{2}\right]\left[n+\nu+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)}, \qquad \beta_n^3=\beta_n^1,\\
&\gamma_n^3=\frac{2i\omega z_0\left[n+\nu-1+\frac{B_2}{2}\right]\left[n+\nu-\frac{B_2}{2}-\frac{B_1}{z_0}\right](n+\nu+i\eta)(n+\nu-i\eta)}{(2n+2\nu-1)(2n+2\nu)}.
\end{aligned} \tag{26}$$

The fourth set, obtained by applying T_1 on $\mathbb{U}_3$, reads

$$\begin{aligned}
&U_4=f_4e^{i\omega[z-z_0]}\sum_n\frac{b_n^4\left[2i\omega(z-z_0)\right]^n}{\Gamma[2n+2\nu+2]}\Phi\left[n+\nu+1+i\eta,2n+2\nu+2;-2i\omega(z-z_0)\right],\\
&U_4^{\pm}=f_4e^{\pm i\omega[z-z_0]}\sum_n\frac{b_n^4\left[2i\omega(z_0-z)\right]^n}{\Gamma[n+\nu+1\mp i\eta]}\times\\
&\qquad\qquad\qquad\Psi\left[n+\nu+1\pm i\eta,2n+2\nu+2;\mp2i\omega(z-z_0)\right],
\end{aligned} \tag{27}$$

where $f_4=f_4(z)=z^{1+(B_1/z_0)}\left(z-z_0\right)^{\nu-(B_2/2)-(B_1/z_0)}$ and, in recurrence relations for b_n^4,

$$\begin{aligned}
&\alpha_n^4=-\frac{2i\omega z_0\left[n+\nu+\frac{B_2}{2}\right]\left[n+\nu+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)}, \qquad \beta_n^4=\beta_n^1,\\
&\gamma_n^4=\frac{2i\omega z_0\left[n+\nu+1-\frac{B_2}{2}\right]\left[n+\nu+\frac{B_2}{2}+\frac{B_1}{z_0}\right](n+\nu+i\eta)(n+\nu-i\eta)}{(2n+2\nu-1)(2n+2\nu)}.
\end{aligned} \tag{28}$$

If there is no free parameter in the CHE, ν must be determined as solutions of a characteristic equation. However, by considering the form of the solutions and respective recurrence relations for the series coefficients, we find that

$$2\nu \ \text{ and } \ \nu\mp i\eta \ \text{ cannot be integers} \tag{29}$$

for two-sided series. The restriction $\nu\mp i\eta\neq$integer assures that factors $1/\Gamma(n+\nu+1\pm i\eta)$ which appear in $U_i^{\pm}(z)$ are not zero for any value of n; assures as well that the factors $(n+\nu+i\eta)(n+\nu-i\eta)$ in γ_n^i do not vanish for any n. In fact, such restriction is necessary to have two-sided infinite series for the three solutions in each of the four sets $\mathbb{U}_i$.

The condition $2\nu\neq$integer is necessary in order to avoid two terms linearly dependent in the series of $U_i^{\pm}(z)$. Indeed, suppose that $2\nu=$integer in the solutions $U_1^{\pm}(z)$. These are series expansions in terms of

$$B_n^{\pm}(z)=[-2i\omega z]^{n+\nu+1}\Psi[n+\nu+1\pm i\eta,2n+2\nu+2;\mp2i\omega z].$$

By setting $n=n_1$ and using (A.3), we find

$$B_{n_1}^{\pm}(z)=\pm(-1)^{\nu\mp\nu}[-2i\omega z]^{-n_1-\nu}\Psi[-n_1-\nu\pm i\eta,-2n_1-2\nu;\mp2i\omega z].$$

Hence, $B_{n_1}^{\pm}$ and $B_{n_2}^{\pm}$ are proportional to each other for some $n=n_2$ such that $n_1+n_2+1=-2\nu$. Similar results are found for the other solutions $U_i^{\pm}(z)$. On the other side, by supposing that $2\nu\neq$integer, the functions $\Phi(a,c;y)$ which appear in $U_i(z)$ are well defined because the parameter $c=2n+2\nu+2$ cannot be a negative integer. Nevertheless, see in the paragraph containing Eqs. (55) and (56) some remarks concerning the case $\eta=0$.

According to Eqs. (A.4), if the conditions (29) are true, the three hypergeometric functions are linearly independent and each one can be written as a combination of the others by means of (A.5). In this case, in a common region of validity, we can write one solution of a given set as a superposition of the others. However, the three series are really doubly infinite ($-\infty < n < \infty$) if, in addition to (29), ν satisfies the restrictions

$$\nu \pm \tfrac{B_2}{2} \text{ and } \nu \pm \left(\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right) \text{ are not integers.} \tag{30}$$

These conditions assure that α_n^i and γ_n^i do not vanish for any value of n. In effect, if $\alpha_n^i=0$ for some $n=N_1$, the series should begin at $n=N_1+1$ in order to assure the validity of the theory of the three-term recurrence relations; by the same reason, if $\gamma_n^i=0$ for some $n=N_2$, the series should terminate at $n=N_2-1$.

Notice that, for two-side solutions,

$$\beta_n^i=\beta_n^1, \qquad \alpha_n^i\gamma_{n+1}^i=\alpha_n^1\gamma_{n+1}^1, \qquad [i=2,3,4]. \tag{31}$$

Thence, Eq. (16) implies that all the solutions $\mathbb{U}_i$ satisfy the same characteristic equation and, consequently, the parameter ν takes the same values in all solutions. In addition, as noticed by Leaver, the characteristic equations are periodic in ν with period 1. In effect, in order to indicate that the coefficients depend on ν we rewrite the recurrence relations as $\alpha_n^\nu b_{n+1}^\nu+\beta_n^\nu b_n^\nu+\gamma_n^\nu b_{n-1}^\nu=0$ or as the following tridiagonal matrix equation:

$$\begin{bmatrix} \cdot & \cdot & \cdot & & & & \\ & \gamma_n^\nu & \beta_n^\nu & \alpha_n^\nu & & & \\ & & \gamma_{n+1}^\nu & \beta_{n+1}^\nu & \alpha_{n+1}^\nu & & \\ & & & \gamma_{n+2}^\nu & \beta_{n+2}^\nu & \alpha_{n+2}^\nu & \\ & & & & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ b_{n-1}^\nu \\ b_n^\nu \\ b_{n+1}^\nu \\ \cdot \end{bmatrix} = \mathbf{0} \qquad [-\infty<n<\infty] \tag{32}$$

where $\mathbf{0}$ denotes the null column vector. Thence, the values for ν may be determined by requiring that the determinant of the above matrix vanishes. However, as

$$\gamma_n^{\nu+1}=\gamma_{n+1}^\nu, \qquad \beta_n^{\nu+1}=\beta_{n+1}^\nu, \qquad \alpha_n^{\nu+1}=\alpha_{n+1}^\nu, \qquad \cdots$$

and $-\infty<n<\infty$, the matrix and its determinant are not modified by the replacement $\nu\to\nu+1$ (or $\nu\to\nu+N$, where N is any integer).

Further, if (29) and (30) are fulfilled, all coefficients can be written in terms of b_n^1. Up to multiplicative constants independent of n, we have

$$b_n^2=\frac{\Gamma\left[n+\nu+2-\frac{B_2}{2}\right]\,\Gamma\left[n+\nu+1-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{\Gamma\left[n+\nu+\frac{B_2}{2}\right]\,\Gamma\left[n+\nu+1+\frac{B_1}{z_0}+\frac{B_2}{2}\right]}\,b_n^1, \qquad b_n^3=\frac{(-1)^n\,\Gamma\left[n+\nu+1-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{\Gamma\left[n+\nu+1+\frac{B_1}{z_0}+\frac{B_2}{2}\right]}\,b_n^1,$$
$$b_n^4=\frac{(-1)^n\,\Gamma\left[n+\nu+2-\frac{B_2}{2}\right]}{\Gamma\left[n+\nu+\frac{B_2}{2}\right]}\,b_n^1. \tag{33}$$

As an example, we consider the solutions $W(u)$ for the WHE (7). These may be obtained from the solutions $U(z)$ of the CHE (1) by taking

$$\left.\begin{aligned} &W(u)=U(z), \quad z=\cos^2(\varsigma u), \quad [\varsigma=1,i] \quad \Rightarrow \quad z_0=1,\\ &B_1=-\tfrac{1}{2}, \quad B_2=1, \quad B_3=\tfrac{(p+1)\xi-\vartheta}{4}, \quad i\omega=\tfrac{\xi}{2}, \quad i\eta=\tfrac{p+1}{2} \end{aligned}\right\} \begin{array}{l}\text{WHE as}\\ \text{a CHE.}\end{array} \tag{34}$$

Thus, the solutions $\mathbb{U}_i=\left(U_i,U_i^+,U_i^-\right)$ lead to four sets of solutions

$$\mathbb{W}_i(u)=\left[W_i(u)=U_i(z),\ W_i^+(u)=U_i^+(z),\ W_i^-(u)=U_i^-(z)\right]. \tag{35a}$$

In this case, the coefficients of the recurrence relations for b_n^1 simplify to

$$\alpha_n^1=\tfrac{i\omega}{2}, \quad \beta_n^1=-B_3-\eta\omega-\left[n+\nu+\tfrac{1}{2}\right]^2, \quad \gamma_n^1=-\tfrac{i\omega}{2}[n+\nu+i\eta][n+\nu-i\eta], \tag{35b}$$

whereas Eqs. (33) reduce to

$$b_n^2=\left(n+\nu+\tfrac{1}{2}\right)b_n^1, \qquad b_n^3=(-1)^n b_n^1, \qquad b_n^4=(-1)^n\left(n+\nu+\tfrac{1}{2}\right)b_n^1. \tag{35c}$$

2.1.2 Convergence and an asymptotic behaviour

The D'Alembert test implies two subgroups of solutions since $\mathbb{U}_1$ and $\mathbb{U}_2$ converge for any finite z such that $|z|>|z_0|$, whereas $\mathbb{U}_3$ and $\mathbb{U}_4$ converge for $|z-z_0|>|z_0|$. However, by the Raabe test they may converge also at $|z|=|z_0|$ and $|z-z_0|=|z_0|$ under the conditions

$$|z|\geq|z_0| \text{ if } \begin{cases}\text{Re}\left[B_2+\frac{B_1}{z_0}\right]<1 \text{ in } \mathbb{U}_1,\\ \text{Re}\left[B_2+\frac{B_1}{z_0}\right]>1 \text{ in } \mathbb{U}_2;\end{cases} \qquad |z-z_0|\geq|z_0| \text{ if } \begin{cases}\text{Re}\left[\frac{B_1}{z_0}\right]>-1 \text{ in } \mathbb{U}_3,\\ \text{Re}\left[\frac{B_1}{z_0}\right]<-1 \text{ in } \mathbb{U}_4,\end{cases} \tag{36}$$

where the restrictions on parameters of the equation are necessary only to assure convergence at $|z|=|z_0|$ or $|z-z_0|=|z_0|$. In particular, for the solutions of the WHEs we find

$$\begin{aligned}&|\cos(\varsigma u)|\geq 1 \text{ in } \mathbb{W}_1, \qquad |\cos(\varsigma u)|>1 \text{ in } \mathbb{W}_2,\\ &|\sin(\varsigma u)|\geq 1 \text{ in } \mathbb{W}_3, \qquad |\sin(\varsigma u)|>1 \text{ in } \mathbb{W}_4.\end{aligned} \tag{37}$$

Thence the two-sided solutions are useless for the WHE (that is, for $\varsigma=1$ and u=real), but may be useful for the modified WHE ($\varsigma=i$, u=real). If $\text{Re}[B_2+(B_1/z_0)]=1$ and $\text{Re}[B_1/z_0]=-1$ in (36), the Raabe test becomes inconclusive in the sense that the solutions may converge or diverge at $|z|=|z_0|$ or $|z-z_0|=|z_0|$.

To obtain the conditions (36) it is sufficient to consider the convergence of the first set of solutions. The results for the other sets arise from transformations (13) applied in the order given in (22). Thus, by using the form (12) for the first set, the domains of convergence follow from the ratios

$$\lim_{n\to\infty}\left|\frac{b^1_{n+1}\mathscr{U}_{n+\nu+1}(\eta,\omega z)}{b^1_n\mathscr{U}_{n+\nu}(\eta,\omega z)}\right| \quad \text{and} \quad \lim_{n\to-\infty}\left|\frac{b^1_{n-1}\mathscr{U}_{n+\nu-1}(\eta,\omega z)}{b^1_n\mathscr{U}_{n+\nu}(\eta,\omega z)}\right|$$

The ratios b^1_{n+1}/b^1_n and b^1_{n-1}/b^1_n come from the relations $\alpha^1_n b^1_{n+1}+\beta^1_n b^1_n+\gamma^1_n b^1_{n-1}=0$ which, when $n\to\pm\infty$, yield

$$\begin{aligned}&i\omega z_0\left[1-\tfrac{1}{n}\left(B_2+\tfrac{B_1}{z_0}-\tfrac{1}{2}\right)+O\left(\tfrac{1}{n^2}\right)\right]\tfrac{b^1_{n+1}}{b^1_n}-2n\left[n+2\nu+1+O\left(\tfrac{1}{n}\right)\right]\\ &-i\omega z_0 n\left[n+2\nu+B_2+\tfrac{B_1}{z_0}-\tfrac{1}{2}+O\left(\tfrac{1}{n}\right)\right]\tfrac{b^1_{n-1}}{b^1_n}=0.\end{aligned} \tag{38}$$

Hence, the minimal solution for b^1_{n+1}/b^1_n when $n\to\infty$ is

$$\frac{b^1_{n+1}}{b^1_n}\sim\frac{\omega z_0}{2i}\left[1+\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-\frac{3}{2}\right)\right] \Rightarrow \frac{b^1_{n-1}}{b^1_n}\sim\frac{2i}{\omega z_0}\left[1-\frac{1}{n}\left(B_2+\frac{B_1}{z_0}-\frac{3}{2}\right)\right], \tag{39a}$$

and the minimal solution for b^1_{n-1}/b^1_n when $n\to-\infty$ is

$$\frac{b^1_{n-1}}{b^1_n}\sim\frac{i\omega z_0}{2n^2}\left[1-\frac{1}{n}\left(2\nu+B_2+\frac{B_1}{z_0}-\frac{3}{2}\right)\right]\Rightarrow\frac{b^1_{n+1}}{b^1_n}\sim\frac{2n^2}{i\omega z_0}\left[1+\frac{1}{n}\left(2\nu+B_2+\frac{1}{2}+\frac{B_1}{z_0}\right)\right]. \tag{39b}$$

On the other hand, from relations (A.19) and (A.20) we find that, for finite values of z,

$$\begin{aligned}&n\to\infty: \quad \frac{\phi_{n+\nu+1}}{\phi_{n+\nu}}\sim\frac{i\omega z}{2n^2}\left[1-\frac{1}{n}\left(2\nu+\frac{5}{2}\right)\right], \qquad \frac{\psi^{\pm}_{n+\nu+1}}{\psi^{\pm}_{n+\nu}}\sim\frac{2i}{\omega z}\left[1-\frac{1}{2n}\right],\\ &n\to-\infty: \quad \frac{\mathscr{U}_{n+\nu-1}}{\mathscr{U}_{n+\nu}}\sim\frac{2n^2}{i\omega z}\left[1+\frac{1}{n}\left(2\nu+\frac{1}{2}\right)\right], \qquad \mathscr{U}_{n+\nu}=\left(\phi_{n+\nu},\psi^{(\pm)}_{n+\nu}\right).\end{aligned} \tag{40}$$

Thence, by means of (39a), we find

$$n\to\infty: \begin{cases}\dfrac{b^1_{n+1}\phi_{n+\nu+1}}{b^1_n\ \phi_{n+\nu}}\sim\dfrac{\omega^2 z_0 z}{4n^2}\left[1+\dfrac{1}{n}\left(B_2+\dfrac{B_1}{z_0}-2\nu-4\right)\right],\\ \dfrac{b^1_{n+1}\psi^{\pm}_{n+\nu+1}}{b^1_n\psi^{\pm}_{n+\nu}}\sim\dfrac{z_0}{z}\left[1+\dfrac{1}{n}\left(B_2-2+\dfrac{B_1}{z_0}\right)\right],\end{cases} \tag{41a}$$

and, by means of (39b),

$$n \to -\infty: \quad \frac{b^1_{n-1}\mathscr{U}_{n+\nu-1}}{b^1_n \mathscr{U}_{n+\nu}} \sim \frac{z_0}{z}\left[1-\frac{1}{n}\left(B_2-2+\frac{B_1}{z_0}\right)\right], \qquad \mathscr{U}_{n+\nu}=\left(\phi_{n+\nu},\psi^{\pm}_{n+\nu}\right). \tag{41b}$$

¿From these limits we get

$$\lim_{n\to\infty}\left|\frac{b^1_{n+1}\phi_{n+\nu+1}}{b^1_n\phi_{n+\nu}}\right|=\left|\frac{\omega^2 z_0 z}{4n^2}\right|, \quad \lim_{n\to-\infty}\left|\frac{b^1_{n-1}\phi_{n+\nu-1}}{b^1_n\phi_{n+\nu}}\right|=\frac{|z_0|}{|z|}\left[1+\frac{1}{|n|}\mathrm{Re}\left(B_2-2+\frac{B_1}{z_0}\right)\right] \tag{42}$$

and

$$\lim_{n\to\infty}\left|\frac{b^1_{n+1}\psi^{\pm}_{n+\nu+1}}{b^1_n\psi^{\pm}_{n+\nu}}\right|=\lim_{n\to-\infty}\left|\frac{b^1_{n-1}\psi^{\pm}_{n+\nu-1}}{b^1_n\psi^{\pm}_{n+\nu}}\right|=\frac{|z_0|}{|z|}\left[1+\frac{1}{|n|}\mathrm{Re}\left(B_2-2+\frac{B_1}{z_0}\right)\right]. \tag{43}$$

So, by the D'Alembert test the series converge absolutely for $|z|>|z_0|$ because the right-hand sides of (42) and (43) are <1. However, if $|z|=|z_0|$, by the expressions (11) for the Raabe test, the series converge even for $|z|=|z_0|$ provided that the numerators of $|n|$ in (42) and (43) are <-1, that is,

$$\text{if} \quad \mathrm{Re}\left[B_2+(B_1/z_0)\right]<1, \quad \text{the series in } \mathbb{U}_1(z) \text{ converge for } |z|\geq|z_0|.$$

If $\mathrm{Re}\left[B_2+(B_1/z_0)\right]>1$, the series diverge and, if $\mathrm{Re}\left[B_2+(B_1/z_0)\right]=1$, the test is inconclusive. The convergence regions (36) for the other sets of solutions are obtained by transforming the parameters and the variable z of $\mathbb{U}_1$ in accordance with Eqs. (22). Only the limit $n\to\infty$ is relevant for one-sided series ($n\geq 0$) and, then, the solutions $\mathring{U}_i$ converge for any finite value z in virtue of the first limit given in (42). The convergence of $\mathring{U}^{\pm}_i$ is similar to that of $U^{\pm}_i$.

Since the previous regions of convergence were derived by supposing that z is finite, now we consider the behaviour of the solutions at $z=\infty$. By using (A.7) we find that, when $z\to\infty$,

$$U_1(z)\sim \ e^{i\omega z}[i\omega z]^{-i\eta-\frac{B_2}{2}}\sum\frac{b^1_n}{\Gamma[n+\nu+1-i\eta]}+e^{-i\omega z}[-2i\omega z]^{i\eta-\frac{B_2}{2}}\sum\frac{(-1)^{n-\nu-1+i\eta}b^1_n}{\Gamma[n+\nu+1+i\eta]}. \tag{44}$$

Thus, U_1 may be unbounded by virtue of the exponential factors. This is consistent with the fact that, if conditions (29) and (30) are satisfied, then $U_1(z)$ can be written as a linear combination of $U_1^+(z)$ and $U_1^-(z)$. In fact, when $z\to\infty$, Eq. (A.6) gives

$$\begin{aligned} U_1^+(z)&\sim e^{i\omega z}[-2i\omega z]^{-i\eta-\frac{B_2}{2}}\sum\frac{b^1_n}{\Gamma[n+\nu+1-i\eta]}, && -\tfrac{3\pi}{2}<\arg(-2i\omega z)<\tfrac{3\pi}{2};\\ U_1^-(z)&\sim e^{-i\omega z}[-2i\omega z]^{i\eta-\frac{B_2}{2}}\sum\frac{(-1)^{n-\nu-1+i\eta}\ b^1_n}{\Gamma[n+\nu+1+i\eta]}, && -\tfrac{3\pi}{2}<\arg(2i\omega z)<\tfrac{3\pi}{2}. \end{aligned} \tag{45}$$

Thus, the series in $U_1^{\pm}$ converge at $z=\infty$ but one of them may be unbounded depending on the exponential factors. For instance, if $\mathrm{Re}(i\omega z)\to\infty$, $U_1^+\to\infty$ but U_1^- is bounded.

2.1.3 The case $\eta=0$, the spheroidal and Mathieu equations

Taking $\eta=0$ and keeping fixed the other parameters, the previous solutions are rewritten in series of Bessel functions of the first kind, $J_\kappa(y)$, and in series of the first and the second Hankel functions, $H^{(1)}_\kappa(y)$ and $H^{(2)}_\kappa(y)$. We also include the Bessel functions $Y_\kappa(y)$ of the second kind. These four functions are denoted by $Z^{(j)}_\kappa(y)$ – or by $\mathscr{C}^{(j)}_\kappa(y)$ – according as [7, 18]

$$Z^{(1)}_\kappa(y)=J_\kappa(y), \quad Z^{(2)}_\kappa(y)=Y_\kappa(y), \quad Z^{(3)}_\kappa(y)=H^{(1)}_\kappa(y), \quad Z^{(4)}_\kappa(y)=H^{(2)}_\kappa(y). \tag{46}$$

There are connections among these functions. For example, the relation $Y_\kappa = [H_\kappa^{(1)} - H_\kappa^{(2)}]/(2i)$ permits to obtain the expansion in series of Y_κ as a linear combination of the expansions in series of Hankel functions. Thus, we get four sets of solutions, each containing four solutions. These sets are written as

$$U_i^{(j)}(z) = \left[U_i^{(1)}(z), U_i^{(2)}(z), U_i^{(3)}(z), U_i^{(4)}(z)\right], \quad i = 1, 2, 3, 4, \tag{47}$$

where the right-hand side corresponds to the Bessel functions (46). For one-sided series there are eight sets $\mathring{U}_i^{(j)}$. The solutions $\mathbb{U}_1$ lead to $U_1^{(j)}$ which, in turn, give the other sets by means of the transformations (22), that is,

$$U_2^{(j)}(z) = T_2 U_1^{(j)}(z); \qquad U_3^{(j)}(z) = T_4 U_1^{(j)}(z), \qquad U_4^{(j)}(z) = T_1 U_3^{(j)}(z). \tag{48}$$

Thus, we put $\eta = 0$ in $\mathbb{U}_1$ given in (19), use the relations (A.22) together with [18]

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left[z + (1/2)\right]/\sqrt{\pi},$$

and redefine the coefficients as $a_n^1 = i^n b_n^1/\Gamma(n+\nu+1)$. So, we find

$$U_1^{(j)}(z) = z^{\frac{1}{2}-\frac{B_2}{2}} \sum_n a_n^1 Z_{n+\nu+\frac{1}{2}}^{(j)}(\omega z), \quad [2\nu \neq 0, \pm 1, \pm 2, \cdots] \tag{49a}$$

In the recurrence relations (15) for a_n^1, we have

$$\begin{aligned}
\alpha_n^1 &= \frac{\omega z_0\left[n+\nu+2-\frac{B_2}{2}\right]\left[n+\nu+1-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{(2n+2\nu+3)},\\
\beta_n^1 &= -\left(n+\nu+1-\tfrac{B_2}{2}\right)\left(n+\nu+\tfrac{B_2}{2}\right) - B_3,\\
\gamma_n^1 &= \frac{\omega z_0\left[n+\nu+\frac{B_2}{2}-1\right]\left[n+\nu+\frac{B_2}{2}+\frac{B_1}{z_0}\right]}{(2n+2\nu-1)}.
\end{aligned} \tag{49b}$$

The other sets are given by ($\beta_n^i = \beta_n^1,\ 2\nu \neq 0, \pm 1, \pm 2, \cdots$):

$$\begin{aligned}
&U_2^{(j)}(z) = z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}}[z-z_0]^{1-B_2-\frac{B_1}{z_0}} \sum_n a_n^2 Z_{n+\nu+\frac{1}{2}}^{(j)}(\omega z),\\
&\alpha_n^2 = \frac{\omega z_0\left[n+\nu+\frac{B_2}{2}\right]\left[n+\nu+1+\frac{B_1}{z_0}+\frac{B_2}{2}\right]}{(2n+2\nu+3)}, \qquad \gamma_n^2 = \frac{\omega z_0\left[n+\nu+1-\frac{B_2}{2}\right]\left[n+\nu-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{(2n+2\nu-1)};
\end{aligned} \tag{50}$$

$$\begin{aligned}
&U_3^{(j)}(z) = [z-z_0]^{\frac{1}{2}-\frac{B_2}{2}} \sum_n a_n^3 Z_{n+\nu+\frac{1}{2}}^{(j)}[\omega(z-z_0)],\\
&\alpha_n^3 = -\frac{\omega z_0\left[n+\nu+2-\frac{B_2}{2}\right]\left[n+\nu+1+\frac{B_1}{z_0}+\frac{B_2}{2}\right]}{(2n+2\nu+3)}, \qquad \gamma_n^3 = -\frac{\omega z_0\left[n+\nu-1+\frac{B_2}{2}\right]\left[n+\nu-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{(2n+2\nu-1)};
\end{aligned} \tag{51}$$

$$\begin{aligned}
&U_4^{(j)}(z) = z^{1+\frac{B_1}{z_0}}[z-z_0]^{-\frac{1}{2}-\frac{B_1}{z_0}-\frac{B_2}{2}} \sum_n a_n^4 Z_{n+\nu+\frac{1}{2}}^{(j)}[\omega(z-z_0)],\\
&\alpha_n^4 = -\frac{\omega z_0\left[n+\nu+\frac{B_2}{2}\right]\left[n+\nu+1-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{(2n+2\nu+3)}, \qquad \gamma_n^4 = -\frac{\omega z_0\left[n+\nu+1-\frac{B_2}{2}\right]\left[n+\nu+\frac{B_1}{z_0}+\frac{B_2}{2}\right]}{(2n+2\nu-1)}.
\end{aligned} \tag{52}$$

By using the relations [18]

$$\begin{aligned}
&J_\kappa\left(ye^{i\pi}\right) = e^{i\pi\kappa}J_\kappa(y), && Y_\kappa\left(ye^{i\pi}\right) = e^{-i\pi\kappa}Y_\kappa(y) + 2i\cos(\pi\kappa)J_\kappa(y),\\
&H_\kappa^{(1)}\left(ye^{i\pi}\right) = -e^{-i\pi\kappa}H_\kappa^{(2)}(y), && H_\kappa^{(2)}\left(ye^{i\pi}\right) = e^{i\pi\kappa}H_\kappa^{(1)}(y) + 2\cos(\pi\kappa)H_\kappa^{(2)}(y)
\end{aligned} \tag{53}$$

with $\kappa = n+\nu+(1/2)$, we find that the change $\omega\to-\omega$ does not lead to new independent solutions. In this sense, once more the transformation T_3 is ineffective.

Also in the present case ($\eta=0$) conditions (30), that is,

$$\nu\pm\tfrac{B_2}{2}\ \text{and}\ \nu\pm\left(\tfrac{B_1}{z_0}+\tfrac{B_2}{2}\right)\ \text{are not integers,}\qquad [\text{ see relations (30)}]\tag{54a}$$

are necessary in order to have two-sided infinite series, and relations (33) hold for the coefficients a_n^1. On the other side, the restrictions (29) are replaced by conditions $2\nu\neq 0,\pm1,\pm2\cdots$ which assure the independence of the Bessel function:

$$2\nu\neq 0,\pm1,\pm2,\cdots,\qquad [\text{independence of Bessel functions}].\tag{54b}$$

In fact, it is necessary that $\nu\neq\pm1/2,\pm3/2,\cdots$ in order to avoid two linearly dependent functions of integer order, like $Z_\ell^{(j)}(y)$ and $Z_{-\ell}^{(j)}(y)$ $[Z_\ell=(-1)^\ell Z_{-\ell}]$, where ℓ is zero or positive integer. In addition, $\nu\neq 0,\pm1,\pm2,\cdots$ assures the independence of the Hankel functions in the same series: on the contrary, we would have functions like $H_{\ell+(1/2)}$ and $H_{-\ell-(1/2)}$ which are proportional to each other since [19]

$$H^{(1)}_{-\ell-(1/2)}(y)=i(-1)^\ell H^{(1)}_{\ell+(1/2)}(y),\qquad H^{(2)}_{-\ell-(1/2)}(y)=-i(-1)^\ell H^{(2)}_{\ell+(1/2)}(y).\tag{55}$$

However, for series of Bessel functions of the first and second kind, we have

$$J_{-\ell-(1/2)}(y)=(-1)^{\ell+1}Y_{\ell+(1/2)}(y),\qquad Y^{(2)}_{-\ell-(1/2)}(y)=(-1)^\ell J^{(2)}_{\ell+(1/2)}(y),\tag{56}$$

that is, for $\nu=0,\pm1,\pm2,\cdots$ the functions $J_{\ell+(1/2)}$ and $J_{-\ell-(1/2)}$ (or, $Y_{\ell+(1/2)}$ and $Y_{-\ell-(1/2)}$) are linearly independent. In spite of this, by assuming that $\nu\neq 0,\pm1,\pm2,\cdots$ also for J and Y we guarantee that all of the solutions (60) and (61) for the spheroidal equation are two-sided since α_n^1 and γ_n^1 do not vanish for $-\infty<n<\infty$.

On the other side, for two-sided series the domains of convergence are again given by (36) with $U_i^{(j)}$ substituted for $\mathbb{U}_i$. For the one-sided series the solutions $\mathring{U}_i^{(1)}$, in series of Bessel functions of the first kind, converge for any finite z. The behaviour of the solutions at $z=\infty$ can be found from the fact that, for κ fixed and $|y|\to\infty$ [18],

$$\begin{aligned}
&J_\kappa(y)\sim\sqrt{\tfrac{2}{\pi y}}\cos\left[y-\tfrac{\kappa\pi}{2}-\tfrac{\pi}{4}\right],\quad Y_\kappa(y)\sim\sqrt{\tfrac{2}{\pi y}}\sin\left[y-\tfrac{\kappa\pi}{2}-\tfrac{\pi}{4}\right]:\ |\arg y|<\pi;\\
&H_\kappa^{(1)}(y)\sim\sqrt{\tfrac{2}{\pi y}}\ e^{i\left[y-\frac{\kappa\pi}{2}-\frac{\pi}{4}\right]}:\qquad -\pi<\arg y<2\pi;\\
&H_\kappa^{(2)}(y)\sim\sqrt{\tfrac{2}{\pi y}}\ e^{-i\left[y-\frac{\kappa\pi}{2}-\frac{\pi}{4}\right]}:\qquad -2\pi<\arg y<\pi.
\end{aligned}\tag{57}$$

Now we consider the Meixner solutions. The substitutions

$$y=1-2z,\qquad S(y)=z^{\frac{\mu}{2}}\ [z-1]^{\frac{\mu}{2}}\,U(z)\ \ \Leftrightarrow\ \ S(y)\propto[y^2-1]^{\frac{\mu}{2}}\,U\left(z=\tfrac{1-y}{2}\right)\tag{58a}$$

transform the spheroidal wave equation (8) into

$$z(z-1)\frac{d^2U}{dz^2}+\left[-(\mu+1)+(2\mu+2)\,z\right]\frac{dU}{dz}+\left[\mu\,(\mu+1)-\lambda+4\gamma^2z(z-1)\right]U=0,$$

which is the CHE (1) with parameters

$$z_0=1,\quad B_1=-\mu-1,\quad B_2=2\mu+2,\quad B_3=\mu(\mu+1)-\lambda,\quad \omega=\pm2\gamma,\quad \eta=0.\tag{58b}$$

Instead of $Z_{\kappa}^{(j)}(v)$, Meixner used the functions $\psi_{\kappa}^{(j)}(v)$ which are given by [9]

$$\psi_{\kappa}^{(j)}(v)=\sqrt{\pi/(2v)}\;Z_{\kappa+(1/2)}^{(j)}(v), \tag{59}$$

in analogy with the definitions of the spherical Bessel functions j_κ, y_κ, $h_{\kappa}^{(1)}$ and $h_{\kappa}^{(2)}$ [19]. So, by taking $\omega=-2\gamma$ and using this notation, we get

$$\begin{aligned} &S_1^{(j)}(\mu,y)=\left[\frac{y+1}{y-1}\right]^{\frac{\mu}{2}}\sum_n a_n^1\psi_{n+\nu}^{(j)}[\gamma(y-1)], \qquad S_2^{(j)}(\mu,y)=S_1^{(j)}(-\mu,y),\\ &\alpha_n^1=\tfrac{2\gamma(n+\nu+1-\mu)(n+\nu+1)}{(2n+2\nu+3)},\quad \beta_n^1=(n+\nu)(n+\nu+1)-\lambda,\quad \gamma_n^1=\tfrac{2\gamma(n+\nu+\mu)(n+\nu)}{(2n+2\nu-1)}; \end{aligned} \tag{60}$$

and

$$S_3^{(j)}(\mu,y)=\left[\frac{y-1}{y+1}\right]^{\frac{\mu}{2}}\sum_n(-1)^n a_n^1\psi_{n+\nu}^{(j)}[\gamma(y+1)], \qquad S_4^{(j)}(\mu,y)=S_3^{(j)}(-\mu,y). \tag{61}$$

For these solutions, conditions (54a) and (54b) reduces to

$$2\nu\neq 0,\pm1,\pm2,\cdots, \qquad \nu\pm(\mu+1)\neq \text{integer}.$$

The Meixner solutions are given by $S_2^{(j)}(\mu,y)$ and $S_4^{(j)}(\mu,y)$. By the D'Alembert test $S_i^{(j)}$ converge for $|y-1|>2$ (if $i=1,2$) or for $|y+1|>2$ (if $i=3,4$), as stated in [9, 20]. However, by the Raabe test they may converge at $|y-1|=2$ or $|y+1|=2$ because relations (36) and (58b) yield

$$|y-1|\geq 2,\ \text{if}\ \begin{cases}\text{Re}\,(\mu)<0 \text{ in } S_1^{(j)},\\ \text{Re}\,(\mu)>0 \text{ in } S_2^{(j)};\end{cases} \qquad |y+1|\geq 2 \text{ if } \begin{cases}\text{Re}\,(\mu)<0 \text{ in } S_3^{(j)},\\ \text{Re}\,(\mu)>0 \text{ in } S_4^{(j)}\end{cases} \tag{62}$$

(if Re $(\mu)=0$, the test is inconclusive).

On the other side, the solutions $w(u)$ for the Mathieu equation (6), considered as a particular case of the CHE, may be obtained by setting

$$\left.\begin{aligned} &w(u)=U(z),\quad z=\cos^2(\tfrac{\sigma u}{2}),\quad [\sigma=1,i]\quad \Rightarrow\quad z_0=1,\\ &B_1=-1/2,\quad B_2=1,\quad B_3=2k^2-\text{a},\quad \omega=4k,\quad \eta=0 \end{aligned}\right\}\ \begin{aligned}&\text{Mathieu eq.}\\ &\text{as a CHE,}\end{aligned} \tag{63}$$

where $U(z)$ are the solutions for CHE with $\eta=0$. Thus, from the previous solutions $U_i^{(j)}(z)$ we get four sets of two-sided solutions $w_i^{(j)}(u)$

$$\begin{aligned} &w_1^{(j)}(u)=\sum_n a_n\ Z_{n+\nu+\frac12}^{(j)}\left[4k\cos^2\tfrac{\sigma u}{2}\right], && \left|\cos\tfrac{\sigma u}{2}\right|\geq 1,\\ &w_2^{(j)}(u)=\tan\tfrac{\sigma u}{2}\sum_n\ \left(n+\nu+\tfrac12\right)a_n Z_{n+\nu+\frac12}^{(j)}\left[4k\cos^2\tfrac{\sigma u}{2}\right], && \left|\cos\tfrac{\sigma u}{2}\right|>1, \end{aligned} \tag{64a}$$

$$\begin{aligned} &w_3^{(j)}(u)=\sum_n(-1)^n a_n\ Z_{n+\nu+\frac12}^{(j)}\left[-4k\sin^2\tfrac{\sigma u}{2}\right], && \left|\sin\tfrac{\sigma u}{2}\right|\geq 1,\\ &w_4^{(j)}(u)=\cot\tfrac{\sigma u}{2}\sum_n\ (-1)^n\left(n+\nu+\tfrac12\right)a_n Z_{n+\nu+\frac12}^{(j)}\left[-4k\sin^2\tfrac{\sigma u}{2}\right], && \left|\sin\tfrac{\sigma u}{2}\right|>1, \end{aligned} \tag{64b}$$

where the coefficients a_n satisfy the relations

$$2k(n+\nu+1)a_{n+1}+\left[\text{a}-2k^2-\left(n+\nu+\tfrac12\right)^2\right]a_n+2k(n+\nu)a_{n-1}=0. \tag{64c}$$

For any solutions the behaviour when $z=\cos^2(\sigma u/2)\to\infty$ must be determined by using (57). If $\sigma=1$ and u =real (Mathieu equation) the previous solutions are useless. Notice, however, that the one-sided solutions $\mathring{w}_i^{(j)}$ in series of Bessel functions of the first kind are convergent for all finite values of z.

2.2 Solutions for the Whittaker-Ince limit of the CHE

To obtain solutions to the Whittaker-Ince limit of the CHE (4), we apply [21]

$$\lim_{a\to\infty}\Phi\left(a,c;-\frac{y}{a}\right)=\Gamma(c)\;y^{(1-c)/2}J_{c-1}\big(2\sqrt{y}\big),$$
$$\lim_{a\to\infty}\Big[\Gamma(a+1-c)\;\Psi\left(a,c;-\frac{y}{a}\right)\Big]=\begin{cases}-i\pi e^{i\pi c}y^{(1-c)/2}H^{(1)}_{c-1}\big(2\sqrt{y}\big), & \mathrm{Im}\;y>0,\\ i\pi e^{-i\pi c}y^{(1-c)/2}H^{(2)}_{c-1}\big(2\sqrt{y}\big), & \mathrm{Im}\;y<0,\end{cases} \tag{65}$$

on the hypergeometric functions used as basis for the expansions of the solutions for the CHE. For this it is necessary to rewrite the latter solutions in a suitable form and keep n fixed. Expansions in series of Y_{c-1} are obtained as a linear combination of the expansions in series of Hankel functions. In this manner, from the first set (19), we get a set of four solutions for Eq. (4). These are again denoted by $U_1^{(j)}$ ($j=1,\ldots,4$). In fact, we will compute only the limit of $U_1(z)$: the other solutions follow from the fact that the four Bessel functions satisfy the same differential and difference equations.

On the other side, if $U(z)=U(B_1,B_2,B_3;z_0,q;z)$ represents an arbitrary solution for Eq. (4), then other solutions are generated by the transformations $\mathscr{T}_1$, $\mathscr{T}_2$ and $\mathscr{T}_3$ given by [3]

$$\begin{aligned}&\mathscr{T}_1U(z)=z^{1+B_1/z_0}U(C_1,C_2,C_3;z_0,q;z),\\&\mathscr{T}_2U(z)=(z-z_0)^{1-B_2-B_1/z_0}U(B_1,D_2,D_3;z_0,q;z),\\&\mathscr{T}_3U(z)=U(-B_1-B_2z_0,B_2,B_3-qz_0;z_0,-q;z_0-z),\end{aligned} \tag{66}$$

where C_i and D_i are defined in Eqs. (14). Thus, it is sufficient to take the limit of the first set of solutions (19) and of the coefficients (20). The other sets are obtained from $U_1^{(j)}$ through

$$U_2^{(j)}=\mathscr{T}_2U_1^{(j)},\qquad U_3^{(j)}=\mathscr{T}_3U_2^{(j)},\qquad U_4^{(j)}=\mathscr{T}_1U_3^{(j)}. \tag{67}$$

First we find the four sets of solutions and use the Raabe test to study their convergence. In the second place, we write the solution for the Mathieu equation.

2.2.1 The four sets of solutions

To find the limit of $U_1(z)$ given in (19), we rewrite that solution as

$$U_1(z)=e^{i\omega z}\sum_n\frac{(-1)^nc_n[qz]^{n+\nu+1-(B_2/2)}}{\Gamma[2n+2\nu+2]}\Phi\left[n+\nu+1+i\eta,2n+2\nu+2;-\frac{qz}{i\eta}\right], \tag{68a}$$

where $c_n=b_n^1[-i\eta]^{-n}$ and $q=-2\eta\omega$. ¿From $\alpha_n^1b_{n+1}^1+\beta_n^1b_{n+1}^1+\gamma_n^1b_{n-1}^1=0$, we get

$$-i\eta\;\alpha_n^1c_{n+1}+\beta_n^1c_n+(-i\eta)^{-1}\gamma_n^1c_{n-1}=0, \tag{68b}$$

where α_n^1, β_n^1 and γ_n^1 are given in (20). By supposing that n is fixed and using (65), we obtain the solution $U_1^{(1)}$ in series of Bessel functions of first kind. In fact, we may verify directly that the four solutions $U_1^{(j)}$ given below satisfy Eq. (5).

Then, the first set is given by

$$U_1^{(j)}(z)=z^{(1-B_2)/2}\sum_n(-1)^nc_n^1Z^{(j)}_{2n+2\nu+1}\left(2\sqrt{qz}\right), \tag{69a}$$

where, in the recurrence relations $\alpha_n^1 c_{n+1}^1 + \beta_n^1 c_{n+1}^1 + \gamma_n^1 c_{n-1}^1 = 0$, we have

$$\begin{aligned} &\alpha_n^1 = \frac{qz_0\left[n+\nu+2-\frac{B_2}{2}\right]\left[n+\nu+1-\frac{B_1}{z_0}-\frac{B_2}{2}\right]}{(2n+2\nu+2)(2n+2\nu+3)},\\ &\beta_n^1 = B_3 - \frac{qz_0}{2} + \left(n+\nu+1-\frac{B_2}{2}\right)\left(n+\nu+\frac{B_2}{2}\right) - \frac{qz_0[B_2-2]\left[B_2+\frac{2B_1}{z_0}\right]}{2(2n+2\nu)(2n+2\nu+2)},\\ &\gamma_n^1 = \frac{qz_0\left[n+\nu-1+\frac{B_2}{2}\right]\left[n+\nu+\frac{B_1}{z_0}+\frac{B_2}{2}\right]}{(2n+2\nu-1)(2n+2\nu)}. \end{aligned} \tag{69b}$$

The transformations (66), applied on $U_1^{(j)}$ according to (67), generate the other sets, that is, $(\beta_n^i = \beta_n^1)$

$$\begin{aligned} &U_2^{(j)}(z) = (z-z_0)^{1-B_2-\frac{B_1}{z_0}}\; z^{\frac{B_1}{z_0}+\frac{B_2}{2}-\frac{1}{2}} \sum_n (-1)^n c_n^2 Z_{2n+2\nu+1}^{(j)}\left(2\sqrt{qz}\right),\\ &\alpha_n^2 = \frac{qz_0\left[n+\nu+\frac{B_2}{2}\right]\left[n+\nu+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)}, \qquad \gamma_n^2 = \frac{qz_0\left[n+\nu+1-\frac{B_2}{2}\right]\left[n+\nu-\frac{B_2}{2}-\frac{B_1}{z_0}\right]}{(2n+2\nu-1)(2n+2\nu)}; \end{aligned} \tag{70}$$

$$\begin{aligned} &U_3^{(j)}(z) = (z-z_0)^{(1-B_2)/2} \sum_n (-1)^n c_n^3 Z_{2n+2\nu+1}^{(j)}\left[2\sqrt{q(z-z_0)}\right],\\ &\alpha_n^3 = -\frac{qz_0\left[n+\nu+2-\frac{B_2}{2}\right]\left[n+\nu+1+\frac{B_2}{2}+\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)}, \qquad \gamma_n^3 = -\frac{qz_0\left[n+\nu-1+\frac{B_2}{2}\right]\left[n+\nu-\frac{B_2}{2}-\frac{B_1}{z_0}\right]}{(2n+2\nu-1)(2n+2\nu)}; \end{aligned} \tag{71}$$

$$\begin{aligned} &U_4^{(j)}(z) = z^{1+\frac{B_1}{z_0}}\; (z-z_0)^{-\frac{1}{2}-\frac{B_1}{z_0}-\frac{B_2}{2}} \sum_n (-1)^n c_n^4 Z_{2n+2\nu+1}^{(j)}\left[2\sqrt{q(z-z_0)}\right],\\ &\alpha_n^4 = -\frac{qz_0\left[n+\nu+\frac{B_2}{2}\right]\left[n+\nu+1-\frac{B_2}{2}-\frac{B_1}{z_0}\right]}{(2n+2\nu+2)(2n+2\nu+3)}, \qquad \gamma_n^4 = -\frac{qz_0\left[n+\nu+1-\frac{B_2}{2}\right]\left[n+\nu+\frac{B_2}{2}+\frac{B_1}{z_0}\right]}{(2n+2\nu-1)(2n+2\nu)}. \end{aligned} \tag{72}$$

For the these four sets of solutions, the conditions (29) and (30) are replaced by

$$2\nu, \quad \nu \pm \frac{B_2}{2} \quad \text{and} \quad \nu \pm \left(\frac{B_1}{z_0}+\frac{B_2}{2}\right) \quad \text{are not integers,} \tag{73}$$

while the relations (33) remain valid for the coefficients c_n^i.

The previous list completes the list given in Ref. [2] where the expansions $U_i^{(1,2)}$ in series of J_κ and Y_κ have not been taken into account, whereas the expansions $U_i^{(3,4)}$ have been written in terms of the modified Bessel functions $K_{2n+2\nu+1}[\pm 2i\sqrt{qz}]$ and $K_{2n+2\nu+1}[\pm 2i\sqrt{q(z-z_0})]$. Moreover, now the regions of convergence are modified by the use of the Raabe test.

2.2.2 Convergence of the solutions

As in the case of the CHE, by the D'Alembert test the two-sided expansions $U_1^{(j)}$ and $U_2^{(j)}$ converge absolutely for $|z| > |z_0|$ while $U_3^{(j)}$ and $U_4^{(j)}$ converge for $|z-z_0| > |z_0|$. However, by the Raabe test the solutions also converge at $|z| = |z_0|$ and $|z-z_0| = |z_0|$ under the conditions similar (36), that is,

$$|z| \geq |z_0| \text{ if } \begin{cases} \mathrm{Re}\left[B_2 + \frac{B_1}{z_0}\right] < 1 \text{ in } U_1^{(j)}, \\ \mathrm{Re}\left[B_2 + \frac{B_1}{z_0}\right] > 1 \text{ in } U_2^{(j)}, \end{cases} \qquad |z-z_0| \geq |z_0| \text{ if } \begin{cases} \mathrm{Re}\left[\frac{B_1}{z_0}\right] > -1 \text{ in } U_3^{(j)}, \\ \mathrm{Re}\left[\frac{B_1}{z_0}\right] < -1 \text{ in } U_4^{(j)}. \end{cases} \tag{74}$$

The test does not assure convergence at $z = \infty$ and, so, the behaviour at $z = \infty$ again deserves special attention. We will find that one-sided infinite series $\mathring{U}_i^{(1)}$ in series of Bessel functions of first kind converge for any finite value of z.

Relations (74) correspond to (36) with the replacements $U_i^{(j)} \leftrightarrow \mathbb{U}_i$. In fact, a few modifications in the previous demonstration lead to (74). Thus, if $n \to \pm\infty$, we find

$$
\begin{aligned}
&qz_0\left[1-\tfrac{1}{n}\left(B_2+\tfrac{B_1}{z_0}-\tfrac{1}{2}\right)\right]\tfrac{c_{n+1}^1}{c_n^1}+\left[4n(n+2\nu+1)\right]+\\
&qz_0\left[1+\tfrac{1}{n}\left(B_2+\tfrac{B_1}{z_0}-\tfrac{1}{2}\right)\right]\tfrac{c_{n-1}^1}{c_n^1}=0.
\end{aligned}
$$

When $n \to \infty$ the minimal solution for c_{n+1}/c_n is

$$
\begin{aligned}
n\to\infty:\quad &\tfrac{c_{n+1}^1}{c_n^1}\sim-\tfrac{qz_0}{4n^2}\left[1-\tfrac{1}{n}\left(2\nu-B_2-\tfrac{B_1}{z_0}+\tfrac{7}{2}\right)\right]\Rightarrow\\
&\tfrac{c_{n-1}^1}{c_n^1}\sim-\tfrac{4n^2}{qz_0}\left[1+\tfrac{1}{n}\left(2\nu-B_2-\tfrac{B_1}{z_0}+\tfrac{3}{2}\right)\right],
\end{aligned}
\tag{75}
$$

while the minimal solution for c_{n-1}/c_n, when $n \to -\infty$, is

$$
\begin{aligned}
n\to-\infty:\quad &\tfrac{c_{n-1}^1}{c_n^1}\sim-\tfrac{qz_0}{4n^2}\left[1-\tfrac{1}{n}\left(2\nu+B_2+\tfrac{B_1}{z_0}-\tfrac{3}{2}\right)\right]\Rightarrow\\
&\tfrac{c_{n+1}^1}{c_n^1}\sim-\tfrac{4n^2}{qz_0}\left[1+\tfrac{1}{n}\left(2\nu+B_2+\tfrac{B_1}{z_0}+\tfrac{1}{2}\right)\right].
\end{aligned}
\tag{76}
$$

On the other side, the behaviours (A.24) and (A.25) for the Bessel functions lead to

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{J_{2n+2\nu+3}}{J_{2n+2\nu+1}}=\frac{qz}{4n^2}\left[1-\frac{1}{n}\left(2\nu+\frac{5}{2}\right)\right],\\
&\lim_{n\to\infty}\frac{Z_{2n+2\nu+3}^{(2,3,4)}}{Z_{2n+2\nu+1}^{(2,3,4)}}=\frac{4n^2}{qz}\left[1+\frac{1}{n}\left(2\nu+\frac{3}{2}\right)\right];\\
&\lim_{n\to-\infty}\frac{Z_{2n+2\nu-1}^{(j)}}{Z_{2n+2\nu+1}^{(j)}}=\frac{4n^2}{qz}\left[1+\frac{1}{n}\left(2\nu+\frac{1}{2}\right)\right],\qquad [j=1,2,3,4].
\end{aligned}
\tag{77}
$$

Thus, when n tends to $+\infty$

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\frac{c_{n+1}^1 J_{2n+2\nu+3}}{c_n^1 J_{2n+2\nu+1}}\right]=-\frac{q^2z_0z}{16n^4}\left[1-\frac{1}{n}\left(4\nu+6-B_2-\frac{B_1}{z_0}\right)\right],\\
&\lim_{n\to\infty}\left[\frac{c_{n+1}^1 Z_{2n+2\nu+3}^{(j)}}{c_n^1 Z_{2n+2\nu+1}^{(j)}}\right]=-\frac{z_0}{z}\left[1-\frac{1}{n}\left(2-B_2-\frac{B_1}{z_0}\right)\right],\qquad [j=2,3,4]
\end{aligned}
\tag{78}
$$

and, when $n \to -\infty$,

$$
\lim_{n\to-\infty}\left[\frac{c_{n-1}^1 Z_{2n+2\nu-1}^{(j)}}{c_n^1 Z_{2n+2\nu+1}^{(j)}}\right]=-\frac{z_0}{z}\left[1+\frac{1}{n}\left(2-B_2-\frac{B_1}{z_0}\right)\right],\qquad [j=1,2,3,4]. \tag{79}
$$

Hence, by the Raabe test the expansions $U_1^{(j)}$ are convergent for $|z| \geq |z_0|$ as indicated in (74). For the other sets of solutions the domains of convergence follow from the transformations (66) applied on $U_1^{(j)}$ according to (67). Moreover, from the first limit given in (78) we see that the one-sided infinite series $\mathring{U}_i^{(1)}$ converge for all z excepting probably the point $z = \infty$. The behaviour any solution when $z \to \infty$ must be determined by using (57).

2.2.3 Heine's solutions for the Mathieu equation

¿From the previous solutions we recover the usual solutions in series of Bessel functions for Mathieu's equation (called Heine's solutions [22]) by means of the substitutions

$$\left.\begin{array}{l} w(u)=U(z), \quad z=\cos^2(\sigma u) \quad \Rightarrow \quad z_0=1, \\ B_1=-\frac{1}{2}, \quad B_2=1, \quad B_3=\frac{k^2}{2}-\frac{\mathrm{a}}{4}, \quad q=k^2 \end{array}\right\} \begin{array}{l}\text{Mathieu eq. as Whittaker-}\\ \text{Ince limit of the CHE.}\end{array} \tag{80}$$

However, the regions of convergence for some solutions turn out to be improved by the Raabe test. This fact is useful for some applications, as we will see in Sec. IV.

Relations (33), with b_n^i replaced by c_n^i, yield

$$c_n^2=\left(n+\nu+\tfrac{1}{2}\right)c_n^1, \qquad c_n^3=(-1)^n c_n^1, \qquad c_n^4=(-1)^n\left(n+\nu+\tfrac{1}{2}\right)c_n^1.$$

Then, by writing $c_n=c_n^1$, $w_i^{(j)}(u)=U_i^{(j)}(z)$ and setting $\sqrt{k^2}=k$ we find

$$\begin{aligned} w_1^{(j)}(u)&=\sum_n (-1)^n c_n Z^{(j)}_{2n+2\nu+1}\left[2k\cos(\sigma u)\right], && |\cos(\sigma u)|\geq 1,\\ w_2^{(j)}(u)&=\tan[\sigma u]\sum_n (-1)^n\left(n+\nu+\tfrac{1}{2}\right)c_n Z^{(j)}_{2n+2\nu+1}\left[2k\cos(\sigma u)\right], && |\cos(\sigma u)|>1, \end{aligned} \tag{81a}$$

$$\begin{aligned} w_3^{(j)}(u)&=\sum_n c_n Z^{(j)}_{2n+2\nu+1}\left[2ki\sin(\sigma u)\right], && |\sin(\sigma u)|\geq 1\\ w_4^{(j)}(u)&=\cot(\sigma u)\sum_n \left(n+\nu+\tfrac{1}{2}\right)c_n Z^{(j)}_{2n+2\nu+1}\left[2ki\sin(\sigma u)\right], && |\sin(\sigma u)|>1, \end{aligned} \tag{81b}$$

where the coefficients c_n satisfy the relations

$$k^2c_{n+1}+\left[(2n+2\nu+1)^2-\mathrm{a}\right]c_n+k^2c_{n-1}=0. \tag{81c}$$

As in the case of the two-sided solutions (64a) and (64b), obtained from the CHE, the above solutions are useless for $\sigma=1$ and $u=$real (Mathieu equation).

The conditions (73) reduce to $2\nu\notin\mathbb{Z}$ and assure the linear independence of the terms in a given series. In addition, the above notation for solutions of the Mathieu equation is similar to the one used by Erdélyi [22]. However, in Ref. [18, 19] the coefficients (c_{n+1},c_n,c_{n-1}) are replaced by $(c_{2n+2},c_{2n},c_{2n-2})$ and the Bessel functions $Z_{2n+2\nu+1}$ written as $Z_{2n+\bar{\nu}}$: this is equivalent to put $2\nu+1=\bar{\nu}$ with $\bar{\nu}\notin\mathbb{Z}$. The above domains of convergence may be compared with the ones of solutions 28.23.2-28.23.5 of [19].

By the Raabe test, the the two-sided solutions $w_1^{(j)}(u)$ and $w_3^{(j)}(u)$ are absolutely convergent also at $|\cos(\sigma u)|=1$ and $|\sin(\sigma u)|=1$, respectively. In Refs. [7, 18, 20, 23] these points are not included in the domains of convergence due to the use of the D'Alembert test. The one-sided solutions $\mathring{w}_i^{(1)}$ converge for any finite u; in Ref. [19] it is stated that this property holds also for two-sided solutions (a misprint, for certain).

3 Possible Applications

In this section we consider two examples which use solutions for the CHE. In the first example we discuss solutions for the Klein-Gordon equation for a scalar test field Φ in the gravitational background of a singular and a non-singular spacetimes. As the time dependence of Φ obeys Mathieu equations without any arbitrary constant, we have to use two-side series solutions. The parameter ν must be determined from the characteristic equation. Then, the Raabe test assures that in both cases the solutions of the Mathieu equations are bounded and convergent for all values of the time variable. However, the full wavefunction is bounded everywhere only for the nonsingular spacetime.

The second example deals with the one-dimensional Schrödinger equation for a family of quasi-exactly solvable potentials. In addition to the expected solutions given by finite series, for a subfamily of the potentials we find infinite-series solutions which, due to the Raabe test, are bounded and convergent for all values of the independent variable.

3.1 Klein-Gordon equation in curved spacetimes

In its conformally static form, the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ for nonflat Friedmann-Robertson-Walker spacetimes is written as [24]

$$ds^2 = [A(\tau)]^2 \left[d\tau^2 - d\chi^2 - \frac{\sin^2(\sqrt{\epsilon}\chi)}{\epsilon}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \quad x^\mu = (\tau, \chi, \theta, \varphi) \tag{82}$$

where $\epsilon = \pm 1$ is the spatial curvature, τ is the time variable, whereas χ, θ and φ are the spatial coordinates. The Klein-Gordon equation for a field Φ with mass M ($\hbar = c = 1$) is

$$\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) + \sqrt{-g}\left(M^2 + \varrho R\right)\Phi = 0, \qquad \partial_\mu = \partial/\partial x^\mu,$$

where g is the determinant associated with $g_{\mu\nu}$, R is the Ricci scalar, $\varrho = 1/6$ for conformal coupling, and $\varrho = 0$ for minimal coupling. By performing the separation of variables

$$\Phi(\chi, \theta, \varphi, \tau) = [A(\tau)]^{-1}\, T(\tau)\, X(\chi)\, \Theta(\theta)\, e^{im\varphi}, \qquad m = 0, \pm 1, \pm 2, \cdots, \tag{83}$$

one finds that X and Θ are given by the same special functions for any scale factor $A(\tau)$ [24], while T obeys the equation

$$\frac{d^2T}{d\tau^2} + \left[\kappa^2 + M^2A^2 + (6\varrho - 1)\left(\frac{1}{A}\frac{d^2A}{d\tau^2} + \epsilon\right)\right]T = 0, \tag{84}$$

The constant of separation κ, determined from the spatial dependence of Φ, is given by

$$\kappa = 1, 2, 3, \cdots \quad \text{if } \epsilon = 1, \qquad \text{and } 0 < \kappa < \infty \text{ if } \epsilon = -1.$$

For a nonsingular model of universe and for (singular) radiation-dominated models, Eq. (84) reduces to Mathieu equations.

3.1.1 Nonsingular metric

For the nonsingular case, the scale factor $A(\tau)$ is given by [25]

$$A(\tau) = \mathrm{a}_0 \cosh\tau, \qquad \epsilon = -1, \qquad -\infty < \tau < \infty,$$

where a_0 is a positive constant, leads to the modified Mathieu equation

$$\frac{d^2T}{d\tau^2} + \left[\kappa^2 + \frac{1}{2}M^2\mathrm{a}_0^2 + \frac{1}{2}M^2\mathrm{a}_0^2\cosh(2\tau)\right]T = 0.$$

So, in Eq. (6) we have

$$\sigma = i, \qquad \mathrm{a} = -\kappa^2 - \left(M^2\mathrm{a}_0^2/2\right), \qquad k = M\mathrm{a}_0/2, \qquad u = \tau.$$

Solutions for this problem have already been given in Ref. [26] where the convergence at $\tau = 0$ is not discussed. Here this question is solved by using the Raabe test.

¿From the Heine-type solutions, only $w_1^{(j)}$ given in equation (81a) afford convergent and bounded wave functions for all $\tau \in (-\infty, \infty)$. We find the solutions

$$T^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n c_n Z_{2n+2\nu+1}^{(j)} \left(M \mathrm{a}_0 \cosh \tau \right), \qquad [2\nu \notin \mathbb{Z}] \tag{85a}$$

where the recurrence relations for c_n are

$$M^2 \mathrm{a}_0^2 \, c_{n+1} + \left[(4n + 4\nu + 2)^2 + 4\kappa^2 + 2M^2 \mathrm{a}_0^2 \right] c_n + M^2 \mathrm{a}_0^2 \, c_{n-1} = 0. \tag{85b}$$

The relations among the Bessel functions [19] imply that only two of the four solutions (85a) are linearly independent. Similar results are found by treating the Mathieu equation as a CHE. In effect, by using $w_1^{(j)}$ given in (64a) we obtain

$$\mathrm{T}^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} a_n Z_{n+\nu+(1/2)}^{(j)} \left[2M \mathrm{a}_0 \cosh^2 (\tau/2) \right], \qquad [2\nu \notin \mathbb{Z}] \tag{86a}$$

where the recurrence relations for a_n are

$$M \mathrm{a}_0 [n + \nu + 1] a_{n+1} - \left[\left(n + \nu + \tfrac{1}{2} \right)^2 + \kappa^2 + M^2 \mathrm{a}_0^2 \right] a_n + M \mathrm{a}_0 [n + \nu] a_{n-1} = 0. \tag{86b}$$

Thus, the Raabe test assures that $T^{(j)}$ and $\mathrm{T}^{(j)}$, as well as the corresponding wavefunctions (83), are bounded and convergent for all $\tau \in (-\infty, \infty)$.

3.1.2 Singular metric

For radiation-dominated spacetimes, $A(\tau) = \mathrm{a}_0 \sin(\epsilon \tau)/\sqrt{\epsilon}$ ($\tau \geq 0$) and, so,

$$\frac{d^2 T}{d\tau^2} + \left[\kappa^2 + \frac{\epsilon}{2} M^2 \mathrm{a}_0^2 - \frac{\epsilon}{2} M^2 \mathrm{a}_0^2 \cos\left(2\sqrt{\epsilon}\tau \right) \right] T = 0. \tag{87}$$

We consider only the case $\epsilon = -1$. We take $\sigma = i$, $\mathrm{a} = (M^2 \mathrm{a}_0^2/2) - \kappa^2$, $k = M \mathrm{a}_0/2$ and $u = \tau$. Then the Heine solutions $w_1^{(j)}(u)$ given in equation (81a) lead

$$T^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n c_n Z_{2n+2\nu+1}^{(j)} \left(M \mathrm{a}_0 \cosh \tau \right), \qquad [2\nu \notin \mathbb{Z}] \tag{88a}$$

where the recurrence relations are

$$M^2 \mathrm{a}_0^2 \, c_{n+1} + \left[(4n + 4\nu + 2)^2 + 4\kappa^2 - 2M^2 \mathrm{a}_0^2 \right] c_n + M^2 \mathrm{a}_0^2 \, c_{n-1} = 0. \tag{88b}$$

On the other side, from the solutions $w_1^{(j)}(u)$ given in (64a) (CHE) we find

$$\mathrm{T}^{(j)}(\tau) = \sum_{n=-\infty}^{\infty} a_n Z_{n+\nu+(1/2)}^{(j)} \left[2M \mathrm{a}_0 \cosh^2 (\tau/2) \right], \qquad [2\nu \notin \mathbb{Z}] \tag{89a}$$

where the relations for a_n are

$$M \mathrm{a}_0 [n + \nu + 1] a_{n+1} - \left[\left(n + \nu + \tfrac{1}{2} \right)^2 + \kappa^2 \right] a_n + M \mathrm{a}_0 [n + \nu] a_{n-1} = 0. \tag{89b}$$

Once more, $T^{(j)}$ and $\mathrm{T}^{(j)}$ are convergent and bounded for all $\tau \geq 0$ but now the wavefunctions (83) become unbounded at $\tau = 0$ due to the factor $1/A(\tau) = 1/(\mathrm{a}_0 \sinh \tau)$.

Therefore, if $\epsilon = -1$ the solutions for the modified Mathieu obtained from the CHE and its Whittaker-Hill limit are suitable for the singular and nonsingular metrics. For the singular metric, the unboundedness of the solutions (83) at $\tau = 0$ is expected since at this point there is a physical singularity in the sense that the pressure and density energy diverge.

3.2 Schrödinger equation for quasi-exactly solvable potentials

Now we consider problems involving solutions given by finite and infinite series for the CHE. For this end, we write the one-dimensional stationary Schrödinger equation for a particle with mass M and energy E as

$$\frac{d^2\psi}{du^2}+\big[\mathcal{E}-\mathcal{V}(u)\big]\psi=0,\quad u=ax,\qquad \mathcal{E}=\frac{2M}{\hbar^2a^2}E,\qquad \mathcal{V}(u)=\frac{2M}{\hbar^2a^2}\mathrm{V}(x),\tag{90}$$

where a is a constant with inverse-length dimension, $\hbar$ is the Plank constant divided by 2π, x is the spatial coordinate and $V(x)$ is the potential. For $\mathcal{V}(u)$ we choose the Ushveridze quasi-exact solvable potential [12]

$$\begin{aligned}\mathcal{V}(u) \;&=\; 4\beta^2\sinh^4 u+4\beta\big[\beta-2(\gamma+\delta)-2\ell\big]\sinh^2 u+4\left[\delta-\tfrac{1}{4}\right]\left[\delta-\tfrac{3}{4}\right]\tfrac{1}{\sinh^2 u}\\ &-\; 4\left[\gamma-\tfrac{1}{4}\right]\left[\gamma-\tfrac{3}{4}\right]\tfrac{1}{\cosh^2 u},\qquad [\ell=0,1,2,\cdots]\end{aligned}\tag{91}$$

where β, γ and δ are real constants with $\beta>0$ and $\delta\geq 1/4$.

When $\delta\geq 1/4$ and ℓ is zero or a natural number, the above family of potentials is quasi-exactly solvable because it admits bounded wavefunctions given by finite series which allow to determine only a finite number of energy levels. However, for $1/4\leq\delta<1/2$ and $1/2<\delta\leq 3/4$ we also find infinite-series solutions which are convergent and bounded for all values of the independent variable: this suggests the possibility of determining the remaining part of the energy spectra as solutions of a characteristic equation. For $\delta>3/4$ we find no solutions like these.

Notice that Ushveridze supposed that $\ell=0,1,2,\cdots$. However we will see that, for

$$(\gamma,\delta)=\left(\tfrac{1}{4},\tfrac{1}{4}\right),\ \left(\tfrac{1}{4},\tfrac{3}{4}\right),\ \left(\tfrac{3}{4},\tfrac{1}{4}\right),\ \left(\tfrac{3}{4},\tfrac{3}{4}\right),\tag{92}$$

the potential is quasisolvable even when ℓ is a positive half-integer. In addition, Ushveridze supposed that $u\in(-\infty,\infty)$. However, we get

$$\lim_{u\to\pm\infty}\mathcal{V}(u)=\infty,\qquad \lim_{u\to 0}\mathcal{V}(u)=\begin{cases}-4\left[\gamma-\frac{1}{4}\right]\left[\gamma-\frac{3}{4}\right],\ \text{if } \delta=\frac{1}{4}\text{ or }\delta=\frac{3}{4};\\ -\infty,\ \text{if } \delta\in\left(\frac{1}{4},\frac{3}{4}\right);\\ +\infty,\ \text{if } \delta\notin\left[\frac{1}{4},\frac{3}{4}\right].\end{cases}\tag{93}$$

Hence, for $\delta\notin[1/4,3/4]$ there is an infinite barrier at $u=0$ and, so, we can suppose that $u\geq 0$ or $u\leq 0$.

3.2.1 Wavefunctions for the Whittaker-Hill equation

If γ and δ take the values (92), the potential (91) reduces to

$$\mathcal{V}(u)=4\beta^2\sinh^4 u+4\beta\big[\beta-2(\gamma+\delta)-2\ell\big]\sinh^2 u,\qquad \left[\ell=0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\cdots\right]\tag{94a}$$

where $u\in(-\infty,\infty)$. Thence, by using $\sinh^2 u=[\cosh(2u)-1]/2$ and $\sinh^4 u=[\cosh(4u)-4\cosh(2u)+3]/8$, Eq. (90) becomes a modified WHE (7) with the parameters

$$\sigma=i,\qquad \vartheta=-\mathcal{E}+4\beta(\ell+\gamma+\delta),\qquad p+1=2(\ell+\gamma+\delta),\qquad \xi=2\beta.\tag{94b}$$

In fact, the WHE also occurs in the cases of the Razavy potential [27] and the symmetric double-Morse potential considered by Zaslavskii and Ulyanov [28].

On the other side, the substitutions

$$z = \cosh^2 u, \qquad \psi(u) = \psi[u(z)] = U(z), \qquad [z \geq 1] \tag{95}$$

transform the Schrödinger equation for the preceding potential into the CHE (1) with

$$z_0 = 1, \quad B_1 = -\tfrac{1}{2}, \qquad B_2 = 1, \qquad B_3 = \tfrac{\mathcal{E}}{4}, \qquad i\omega = \pm\beta, \qquad i\eta = \pm(\ell + \gamma + \delta), \tag{96}$$

where the plus or minus sign must be used throughout. Thus, we can attempt to solve the problem by using known solutions for the CHE. For example, from the Baber-Hassé expansions in power series, the Hylleraas solutions or the Jaffé solutions [1, 29, 30, 31] we obtain even and odd finite-series solutions bounded for $z \geq 1$: such solutions allow to find only a finite number of energy levels. There are also infinite-series solutions which, however, must be discarded because they are not bounded for any admissible value of z.

On the other side, if we use one-sided series solutions $\mathring{\mathbb{U}}_i(z)$ in terms of Coulomb wavefunctions, we may find [16]

- Even and odd finite-series solutions which are convergent and bounded for all $z \geq 1$.
- Even infinite-series solutions which, due to the Raabe test, are convergent and bounded for all $z \geq 1$.
- Odd infinite-series solutions which converge and are bounded only for $z > 1$, and odd solutions which converge and are bounded only for finite values of z. To cover the entire interval $z \geq 1$ it is necessary to consider two of such solutions.

Therefore, in this case we could find additional energy levels by solving a transcendental characteristic equation. This conclusion follows as well from the two-sided infinite series solutions given in Eqs. (100a) and (101a).

3.2.2 Wavefunctions for the cases $1/4 \leq \delta < 1/2$ and $1/2 < \delta \leq 3/4$

For the Ushveridze potential (91), the substitutions

$$z = \cosh^2 u, \qquad \psi(u) = \psi[u(z)] = z^{\gamma-\frac{1}{4}}(z-1)^{\delta-\frac{1}{4}}U(z), \qquad [z \geq 1] \tag{97}$$

transform the Schrödinger equation (90) into a confluent Heun equation with

$$\begin{gathered} z_0 = 1, \qquad B_1 = -2\gamma, \qquad B_2 = 2\gamma + 2\delta, \qquad B_3 = \tfrac{\mathcal{E}}{4} + \left(\gamma + \delta - \tfrac{1}{2}\right)^2, \\ i\omega = \pm\beta, \qquad i\eta = \pm(\ell + \delta + \gamma). \end{gathered} \tag{98}$$

Now we exclude the cases (92) and suppose that ℓ is a non negative integer. We select

$$i\omega = -\beta, \qquad i\eta = -\ell - \gamma - \delta.$$

Then, by using for $U(z)$ the solutions given in Eqs. (29a-b) of Ref. [3], we find

$$\psi_1^{\text{baber}}[u(z)] = e^{-\beta z} z^{\gamma-\frac{1}{4}}(z-1)^{\delta-\frac{1}{4}} \sum_{n=0}^{\ell} \mathrm{a}_n (z-1)^n, \qquad [\ell = 0, 1, 2, \cdots] \tag{99a}$$

where the series coefficients satisfy ($\mathrm{a}_{-1} = 0$)

$$\begin{gathered} (n+1)(n+2\delta)\mathrm{a}_{n+1} + \left[n(n + 2\gamma + 2\delta - 1 - 2\beta) + \tfrac{\mathcal{E}}{4} + \left(\gamma + \delta - \tfrac{1}{2}\right)^2 - 2\beta\delta\right]\mathrm{a}_n + \\ -2\beta(n - \ell - 1)\mathrm{a}_{n-1} = 0. \end{gathered} \tag{99b}$$

According to the theory of three-term recurrence relations [7], the series in (99a) ends at $n = \ell$ because the coefficient of a_{n-1} in (99b) vanishes when $n = \ell + 1$. Since $\beta > 0$, the previous eigenfunctions are bounded for all $z \geq 1$ provided that $\delta \geq 1/4$. In fact, ψ_1^{baber} represents $\ell + 1$ distinct solutions, each one with a different energy [7].

By the other side, we find cases for which there are infinite-series solutions appropriate for any $z \geq 1$. For this we insert into (97) the two-sided solutions U_1^+ and U_2^+ given in (19) and (23), respectively, and use the Raabe test along with the limit (A.6). Thence by scribing convenient values to the parameter ν, we find the solutions ψ_1^+ and ψ_2^+ having the following properties

- The solutions ψ_1^+ are convergent and bounded for all $z \geq 1$ if $1/4 \leq \delta < 1/2$.
- The solutions ψ_2^+ are convergent and bounded for all $z \geq 1$ if $1/2 < \delta \leq 3/4$.
- If $\delta = 1/2$, then $\psi_1^+ = \psi_2^+$. For this case the Raabe test is inconclusive as to the convergence at $z = 1$.

In effect, for $U(z) = U_1^+(z)$ solutions (97) yield

$$\begin{aligned}\psi_1^+(z) &= e^{-\beta z} z^{\nu-\delta+\frac{3}{4}}(z-1)^{\delta-\frac{1}{4}} \sum_{n=-\infty}^{\infty} \frac{b_n^1\ [2\beta z]^n}{\Gamma[n+\nu+1+\ell+\gamma+\delta]} \\ &\times\ \Psi\left[n+\nu+1-\ell-\gamma-\delta, 2n+2\nu+2; 2\beta z\right], \qquad 1/4 \leq \delta < 1/2 \end{aligned} \tag{100a}$$

where, in the recurrence relations (15) for b_n^1 we have

$$\begin{aligned}\alpha_n^1 &= -\tfrac{2\beta[n+\nu+2-\gamma-\delta][n+\nu+1+\gamma-\delta]}{(2n+2\nu+2)(2n+2\nu+3)}, \\ \beta_n^1 &= -\tfrac{\mathcal{E}}{4} + \beta(\ell+\gamma+\delta) - \left[n+\nu+\tfrac{1}{2}\right]^2 - \tfrac{\beta[\ell+\gamma+\delta][\gamma+\delta-1][\gamma-\delta]}{[n+\nu][n+\nu+1]}, \\ \gamma_n^1 &= \tfrac{2\beta[n+\nu-1+\gamma+\delta][n+\nu-\gamma+\delta][n+\nu+\ell+\gamma+\delta][n+\nu-\ell-\gamma-\delta]}{[2n+2\nu-1][2n+2\nu]}.\end{aligned} \tag{100b}$$

By the Raabe test the condition $\delta < 1/2$ assures that the series converge at $z = 1$, while the condition $\delta \geq 1/4$ assures that the factor $(z-1)^{\delta-(1/4)}$ is bounded at $z = 1$. Similarly, for $U(z) = U_2^+(z)$ we obtain

$$\begin{aligned}\psi_2^+(z) &= e^{-\beta z} z^{\nu+\delta-\frac{1}{4}}(z-1)^{-\delta+\frac{3}{4}} \sum_{n=-\infty}^{\infty} \frac{b_n^2\ [2\beta z]^n}{\Gamma[n+\nu+1+\ell+\gamma+\delta]} \\ &\times\ \Psi\left[n+\nu+1-\ell-\gamma-\delta, 2n+2\nu+2; 2\beta z\right], \qquad 1/2 < \delta \leq 4/3 \end{aligned} \tag{101a}$$

where, in the recurrence relations (15) for b_n^2

$$\begin{aligned}\alpha_n^2 &= -\tfrac{2\beta[n+\nu+\gamma+\delta][n+\nu+1-\gamma+\delta]}{(2n+2\nu+2)(2n+2\nu+3)}, \qquad \beta_n^2 = \beta_n^1 \\ \gamma_n^2 &= \tfrac{2\beta[n+\nu+1-\gamma-\delta][n+\nu+\gamma-\delta][n+\nu+\ell+\gamma+\delta][n+\nu-\ell-\gamma-\delta]}{[2n+2\nu-1][2n+2\nu]},\end{aligned} \tag{101b}$$

where $\beta_n^2 = \beta_n^1$ is a functional identity; in fact, β_n^1 and β_n^2 are different of each other because they hold for distinct intervals of δ.

To assure that all the terms of the series are linearly independent and that the summation extends from minus to plus infinity, the parameter ν must be chosen such that

$$2\nu, \quad \nu \pm (\gamma+\delta) \quad \text{and} \quad \nu \pm (\gamma-\delta) \quad \text{are not integers,} \tag{102}$$

where the values for δ are different for solutions (100a) and (101a). The linear independence is assured by requiring that 2ν is not integer, without any restrictions on the parameters of the potential. Thus, for fixed values of γ and δ, we can choose for ν any value in the open interval $(0, 1/2)$ convenient to satisfy the above conditions. The use of one-sided series would lead to restrictions on γ and δ.

4 Conclusion

We have dealt with the convergence of Leaver's expansions in series of Coulomb wave functions for solutions of the CHE. By redefining the Coulomb functions, we have completed the proof of convergence delineated by Leaver and, in addition, have found that the Raabe test improves the regions of convergence for solutions of the CHE and its Whittaker-Ince limit (4) if certain conditions are fulfilled. It is worth noticing that in using the convergence tests we suppose that the independent variable z is finite. So, when z tends to infinite the behaviour of each solution must be analysed carefully.

We have used transformations of variables which lead to solutions with different domains of convergence and/or different behaviors at the singular points. By this procedure, we have recovered all of the Meixner solutions [9] for the spheroidal equation (8) and the Heine solutions [19] for the Mathieu equation (6). In both cases these expansions are given by series of Bessel functions whose convergence regions may be improved by the Raabe test.

We have considered only two-sided solutions for the CHE and its Whittaker-Ince limit. Despite this, due to the validity of the Raabe test, in section 3, we have found that the Klein-Gordon equation in a non-singular model of universe has solutions bounded and convergent for all values of the time variable. In section 3 we have also regarded the Schrödinger equation for a class of quasi-exactly solvable (QES) potentials.

If the real parameter δ satisfies $\delta \geq 1/4$, a part of the energy spectrum of the Schrödinger equation can be computed from finite-series solutions, as expected for any QES potential. However, the remarkable fact is the existence of infinite-series solutions which (by the Raabe test) converge and are bounded for all values of the independent variable if $1/4 \leq \delta \leq 3/4$. These are the solutions which, in principle, permit to find new energy levels as solutions of a transcendental equation. However, here we have not found infinite-series wavefunctions appropriate for $\delta > 3/4$.

Finally, some comments on the solutions for the DCHE as well as on the one-sided solutions for the CHE and DCHE – for details see Ref. [16]. First, the Raabe test is useless for solutions of the DCHE. Second, we find two subgroups of solutions for the DCHE: one is obtained from solutions of the CHE when $z_0 \to 0$; the other follows from that subgroup by a transformation of the DCHE and cannot be derived as limit of expansions in series of Coulomb functions. Thus, we may seek solutions for the CHE which yield such subgroup for the DCHE when $z_0 \to 0$.

On the other hand, to obtain one-sided series solutions we restrict the summation of the two-sided series to $n \geq 0$ by writing the parameter ν as function of the parameters of the differential equations. Thus, each of the four sets $\mathbb{U}_i(z)$ for the CHE gives two expressions for ν and, accordingly, eight sets $\mathring{\mathbb{U}}_i(z)$ of one-sided series solutions. For special values of the parameters these solutions are given by finite series. It must be noticed that: if $\eta \neq 0$, there are three possible types of recurrence relations for the series coefficients; if $\eta = 0$, only two types. In addition, the form of these relations depends on the normalization used for the Coulomb functions.

A Confluent-hypergeometric and Coulomb Functions

Here we write some useful formulas concerning the confluent hypergeometric functions and, in Eqs. (A.12) and (A.13), redefine the Coulomb wave functions. At the end we obtain the relations (A.21) which are important to apply the convergence tests for infinite-series solutions of the CHE.

The regular and irregular confluent hypergeometric functions, $\Phi(a, c; u)$ and $\Psi(a, c; u)$, are solutions of the confluent hypergeometric equation [21]

$$y\tfrac{d^2\varphi}{dy^2}+(c-y)\tfrac{d\varphi}{dy}-a\;\varphi=0. \tag{A.1}$$

The functions $\Phi(a,c;y)$ and $\Psi(a,c;y)$ are also denoted by $M(a,c,y)$ and $U(a,c,y)$, respectively [18]. In fact, the following four solutions for Eq. (A.1)

$$\begin{aligned}&\varphi^{(1)}(y)=\Phi(a,c;y), &&\varphi^{(2)}(y)=\Psi(a,c;y),\\ &\varphi^{(3)}(y)=e^{y}\Psi(c-a,c;-y), &&\varphi^{(4)}(y)=y^{1-c}\Phi(1+a-c,2-c;y),\end{aligned} \tag{A.2}$$

are all of them defined and distinct only if c is not an integer [21]. Different forms for $\varphi^{(i)}$ follow from the Kummer transformations

$$\Phi(a,c;y)=e^{y}\Phi(c-a,c;-y),\qquad \Psi(a,c;y)=y^{1-c}\Psi(1+a-c,2-c;y). \tag{A.3}$$

In this article, we use only $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$. Their Wronskians are [21]

$$\begin{aligned}&\mathscr{W}\left[\varphi^{(1)},\varphi^{(2)}\right]=\mathscr{W}\left[\Phi(a,c;y),\Psi(a,c;y)\right]=-\tfrac{\Gamma(c)}{\Gamma(a)}\;y^{-c}\;e^{y},\\ &\mathscr{W}\left[\varphi^{(1)},\varphi^{(3)}\right]=\mathscr{W}\left[\Phi(a,c;y),e^{y}\Psi(c-a,c;e^{\pm i\pi}y)\right]=\tfrac{\Gamma(c)}{\Gamma(c-a)}\;e^{\mp i\pi c}\;y^{-c}\;e^{y},\\ &\mathscr{W}\left[\varphi^{(2)},\varphi^{(3)}\right]=\mathscr{W}\left[\Psi(a,c;y),e^{y}\Psi(c-a,c;e^{\pm i\pi}y)\right]=e^{\pm i\pi(a-c)}\;y^{-c}\;e^{y}.\end{aligned} \tag{A.4}$$

Therefore, if a, c and $c-a$ are not zero or negative integers, the three solutions are well defined and any two of them form a fundamental system of solutions for confluent hypergeometric equation [21]. If c is not zero or a negative integer, the solutions are connected by [21],

$$\tfrac{\Phi(a,c;y)}{\Gamma(c)}=\tfrac{e^{\mp i\pi a}}{\Gamma(c-a)}\;\Psi(a,c;y)+\tfrac{e^{\pm i\pi(c-a)}}{\Gamma(a)}e^{y}\;\Psi\left(c-a,c;e^{\pm i\pi}y\right), \tag{A.5}$$

When $y\to\infty$, the behaviour of of $\Psi(a,c;y)$ is given by [18]

$$\Psi(a,c;y)\sim y^{-a}\sum_{m=0}^{\infty}\frac{(a)_m(a-c+1)_m}{m!}(-y)^{-m},\qquad -\frac{3\pi}{2}<\arg y<\frac{3\pi}{2}; \tag{A.6}$$

while the behaviour of $\Phi(a,c;y)$ is given by

$$\begin{aligned}\frac{\Phi(a,c;y)}{\Gamma(c)}\sim\frac{e^{y}y^{a-c}}{\Gamma(a)}\sum_{m=0}^{\infty}\frac{(1-a)_m(c-a)_m}{m!}y^{-m}+\\ \tfrac{e^{\pm i\pi a}y^{-a}}{\Gamma(c-a)}\textstyle\sum_{m=0}^{\infty}\tfrac{(a)_m(a-c+1)_m}{m!}(-y)^{-m},\quad a\neq 0,-1,\cdots,\;c-a\neq 0,-1,\cdots,\end{aligned} \tag{A.7}$$

where the upper sign holds for $-\pi/2<\arg y<3\pi/2$ and the lower sign, for $-3\pi/2<\arg y\leq -\pi/2$. In these limits $(\mathrm{x})_m$ denotes the Pochhammer symbol whose definition is

$$(\mathrm{x})_0=1,\qquad (\mathrm{x})_1=\mathrm{x},\qquad (\mathrm{x})_m=\mathrm{x}(\mathrm{x}+1)(\mathrm{x}+2)\cdots(\mathrm{x}+m-1)=\Gamma(\mathrm{x}+m)/\Gamma(\mathrm{x}).$$

Using these definitions, the function $\Phi(a,c:y)$ is written as

$$\Phi(a,c;y)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{y^n}{n!}=1+y+\frac{a}{c}y+\frac{a(a+1)}{c(c+1)}\frac{y}{2!}+\cdots. \tag{A.8}$$

In addition, we have the integral representations [21]

$$\Phi(a,c;y)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{yu}u^{a-1}(1-u)^{c-a-1}du,\qquad \mathrm{Re}(c)>\mathrm{Re}(a)>0, \tag{A.9}$$

and

$$\Psi(a,c;y)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-yu}u^{a-1}(1+u)^{c-a-1}du,\qquad \mathrm{Re}(a)>0. \tag{A.10}$$

On the other hand the Coulomb wave functions are solutions of the equation

$$\tfrac{d^2\mathscr{U}_{n+\nu}}{dy^2}+\left[1-\tfrac{2\eta}{y}-\tfrac{(n+\nu)(n+\nu+1)}{y^2}\right]\mathscr{U}_{n+\nu}=0. \tag{A.11}$$

If $\eta=0$, this can be written in the usual form of the Bessel equation by a substitution of variable. If $\eta\neq 0$ the solutions $\mathscr{U}_{n+\nu}(y)=\mathscr{U}_{n+\nu}(\eta,y)$ are written in terms of one regular confluent hypergeometric function Φ and two irregular functions Ψ, that is,

$$\mathscr{U}_{n+\nu}(\eta,y)=\Big[\phi_{n+\nu}(\eta,y),\ \psi^{+}_{n+\nu}(\eta,y),\ \psi^{-}_{n+\nu)}(\eta,y)\Big] \tag{A.12}$$

where, by definition, we take

$$\begin{aligned}\phi_{n+\nu}(\eta,y)&=\tfrac{e^{iy}}{\Gamma[2n+2\nu+2]}\,[2iy]^{n+\nu+1}\Phi[n+\nu+1+i\eta,2n+2\nu+2;-2iy],\\ \psi^{\pm}_{n+\nu}(\eta,y)&=\frac{\pm 2ie^{\eta\pi}\ e^{\pm iy}}{\Gamma[n+\nu+1\mp i\eta]}[-2iy]^{n+\nu+1}\Psi[n+\nu+1\pm i\eta,2n+2\nu+2;\mp 2iy].\end{aligned} \tag{A.13}$$

In $\psi^{\pm}_{n+\nu}$ the irrelevant factors $\pm 2i\exp(\eta\pi)$ are maintained just to connect the above definitions with the ones used by Leaver. In fact, for $\mathscr{U}_{n+\nu}$ Leaver used the functions $F_{n+\nu}(\eta,y)$ and $G_{n+\nu}(\eta,y)$, defined as

$$\begin{aligned}F_{n+\nu}(\eta,y)=&\tfrac{[\Gamma(n+\nu+1+i\eta)\Gamma(n+\nu+1-i\eta)]^{1/2}}{2e^{\pi\eta/2}\Gamma(2n+2\nu+2)}\times\\ &e^{iy}(2y)^{n+\nu+1}\Phi(n+\nu+1+i\eta,2n+2\nu+2;-2iy),\end{aligned} \tag{A.14}$$

$$\begin{aligned}G_{n+\nu}(\eta,y)\pm iF_{n+\nu}(\eta,y)=&e^{\pi\eta/2}e^{\mp i\pi(n+\nu+1/2)}\left[\tfrac{\Gamma(n+\nu+1\pm i\eta)}{\Gamma(n+\nu+1\mp i\eta)}\right]^{1/2}\times\\ &e^{\pm iy}(2y)^{n+\nu+1}\Psi\left[n+\nu+1\pm i\eta,2n+2\nu+2;\mp 2iy\right].\end{aligned} \tag{A.15}$$

Thus, $\phi_{n+\nu}$ and $\psi^{\pm}_{n+\nu}$ are obtained by dividing the above expressions by Γ_n, defined as

$$\Gamma_n=(1/2)e^{-\eta\pi/2}(-i)^{n+\nu+1}[\Gamma(n+\nu+1+i\eta)\Gamma(n+\nu+1-i\eta)]^{1/2}. \tag{A.16}$$

Inversely, when $\phi_{n+\nu}$ and $\psi^{\pm}_{n+\nu}$ are multiplied by Γ_n, we recover the Leaver normalization.

¿From the properties of the functions $F_{n+\nu}(\eta,y)$ and $G_{n+\nu}(\eta,y)$ given in Eqs. (126) and (125) of Leaver's paper, we find that the functions (A.12) satisfy the equations

$$\begin{aligned}\frac{d\mathscr{U}_{n+\nu}}{dy}=&\frac{i(n+\nu)(n+\nu+1+i\eta)(n+\nu+1-i\eta)}{(n+\nu+1)(2n+2\nu+1)}\mathscr{U}_{n+\nu+1}\\ &-\frac{\eta}{(n+\nu)(n+\nu+1)}\mathscr{U}_{n+\nu}+\frac{i(n+\nu+1)}{(n+\nu)(2n+2\nu+1)}\mathscr{U}_{n+\nu-1}\end{aligned} \tag{A.17}$$

and

$$\begin{aligned}&\tfrac{(n+\nu)(n+\nu+1+i\eta)(n+\nu+1-i\eta)}{(2n+2\nu+1)}\mathscr{U}_{n+\nu+1}-\\ &i\left[\tfrac{(n+\nu)(n+\nu+1)}{y}+\eta\right]\mathscr{U}_{n+\nu}-\tfrac{(n+\nu+1)}{(2n+2\nu+1)}\mathscr{U}_{n+\nu-1}=0.\end{aligned} \tag{A.18}$$

For ν and η fixed, by dividing all terms of (A.18) by $(n^2/2)\mathscr{U}_{n+\nu}$ and letting $n \to \pm\infty$ we find

$$\left[1+\tfrac{1}{n}\left(2\nu+\tfrac{3}{2}\right)\right]\tfrac{\mathscr{U}_{n+\nu+1}}{\mathscr{U}_{n+\nu}}-\tfrac{2i}{y}\left[1+\tfrac{1}{n}\left(2\nu+1\right)\right]-\tfrac{1}{n^2}\left[1+\tfrac{1}{2n}\right]\tfrac{\mathscr{U}_{n+\nu-1}}{\mathscr{U}_{n+\nu}}=0,$$

whose solutions are

$$\tfrac{\mathscr{U}_{n+\nu+1}}{\mathscr{U}_{n+\nu}}\sim\tfrac{iy}{2n^2}\left[1-\tfrac{1}{n}\left(2\nu+\tfrac{5}{2}\right)\right]\ \Leftrightarrow\ \tfrac{\mathscr{U}_{n+\nu-1}}{\mathscr{U}_{n+\nu}}\sim-\tfrac{2in^2}{y}\left[1+\tfrac{1}{n}\left(2\nu+\tfrac{1}{2}\right)\right] \tag{A.19}$$

and

$$\tfrac{\mathscr{U}_{n+\nu+1}}{\mathscr{U}_{n+\nu}}\sim\tfrac{2i}{y}\left[1-\tfrac{1}{2n}\right]\ \Leftrightarrow\ \tfrac{\mathscr{U}_{n+\nu-1}}{\mathscr{U}_{n+\nu}}\sim\tfrac{y}{2i}\left[1+\tfrac{1}{2n}\right], \tag{A.20}$$

provided that $y/n^2=0$ when $n\to\pm\infty$ (this condition is satisfied if y is finite). Thus, there are two possibilities for the ratios between successive Coulomb functions. By demanding that these relations are valid also for $\eta=0$, we find only one ratio: (i) the first expressions in (A.19) and (A.20) hold, respectively, for $\phi_{n+\nu}$ and $\psi^{\pm}_{n+\nu}$ when $n\to\infty$, (ii) the second expression in (A.19) is valid for the three functions when $n\to-\infty$. In other words,

$$\begin{aligned}
&\tfrac{\phi_{n+\nu+1}}{\phi_{n+\nu}}\sim\tfrac{iy}{2n^2}\left[1-\tfrac{1}{n}\left(2\nu+\tfrac{5}{2}\right)\right], &&\tfrac{\psi^{\pm}_{n+\nu+1}}{\psi^{\pm}_{n+\nu}}\sim\tfrac{2i}{y}\left[1-\tfrac{1}{2n}\right], &&[n\to\infty],\\
&\tfrac{\mathscr{U}_{n+\nu-1}}{\mathscr{U}_{n+\nu}}\sim-\tfrac{2in^2}{y}\left[1+\tfrac{1}{n}\left(2\nu+\tfrac{1}{2}\right)\right], &&\mathscr{U}_{n+\nu}=\left(\phi_{n+\nu},\psi^{\pm}_{n+\nu}\right), &&[n\to-\infty].
\end{aligned} \tag{A.21}$$

The above conclusions are obtained as follows. In the first place, if $\eta=0$ the functions $\phi_{n+\nu}$ and $\psi^{\pm}_{n+\nu}$ can be rewritten in terms of Bessel functions since [21]

$$\begin{aligned}
&\Phi(n+\nu+1,2n+2\nu+2;-2iy)=\Gamma\left[n+\nu+(3/2)\right][y/2]^{-n-\nu-\frac{1}{2}}\,e^{-iy}J_{n+\nu+\frac{1}{2}}(y),\\
&\Psi(n+\nu+1,2n+2\nu+2;-2iy)=\tfrac{i\sqrt{\pi}}{2}e^{-iy+i\pi(n+\nu+\frac{1}{2})}(2y)^{-n-\nu-\frac{1}{2}}H^{(1)}_{n+\nu+\frac{1}{2}}(y),\\
&\Psi(n+\nu+1,2n+2\nu+2;+2iy)=-\tfrac{i\sqrt{\pi}}{2}e^{iy-i\pi(n+\nu+\frac{1}{2})}(2y)^{-n-\nu-\frac{1}{2}}H^{(2)}_{n+\nu+\frac{1}{2}}(y),
\end{aligned} \tag{A.22}$$

where J_κ is the Bessel function of the first kind, and $H^{(1)}_\kappa$ and $H^{(2)}_\kappa$ are the first and the second Hankel functions. Thence

$$\phi_{n+\nu}(0,y)=\tfrac{i^n\ C}{\Gamma[n+\nu+1]}\sqrt{y}J_{n+\nu+\frac{1}{2}}(y),\quad \psi^{\pm}_{n+\nu}(0,y)=\tfrac{i^n\ C^{\pm}}{\Gamma[n+\nu+1]}\sqrt{y}H^{(1,2)}_{n+\nu+\frac{1}{2}}(y) \tag{A.23}$$

where the constants C and $C^{\pm}$ do not depend on n. In the second place, if y is bounded and $\kappa\to\infty$ [7]

$$J_\kappa(y)\sim\tfrac{1}{\Gamma(\kappa+1)}\left(\tfrac{y}{2}\right)^{\kappa},\qquad H^{(1)}_\kappa(y)\sim-H^{(2)}_\kappa(y)\sim-\tfrac{i}{\pi}\Gamma(\kappa)\left(\tfrac{2}{y}\right)^{\kappa}. \tag{A.24}$$

Combining (A.23) with (A.24), we establish (A.21) for $\kappa=n+\nu+(1/2)$ when $n\to\infty$ $(\eta=0)$. On the other side, if $\kappa\to-\infty$, we use the previous relations for $H^{(1,2)}_\kappa(y)$ in conjunction with [19]

$$H^{(1)}_{-\kappa}(y)=e^{i\pi\kappa}H^{(1)}_\kappa(y),\qquad H^{(2)}_{-\kappa}(y)=e^{-i\pi\kappa}H^{(2)}_\kappa(y), \tag{A.25}$$

Thus, we find (A.21) for $\mathscr{U}_{n+\nu}=\psi^{\pm}_{n+\nu}$ when $\kappa=n+\nu+(1/2)$ with $n\to-\infty$ $(\eta=0)$. For $\mathscr{U}_{n+\nu}=\phi_{n+\nu}$, if y is bounded and $\kappa\to-\infty$, once more we use the relation given in (A.24) for $J_\kappa(y)$ since

$$\begin{aligned}
J_\kappa(y) &= \left(\frac{y}{2}\right)^{\kappa}\sum_{m=0}^{\infty}\frac{(-1)^m}{m\,!\,\Gamma(\kappa+m+1)}\left(\frac{y}{2}\right)^{2m}\\
&= \left(\frac{y}{2}\right)^{\kappa}\left[\frac{1}{\Gamma(\kappa+1)}+\sum_{m=1}^{\infty}\frac{(-1)^m}{m\,!\,\Gamma(\kappa+m+1)}\left(\frac{y}{2}\right)^{2m}\right].
\end{aligned} \tag{A.26}$$

In this manner, we establish the ratio (A.21) for the three Coulomb functions when $n\to-\infty$.

B Recurrence Relations for the Series Coefficients

Now we present the derivation of the recurrence relations for the series coefficients of the two-sided solutions. The relations for the other sets of solutions may be obtained from these by transformations of variables. Notice that the derivation is formal in the sense that, in each series, we suppose linear independence of all Coulomb wave functions.

The Leaver substitutions [1]

$$U(z) = z^{-B_2/2}H(y), \qquad y = \omega z \tag{B.1}$$

transform the CHE (1) into

$$y(y-\omega z_0)\left[\frac{d^2H}{dy^2}+\left(1-\frac{2\eta}{y}\right)H\right]+\mathsf{C}_1\omega\frac{dH}{dy}+\left[\mathsf{C}_2+\frac{\mathsf{C}_3\omega}{y}\right]H=0 \quad \text{where} \tag{B.2}$$
$$\mathsf{C}_1 = B_1 + B_2 z_0, \qquad \mathsf{C}_2 = B_3 - \frac{B_2}{2}\left[\frac{B_2}{2}-1\right], \qquad \mathsf{C}_3 = -\frac{B_2 z_0}{2}\left[1+\frac{B_2}{2}+\frac{B_1}{z_0}\right].$$

Expanding $H(y)$ as

$$H(y) = \sum_{n=-\infty}^{\infty} b_n^1 \mathscr{U}_{n+\nu}(\eta, y) \quad \Leftrightarrow \quad \mathbb{U}_1(z) = z^{-\frac{B_2}{2}} \sum_{n=-\infty}^{\infty} b_n^1 \mathscr{U}_{n+\nu}(\eta, y) \tag{B.3}$$

and using Eqs. (A.11), (A.17) and (A.18) we find

$$\sum_{n=-\infty}^{\infty} b_n^1\left[\alpha_{n-1}^1\mathscr{U}_{n+\nu-1}(\eta,y)+\beta_n^1\mathscr{U}_{n+\nu}(\eta,y)+\gamma_{n+1}^1\mathscr{U}_{n+\nu+1}(\eta,y)\right]=0, \tag{B.4}$$

where $\alpha_n^{(1)}$, $\beta_n^{(1)}$ and $\gamma_n^{(1)}$ are defined in Eqs. (20).

If ν is such that the summation runs from minus to plus infinity, the preceding equation takes the form

$$\sum_{n=-\infty}^{\infty}\left[\alpha_n^1\, b_{n+1}^1+\beta_n^1\, b_n^1+\gamma_n^1\, b_{n-1}^1\right]\mathscr{U}_{n+\nu}(\eta,y)=0, \tag{B.5}$$

which is satisfied by the three-term recurrence relations (15) provided that all the functions $\mathscr{U}_{n+\nu}(\eta, y)$ are linearly independent.

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