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Stochastic Quantization of Real-Time Thermal Field Theory

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Abstract

We use the stochastic quantization method to obtain the free scalar propagator of a finite temperature field theory formulated in Minkowski spacetime. First we use the Markovian stochastic quantization approach to present the two-point function of the theory. Second, we assume a Langevin equation with a memory kernel and Einstein's relations with colored noise. The convergence of the stochastic processes in the asymptotic limit of the Markov parameter of these Markovian and non-Markovian Langevin equations for a free scalar theory is obtained. Our formalism can be the starting point to discuss systems at finite temperature out of equilibrium.

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1 Introduction

The real time formalism allows one to discuss in detail finite temperature field theory out of equilibrium. The aim of this paper is to present an alternative approach to study finite temperature field theory in the the real time formalism using the stochastic quantization.

The program of stochastic quantization, first proposed by Parisi and Wu [1], and the stochastic regularization were carried out for systems described by fields defined in flat, Euclidean manifolds. A brief introduction to stochastic quantization can be found in Refs. [2] [3] [4], and a complete review is given in Ref. [5]. Recently Menezes and Svaiter [6] [7] implemented the stochastic quantization to study systems with complex valued path integral weights. Since we have a non-positive definite measure, the convergence of the stochastic process in the asymptotic limit of the Markov parameter is not achieved. To circumvent this problem, these authors assumed a Langevin equation with memory kernel and Einstein's relations with colored noise. In the asymptotic limit of the Markov parameter, the equilibrium solution of such Langevin equation was analyzed. It was shown that for a large class of elliptic non-Hermitean operators, which define different models in quantum field theory, the solution converges to the correct equilibrium state in the asymptotic limit of the Markov parameter $\tau \to \infty$.

We would like to remark that such kind of problems of non-convergence of the Langevin equation in the stochastic quantization framework also appears if someone consider the stochastic quantization of classical fields defined in a generic curved manifold. For curved static manifolds, the implementation of the stochastic quantization is straightforward. In this situation it is possible to perform a Wick rotation, i.e., analytically extend the pseudo-Riemannian manifold to the Riemannian domain without problem. Recently, the stochastic quantization for fields defined in a curved spacetime have been studied in the Refs. [8] [9]. Nevertheless, for non-static curved manifolds we have to extend the formalism beyond the Euclidean signature, i.e., to formulate the stochastic quantization in pseudo-Riemannian manifold, instead of formulating it in the Riemannian space, as was originally proposed. See for example the discussion presented by Hüffel and Rumpf [10] and Gozzi [11]. In the first of these papers the authors proposed a modification of the original Parisi-Wu scheme, introducing a complex drift term in the Langevin equation, to implement the stochastic quantization in Minkowski spacetime. Gozzi studied the spectrum of the non-self-adjoint Fokker-Planck Hamiltonian to justify this program. See also the papers [12] [13]. Of course, these situations are special cases of ordinary Euclidean formulation for systems with complex actions.

The main difference between the implementation of the stochastic quantization in Minkowski spacetime and in Euclidean space is the fact that in the latter case the approach to the equilibrium state is a stationary solution of the Fokker-Planck equation. In the Minkowski formulation, the Hamiltonian is non-Hermitian and the eigenvalues of such Hamiltonian are in general complex. The real part of such eigenvalues are important to the asymptotic behavior at large Markov time, and the approach to the equilibrium is achieved only if we can show its semi-positive definiteness. The crucial question is the following: what happens if the Langevin equation describes diffusion around a complex action? Some authors claim that it is possible to obtain meaningful results out of Langevin equation describing diffusion processes around a complex action. Parisi [14] and Klauder and Peterson [15] investigated the complex Langevin equation, where some numerical simulations in one-dimensional systems were presented. See also the papers [16] [17]. We would also like to mention the approach developed by Okamoto et al. [18] where the role of the kernel in the complex Langevin equation was studied. More recently, Guralnik and Pehlevan constructed an effective potential for the complex Langevin equation on a lattice [19]. These authors also investigated a complex Langevin equation and Dyson-Schwinger equations that appear in such situations [20].

We would like to remind that there are many examples in the literature where Euclidean action is complex. We have, for example, QCD with non-vanishing chemical potential at finite temperature; for $SU(N)$ theories with $N > 2$, the fermion determinant becomes complex and also the effective action. Complex terms can also appear in the Langevin equation for fermions, but a suitable kernel can circumvent this problem [21] [22] [23]. Another important case that deserves attention is the stochastic quantization of topological field theories. The simplest case though is, of course, the stochastic quantization in Minkowski spacetime, as we discussed. This situation appears in the case for non-equilibrium problems, which are not amenable to an Euclidean formulation. Recently, Berges and Stamatescu [24] used stochastic quantization techniques to present lattice simulations of non-equilibrium quantum fields in Minkowskian spacetime.

In the perturbation theory in quantum field theory at finite temperature there are three estabilished methods: the Matsubara method [25] [26], the path ordered method [27] [28] and the Thermo Field Dynamics (TFD) approach [29] [30]. For non-equilibrium quantum field systems, the path ordered method or the Thermo Field Dynamics must be used. In the Thermo Field Dynamics approach one can develop finite temperature field theory in real time using the operator formalism, while in the path ordered method the path integral formalism is used. These three methods are related to each other by the analytical continuation of time variables. The motivation of this paper is to present an alternative approach to study finite temperature field theory in the the real time formalism using the Markovian and the non-Markovian stochastic quantization procedures [31]. Basic ideas of the non-Markovian Langevin equation can be found in the Refs. [32] [33] [34] [35].

The outline of the paper is the following. Introduction is given in section I. In section II we present a brief review of the real time formalism in quantum field theory. In section III we use the Markovian stochastic quantization method to study a non-equilibrium thermal field theory formulated in Minkowski spacetime. The non-Markovian approach of the stochastic quantization applied to a thermal scalar field theory is developed in section IV. Conclusions are given in section V. In the appendix, convergence conditions for the stochastic process are derived. In this paper we use $\hbar = c = k_B = 1$.

2 Real-time formalism in finite temperature quantum field theory

In this section we give a brief survey of the formulation of field theory at finite temperature in Minkowski spacetime. Unlike in the imaginary time formalism, in real time formulation, sums over Matsubara frequencies are absent and there is no need to analytically extend the Green functions back to Minkowski spacetime. Moreover, the real time formalism is the starting point for the development of the non-equilibrium quantum field theory, since the investigation of dynamical properties of systems is more naturally performed in this formalism. The real time formalism can describe non-equilibrium processes because the time variable plays a fundamental role and and cannot be traded in for an equilibrium temperature.

For simplicity we work with a neutral scalar field. The field operator in the Heisenberg picture is given by

$$\phi(t, \mathbf{x}) = e^{iHt}\phi(0, \mathbf{x})\, e^{-iHt}, \tag{1}$$

where the time variable t is allowed to be complex. The main quantities to be computed are the thermal Green functions $G_C(x_1, \ldots, x_N)$, defined as

$$G_C(x_1, \ldots, x_N) = \langle T_C\left(\phi(x_1)\ldots\phi(x_N)\right)\rangle_\beta, \tag{2}$$

where the time ordering is taken along a complex time path, yet to be defined. Considering a parametrization $t = z(v)$ of the path, the following expressions:

$$\theta_C(t - t') \;=\; \theta(v - v'), \tag{3}$$

$$\delta_C(t - t') \;=\; \left(\frac{\partial z}{\partial v}\right)^{-1} \delta(v - v'), \tag{4}$$

define the generalized $\theta-$ and $\delta-$functions. The functional differentiation is also extended in the following way:

$$\frac{\delta j(x)}{\delta j(x')} = \delta_C(t - t')\delta^3(\mathbf{x} - \mathbf{x}'), \tag{5}$$

for functions $j(x)$ defined on the path C. The Green functions defined by Eq. (2) can be obtained from a generating functional $Z_C[\beta; j]$ through the expression

$$G_C(x_1, \ldots, x_N) = \frac{(-i)^N}{Z_C[\beta; j]}\frac{\delta^N Z_C[\beta; j]}{\delta j(x_1)\cdots\delta j(x_N)}\bigg|_{j=0}. \tag{6}$$

in the above equation, the generating functional is given by

$$\begin{aligned} Z[\beta; j] &= \mathrm{Tr}\left[e^{-\beta H}\; T_C \exp\left(i\int_C d^4x\; j(x)\phi(x)\right)\right] \\ &= \int \mathcal{D}\phi'\; \langle\phi'(x); t - i\beta| T_C\; \exp\left(i\int_C d^4x\; j(x)\phi(x)\right)|\phi'(x); t\rangle, \end{aligned} \tag{7}$$

where the path C must go through all the arguments of the Green functions. It is also possible to note from this expression that the path C starts from a time $t_i = t$ and ends at a time $t_f = t - i\beta$. We may recast the generating functional into the form

$$Z_C[\beta; j] = \mathcal{N}\exp\left\{-i\int_C\; d^4x\; V\left(\frac{\delta}{i\delta}j(x)\right)\right\}Z_C^F[\beta; j] \tag{8}$$

where $\mathcal{N}$ is a normalization parameter and the free generating functional is given by

$$Z_C^F[\beta; j] = \exp\left\{-\frac{1}{2}\int_C d^4x \int_C d^4y \; j(x) D_C^F(x-y) j(y)\right\}. \tag{9}$$

In Eq. (9), the propagator $D_C^F(x-y)$ is defined through the formula

$$D_C^F(x-x') = \theta_C(t-t') D_C^>(x,x') + \theta_C(t'-t) D_C^<(x,x'), \tag{10}$$

where $D_C^>(x,x')$ and $D_C^<(x,x')$ are, respectively:

$$\begin{aligned} D_C^>(x,x') &= \langle \phi(x)\phi(x')\rangle_\beta, \\ D_C^<(x,x') &= \langle \phi(x')\phi(x)\rangle_\beta. \end{aligned} \tag{11}$$

Since the propagator $D_C^F(x-x')$ is properly defined in the interval $-\beta \leq \mathrm{Im}(t-t') \leq \beta$, one may conclude that the path considered must be such that the imaginary part of the time variable t is non-increasing when the parameter v increases. Furthermore, since we are interested in Green functions whose arguments are real, the path C must contain the real axis. One possible choice for the contour C is described in the following [36]:

1. C starts from a real value t_i, large and negative.
2. The contour follows the real axis up to the large positive value $-t_i$. This part of C is denoted by C_1.
3. The path from $-t_i$ to $-t_i - i\frac{\beta}{2}$, along a vertical straight line. This is denoted by C_3.
4. The path follows a horizontal line C_2 going from $-t_i - i\frac{\beta}{2}$ to $t_i - i\frac{\beta}{2}$.
5. Finally, the path follows a vertical line C_4 from $t_i - i\frac{\beta}{2}$ to $t_i - i\beta$.

Taking $t_i \to -\infty$, the free generating functional can be factorized,

$$Z_C^F[\beta; j] = Z_{C_1\cup C_2}^F[\beta; j] Z_{C_3\cup C_4}^F[\beta; j]. \tag{12}$$

The Green functions with real time arguments can be deduced from $Z_{C_1\cup C_2}^F[\beta; j]$ only. The $Z_{C_3\cup C_4}^F[\beta; j]$ generating functional can be considered a multiplicative constant. Choosing t and t' real, running from $-\infty$ to ∞ and label the sources $j_1(x) = j(t,\mathbf{x})$ and $j_2(x) = j(t - i\beta/2, \mathbf{x})$. Also, one has $\dfrac{\delta j_a(x)}{\delta j_b(x')} = \delta_{ab}\delta^4(x-x')$. With this expressions one may rewrite the free generating functional as

$$Z_C^F[\beta; j] = \mathcal{N}' \exp\left\{-\frac{1}{2}\int d^4x \int d^4x' \; j_a(x) D_{ab}^F j_b(x')\right\}, \tag{13}$$

where, again, $\mathcal{N}'$ is a normalization parameter. The components of the matricial propagator $D_{ab}^F(x-x')$ are given by

$$\begin{aligned} D_{11}^F(x-x') &= D_F(t-t', \mathbf{x}-\mathbf{x}'), \\ D_{22}^F(x-x') &= D_F^*(t-t', \mathbf{x}-\mathbf{x}'), \\ D_{12}^F(x-x') &= D^<(t-t'+i\beta/2, \mathbf{x}-\mathbf{x}'), \\ D_{21}^F(x-x') &= D^>(t-t'-i\beta/2, \mathbf{x}-\mathbf{x}'). \end{aligned} \tag{14}$$

The effective generating functional can be written as

$$Z_C[\beta; j] = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \ \exp\left\{-\frac{1}{2}\int d^4x \ d^4x' \ \phi_a(x)(D_F^{-1})_{ab}(x-x')\phi_b(x')\right\} \tag{15}$$
$$\times \exp\left\{-i\int d^4x \ \left(V(\phi_1) - V(\phi_2)\right) + i\int d^4x \ j_a(x)\phi_a(x)\right\}.$$

The field ϕ_2 may be interpreted as a ghost field on the contour C_2. This doubling of the field degrees of freedom, which does not occur in imaginary time formulation, is unavoidable in the real time formulation. For more details on this subject, the reader is referred to the original paper of Niemi and Semenoff [37] or the Landsmann and van Weert review [38].

3 Real-time finite temperature quantum field theory: the Markovian stochastic quantization approach

The real time formalism is a framework to describe both equilibrium and non-equilibrium systems. Dynamical questions, as for example a weakly interacting Bose gas having a temperature gradient can be studied only in the real time formalism, with a matrix structure of the propagator. Before we study the non-Markovian approach, in this section we will analyze the usual stochastic quantization of a finite temperature field theory formulated in Minkowski space. In Minkowski space, it is well known that the Langevin equation should be written as:

$$\frac{\partial}{\partial\tau}\phi(x,\tau) = i\frac{\delta S}{\delta\phi(x)}\Big|_{\phi(x)=\phi(x,\tau)} + \eta(x,\tau), \tag{16}$$

where $S(\phi)$ is the action for a free scalar field:

$$S(\phi) = \int d^dx \frac{1}{2}\left\{\partial^\mu\phi \ \partial_\mu\phi - m^2\phi^2\right\}, \tag{17}$$

and the correlation functions for the noise field are:

$$\langle \eta(x,\tau)\rangle_\eta = 0, \tag{18}$$
$$\langle \eta(x,\tau)\eta(x',\tau')\rangle_\eta = \delta(|\tau-\tau'|)\delta^d(x-x'). \tag{19}$$

If we consider a complex free scalar field, with an action $S(\phi,\phi^*)$ given by

$$S = \int d^dx \left\{\partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi\right\}, \tag{20}$$

we should have two Langevin equations, one for the scalar field and the other to its complex conjugate. If we write $\Phi = \begin{pmatrix}\phi \\ \phi^*\end{pmatrix}$, and working in Fourier space, we can write the Langevin equations as:

$$\frac{\partial}{\partial\tau}\Phi_a(k,\tau) = i(D_0^{-1})_{ab}\Phi_b(k,\tau) + \eta_a(k,\tau), \tag{21}$$

where we also consider a complex noise field, $\eta = \begin{pmatrix} \eta \\ \eta^* \end{pmatrix}$, $a, b = 1, 2$ and

$$(D_0^{-1})_{ab}(k) = \begin{pmatrix} (k^2 - m^2 + i\epsilon) & 0 \\ 0 & -(k^2 - m^2 - i\epsilon) \end{pmatrix}. \tag{22}$$

The literature emphasizes [5] that the addition of a negative imaginary mass term $-(i/2)\,\epsilon\phi^*\phi$ to the action in Eq. (20) is necessary in order to obtain convergence for the stochastic process being considered. So, it means that we can only take the limit $\epsilon \to 0$ after all calculations have been performed. That explains the presence of the term $-i\epsilon$ in the expression for the quantity $(D_0^{-1})_{ab}$, that appears in Eq. (22). It is straightforward to obtain the two-point correlation functions for this case and to develop the perturbative solution to Eq. (21), with the following discussion on stochastic diagrams. We do not wish to go into details here. For the interested reader, we recommend the references [10] and [12].

Now, let us point our attentions to the non-equilibrium case. As we discussed in the last section, the doubling of the field degrees of freedom is unavoidable in the real time formulation. So, we can also write the field as an isovector $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. The action for this isovector scalar field, in the free case, is given by:

$$S = -\frac{1}{2} \int d^4x \; d^4x' \phi_a(x)(D_F^{-1})_{ab}(x - x')\phi_b(x'), \tag{23}$$

where the components of $(D_F)_{ab}$ are given by Eq. (14). In Fourier space:

$$(D_F)_{ab}(k) = (U^t)_{ac}(\theta)(D_0)_{cd}(k)(U)_{db}(\theta), \tag{24}$$

where $(D_0)_{ab}$ is the inverse of $(D_0^{-1})_{ab}$, given by Eq. (22), and

$$(U)_{ab}(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \tag{25}$$

where:

$$\cosh^2\theta = \frac{e^{\beta|k_0|}}{e^{\beta|k_0|} - 1}. \tag{26}$$

So, we can split $(D_F)_{ab}$ into two parts, $(D_F)_{ab} = (D_0)_{ab} + (D_\beta)_{ab}$, where $(D_0)_{ab}(k)$ is temperature independent

$$(D_0)_{ab}(k) = \begin{pmatrix} \frac{1}{k^2 - m^2 + i\epsilon} & 0 \\ 0 & \frac{-1}{k^2 - m^2 - i\epsilon} \end{pmatrix}, \tag{27}$$

and all temperature dependence appears in $(D_\beta)_{ab}(k)$, which is given by

$$(D_\beta)_{ab}(k) = \frac{-i\epsilon}{(k^2 - m^2)^2 + \epsilon^2} \begin{pmatrix} 2\sinh^2\theta & \sinh 2\theta \\ \sinh 2\theta & 2\sinh^2\theta \end{pmatrix}. \tag{28}$$

Notice that, in the limit $\epsilon \to 0$, we have:

$$\frac{\epsilon}{(k^2 - m^2)^2 + \epsilon^2} \to \pi\delta(k^2 - m^2). \tag{29}$$

As in the zero temperature case, we have to use the expressions above with finite ϵ and take the $\epsilon \to 0$ limit after all the calculations have been done in order to obtain convergence in the limit $\tau \to \infty$. We also know that this is the case for the path integral formalism [37]. The Markovian Langevin equation for the non-equilibrium case is given by

$$\frac{\partial}{\partial \tau}\phi_a(k,\tau) = i(D_F^{-1})_{ab}(k)\phi_b(k,\tau) + \eta_a(k,\tau), \tag{30}$$

where the noise correlation functions are given by:

$$\langle \eta_a(k,\tau)\rangle_\eta = 0, \tag{31}$$

$$\langle \eta_a(k,\tau)\eta_b(k',\tau')\rangle_\eta = (2\pi)^d\,\delta_{ab}\,\delta^d(k+k')\delta(|\tau-\tau'|). \tag{32}$$

The solution for the Eq. (30) is given by:

$$\phi_a(k,\tau) = \int^\infty d\tau'(g(k,\tau-\tau'))_{ab}\;\eta_b(k,\tau'), \tag{33}$$

where $g(k,\tau) = e^{iD_F^{-1}(k)\tau}\theta(\tau)$ is the Green function for the diffusion problem. In order to check convergence, i.e, to analyze if $g(k,\tau)|_{\tau\to\infty} \to 0$, we must first diagonalize the matrix $iD_F^{-1}(k)$. From Eqs. (27) and (28), we have:

$$D_F^{-1}(k) = I(k,\epsilon)\begin{pmatrix} \frac{-1}{k^2-m^2-i\epsilon} - \frac{i\epsilon}{(k^2-m^2)^2+\epsilon^2}\,2\,\sinh^2\theta & \frac{i\epsilon}{(k^2-m^2)^2+\epsilon^2}\,\sinh 2\theta \\ \frac{i\epsilon}{(k^2-m^2)^2+\epsilon^2}\,\sinh 2\theta & \frac{1}{k^2-m^2+i\epsilon} - \frac{i\epsilon}{(k^2-m^2)^2+\epsilon^2}\,2\,\sinh^2\theta \end{pmatrix}, \tag{34}$$

where:

$$I(k,\epsilon) = \frac{-(k^2-m^2)^2 - \epsilon^2(\cosh^4\theta + \sinh^4\theta) - \frac{\epsilon^2}{2}\sinh^2 2\theta}{((k^2-m^2)^2+\epsilon^2)}. \tag{35}$$

Diagonalizing $iD_F^{-1}(k)$, we get the matrix $D'(k)$, given by:

$$D'(k) = i\,I(k,\epsilon)\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \tag{36}$$

where:

$$\lambda_\pm = \frac{\pm\sqrt{(k^2-m^2)^2 - \epsilon^2\,\sinh^2 2\theta} - i\,\epsilon(1+2\sinh^2\theta)}{(k^2-m^2)^2+\epsilon^2}. \tag{37}$$

Since $I(k,\epsilon) < 0$, we notice from the above equations that, indeed, we get $g(k,\tau)|_{\tau\to\infty} \to 0$. We also remark that, as in the zero temperature case, the convergence of the stochastic process was possible because we have maintained in the Eqs. (27) and (28) a finite ϵ. As the reader can easily verify from the Eqs. (36) and (37), if we take the $\epsilon \to 0$ limit in the beginning of the calculations, we should lose the convergence factor $e^{-I(k,\epsilon)\epsilon(1+2\sinh^2\theta)}$. So this limit should be taken after all the calculations have been done in order to obtain convergence in the limit $\tau \to \infty$. We also know that this is the case for the path integral formalism [37].

Now, we are ready to calculate the two point function $\langle \phi_a(k,\tau)\phi_b(k',\tau)\rangle_\eta$. Proceeding with similar calculations as the zero temperature case, it is possible to show that the two-point correlation function is given by:

$$\langle \phi_a(k,\tau)\phi_b(k',\tau)\rangle_\eta = (2\pi)^d\delta^d(k+k')i\,(D_F)_{ac}(k)(1-e^{2iD_F^{-1}(k)\tau})_{cb}, \tag{38}$$

so we see that, in the limit $\tau \to \infty$, we recover the usual result. We are interested now to see the effects of a memory kernel in this non-equilibrium quantum field theory. This is the subject of the next section.

4 Real-time finite temperature quantum field theory: the non-Markovian stochastic quantization approach

The aim of this section is to study finite temperature quantum field theory in Minkowski spacetime, using the non-Markovian stochastic quantization approach. In Minkowski space, the Langevin equation with memory kernel is written as

$$\frac{\partial}{\partial\tau}\phi(x,\tau) = i\int_0^\tau ds\; M_\Lambda(\tau - s)\frac{\delta S}{\delta\phi(x)}\Big|_{\phi(x)=\phi(x,s)} + \eta(x,\tau), \tag{39}$$

where S is the action for the free scalar field, given by Eq. (23). The noise field distribution is such that its first and second momenta are given by

$$\langle \eta_a(x,\tau)\rangle_\eta = 0, \tag{40}$$

$$\langle \eta_a(x,\tau)\eta_b(x',\tau')\rangle_\eta = 2\delta_{ab}\, M_\Lambda(|\tau-\tau'|)\delta^d(x-x'), \tag{41}$$

that is, the distribution is a colored noise Gaussian distribution. We remind the reader that Eq. (39) is to be understood as a matrix equation.

Using a Fourier decomposition for the scalar and noise fields, given by

$$X(x,\tau) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dk\; e^{ikx}X(k,\tau), \tag{42}$$

where the field X represents either the noise field η and the scalar field ϕ, we obtain that each Fourier mode $\phi(k,\tau)$ satisfies a Langevin equation of the form

$$\frac{\partial}{\partial\tau}\phi_a(k,\tau) = i\int_0^\tau ds\; M_\Lambda(\tau-s)\,(D_F^{-1})_{ab}(k)\,\phi_b(k,s) + \eta_a(k,\tau), \tag{43}$$

where $D_F^{-1}(k)$ is the inverse of $D_F(k)$, defined by Eq. (24). With this decomposition, we obtain from Eq. (40) and Eq. (41) the following relations for the noise field Fourier components,

$$\langle \eta_a(k,\tau)\rangle_\eta = 0, \tag{44}$$

$$\langle \eta_a(k,\tau)\eta_b(k',\tau')\rangle_\eta = (2\pi)^d\,\delta_{ab}\, M_\Lambda(|\tau-\tau'|)\delta^d(k+k'). \tag{45}$$

Defining the Laplace transform of the memory kernel as

$$M(z) = \int_0^\infty d\tau\; M_\Lambda(\tau)e^{-z\tau}, \tag{46}$$

we obtain the solution for Eq. (43), subject to the initial condition $\phi_a(k,\tau=0)=0$, $a=1,2$:

$$\phi_a(k,\tau) = \int_0^\infty d\tau'\; G_{ab}(k,\tau-\tau')\eta_a(k,\tau') \tag{47}$$

In Eq. (47) $G_{ab}(k,\tau-\tau') = \Omega_{ab}(k,\tau-\tau')\theta(\tau-\tau')$ is the retarded Green function for the diffusion problem and each component of the Ω-matrix is defined through its Laplace transform,

$$\Omega_{ab} = \left[z\delta_{ab} - i\,M(z)\left(D_F^{-1}\right)_{ab}(k)\right]^{-1}. \tag{48}$$

Thus, the two point correlation function in the Fourier representation is written as

$$\begin{aligned}&\langle\phi_a(k,\tau)\phi_b(k',\tau')\rangle_\eta = D_{ab}(k,\tau,\tau')\\ &= (2\pi)^d\delta^d(k+k')\int_0^\tau ds\int_0^{\tau'} ds'\ \left[\Omega(k,\tau-\tau')\Omega(k,\tau-s')\right]_{ab}\ M_\Lambda(|s-s'|).\end{aligned} \tag{49}$$

The two-dimensional Laplace transform of the above equation is given by

$$\begin{aligned}&\int_0^\infty d\tau\ e^{-z\tau}\ \int_0^\infty d\tau'\ e^{-z'\tau'}\ \int_0^\tau ds\ \int_0^{\tau'} ds'\ \left[\Omega(k,\tau-\tau')\Omega(k,\tau-s')\right]_{ab}\ M_\Lambda(|s-s'|)\\ &= \left[\Omega(k,z)\Omega(k',z')\right]_{ab}\left(\frac{M(z)+M(z')}{z+z'}\right).\end{aligned} \tag{50}$$

Using the Eq. (48) this expression becomes

$$\begin{aligned}&\int_0^\infty d\tau\ e^{-z\tau}\ \int_0^\infty d\tau'\ e^{-z'\tau'}\ \int_0^\tau ds\ \int_0^{\tau'} ds'\ \left[\Omega(k,\tau-\tau')\Omega(k,\tau-s')\right]_{ab}\ M_\Lambda(|s-s'|)\\ &= i\left(\frac{\Omega(k,z)+\Omega(k,z')}{z+z'} - \Omega(k,z)\Omega(k,z')\right)_{ac}(D_F)_{cb}(k).\end{aligned} \tag{51}$$

Applying the inverse transform, we obtain for the two-point function

$$D_{ab}(k,\tau,\tau') = 2i(2\pi)^d\,\delta^d(k+k')\left(\Omega(k,|\tau'-\tau|) - \Omega(k,\tau)\Omega(k,\tau')\right)_{ac}(D_F)_{cb}(k). \tag{52}$$

In order to investigate the convergence of the above equation, we need to specify an expression for the memory kernel M_Λ. We set

$$M_\Lambda(\tau) = \frac{1}{2}\Lambda^2 e^{-\Lambda^2|\tau|}. \tag{53}$$

Substituting the Laplace transform of the Eq. (53) in the Eq. (48), we have that the Ω-matrix is given by

$$\Omega(k,\tau) = \begin{pmatrix}\Omega_{11}(k,\tau) & \Omega_{12}(k,\tau)\\ \Omega_{21}(k,\tau) & \Omega_{22}(k,\tau)\end{pmatrix}, \tag{54}$$

where the components $\Omega_{ab}(k,\tau)$ are given in the Appendix. So, we are in a position to present an expression for the two-point correlation function in the limit $\tau=\tau'\to\infty$:

$$D_{ab}(k,\tau,\tau')|_{\tau=\tau'\to\infty} = i\ (2\pi)^d\delta^d(k+k')(D_F)_{ab}(k), \tag{55}$$

so that, in the limit $\epsilon\to 0$, we have:

$$D_{ab}(k,\tau,\tau')|_{\tau=\tau'\to\infty;\,\epsilon\to 0} = i\ (2\pi)^d\delta^d(k+k')(D_F)_{ab}(k)|_{\,\epsilon\to 0}. \tag{56}$$

A question still remains opened. What are, if any, the advantages of our non-Markovian method over the usual Markovian one? In order to answer such question, we shall apply a Fokker-Planck analysis. As we know, correlation functions are introduced as averages over η:

$$\begin{aligned}&\langle\, \phi(x_1,\tau_1)\phi(x_2,\tau_2)\cdots\phi(x_n,\tau_n)\,\rangle_\eta = \\ &N\int [d\eta]\exp\Big(-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(x,\tau)\Big)\,\phi(x_1,\tau_1)\phi(x_2,\tau_2)\cdots\phi(x_n,\tau_n),\end{aligned} \tag{57}$$

where ϕ obeys Eq. (16) and N is given by:

$$N=\int [d\eta]\exp\Big(-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(x,\tau)\Big) \tag{58}$$

An alternative way to write this average is to introduce the probability density $P[\phi,\tau]$, which is defined as [39]:

$$P[\phi,\tau]\equiv\int [d\eta]\exp\Big(-\frac{1}{4}\int d^dx\int d\tau\,\eta^2(x,\tau)\Big)\prod_y\delta(\phi(y)-\phi(y,\tau)). \tag{59}$$

In terms of P, the correlation functions will read:

$$\langle\, \phi(x_1,\tau_1)\phi(x_2,\tau_2)\cdots\phi(x_n,\tau_n)\,\rangle_\eta = N\int [d\phi]\,\phi(x_1,\tau_1)\phi(x_2,\tau_2)\cdots\phi(x_n,\tau_n)P[\phi,\tau]. \tag{60}$$

The free probability density P satisfies the following Fokker-Planck equation:

$$\frac{\partial}{\partial\tau}P[\phi,\tau]=\int d^dx\frac{\delta}{\delta\,\phi(x)}\Big(\frac{\delta}{\delta\,\phi(x)}-i\frac{\delta\,S}{\delta\,\varphi(x)}\Big)P[\phi,\tau], \tag{61}$$

where S is given by Eq. (17) and with the initial condition:

$$P[\phi,0]=\prod_y\delta(\phi(y)). \tag{62}$$

The stochastic quantization says that we shall have:

$$w.\lim_{\tau\to\infty}P[\phi,\tau]=\frac{\exp(i\,S[\phi])}{\int[d\phi]\exp(i\,S[\phi])}, \tag{63}$$

where the limit is supposed to be taken "weakly" in the sense of the reference [39].

In our real time non-Markovian case, if we notice the resemblance between our retarded Green function $G_{ab}(k,\tau)$ and the one found in Ref. [31], we may follow similar steps to calculate the free probability density. It is given by, in momentum space:

$$P[\phi,\tau]=N^{-1}\exp\Big(\frac{i}{2}\int dk\,\phi_a(k)D^{-1}_{ab}(k,\tau,\tau)\phi_b(-k)\Big) \tag{64}$$

where N^{-1} is a normalization factor and $D^{-1}_{ab}(k,\tau,\tau')$ is the inverse of $D_{ab}(k,\tau,\tau')$, defined by Eq. (52). It is easy to verify that, in the limit $\tau\to\infty$, $P[\phi,\tau]$ will satisfy, up to constants, similar relations as obtained by Ref. [39]. However, for massless scalar theories, those estimations in such reference decay as inverse power of τ. In our approach, even in the massless situation, an exponential behavior is found. Therefore, in the limit $\tau \to \infty$, it seems that we get an improved convergence.

In conclusion, we have used the stochastic quantization method to study thermal field theory formulated in Minkowski spacetime. We have assumed a Langevin equation with a memory kernel and Einstein's relations with colored noise. From the above last equation, we see that the equilibrium solution in the asymptotic Markov time of this non-Markovian Langevin equation for scalar theories can be obtained. Our approach based in stochastic quantization using a non-Markovian Langevin equation proved to be well suited to quantize a classical field out of equilibrium in the real time formalism at finite temperature.

5 Conclusions

In the past several years there have been a lot of interest in quantum field theory at finite temperature. There are three main formulations of finite temperature field theory. The imaginary time approach or the Matsubara formalism and the real time formalism, which can be operatorial or use the path integral approach. In the real time formalism, it is necessary to double the number of the field degrees of freedom. To quantize a classical thermal field theory out of equilibrium, using the stochastic quantization, we are forced to work in the Minkowski spacetime, where naturally a imaginary drift term appears in the Langevin equation. Since in this case the path integral weight is not positive definite, the stochastic quantization in this situation is problematic. Parisi and Klauder proposed complex Langevin equations [14] [15], and some problems of this approach are the following. First of all, complex Langevin simulations do not converge to a stationary distribution in many situations. Besides, if it does, it may converge to many different stationary distributions. The complex Langevin equation also appears when the original method proposed by Parisi and Wu is extended to include theories with fermions [21] [22] [23]. The first question that appears in this context is if make sense the Brownian problem with anticommutating numbers. It can be shown that, for massless fermionic fields, there will not be a convergence factor after integrating the Markovian Langevin equation. Therefore the equilibrium is not reached. One way of avoiding this problem is to introduce a kernel in the Langevin equation describing the evolution of two Grassmannian fields.

In this paper, we have used the method of the stochastic quantization to study thermal field theory formulated in real time. As we discussed, this closed time path method can be used to describe non-equilibrium thermal field theory. First we use the Markovian stochastic quantization approach to present the two-point function of the theory. Second, we assumed a Langevin equation with a memory kernel and Einstein's relation with colored noise. The equilibrium solution of such Langevin equation was analyzed. We have shown that for a large class of elliptic non-Hermitean operators which define different models in quantum field theory converges in the asymptotic limit of the Markov parameter $\tau \to \infty$, and we have obtained the free Green functions of the theory. Although non-trivial, the method proposed can be extended to interacting field theory with complex actions, where a consistent perturbation theory out of equilibrium can be developed.

Finally, the literature has emphasized that the stochastic quantization is only an alternative formalism to quantize a classical field theory, but new results have not been obtained. Nevertheless the stochastic quantization and the Langevin equation can be extremely useful in numerical simulations of field theory models [40] [41]. The implementation of this non-Markovian Langevin equation on the lattice is under investigation by the authors.

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A Appendix

In this appendix, we derive the components $\Omega_{ab}(k,\tau)$. We can write the Ω-matrix as an inverse of the matrix A, whose components are given by

$$A = \begin{pmatrix} z - i\,M(z)\,d' & i\,M(z)\,b' \\ i\,M(z)\,b' & z - i\,M(z)\,a' \end{pmatrix}. \tag{A.1}$$

The quantities that appear in the A-matrix are defined by

$$a' = \frac{a}{ad - b^2}, \tag{A.2}$$

$$b' = \frac{b}{ad - b^2}, \tag{A.3}$$

$$d' = \frac{d}{ad - b^2}, \tag{A.4}$$

and

$$a = \frac{1}{k^2 - m^2 + i\epsilon} - \frac{i\epsilon}{(k^2 - m^2)^2 + \epsilon^2}\, 2\sinh^2\theta, \tag{A.5}$$

$$b = \frac{-i\epsilon}{(k^2 - m^2)^2 + \epsilon^2}\,\sinh 2\theta, \tag{A.6}$$

$$d = \frac{-1}{k^2 - m^2 - i\epsilon} - \frac{i\epsilon}{(k^2 - m^2)^2 + \epsilon^2}\, 2\sinh^2\theta. \tag{A.7}$$

So, we will have

$$(\Omega)_{ab}(k,z) = (A^{-1})_{ab} = \begin{pmatrix} \Omega_{11}(k,z) & \Omega_{12}(k,z) \\ \Omega_{21}(k,z) & \Omega_{22}(k,z) \end{pmatrix}, \tag{A.8}$$

where:

$$\Omega_{11}(k,z) = \frac{z - i\,M(z)\,a'}{(z - i\,M(z)\,a')(z - i\,M(z)\,d') + M^2(z)b'^2}, \tag{A.9}$$

$$\Omega_{12}(k,z) = \Omega_{21}(z) = \frac{-i\,M(z)\,b'}{(z - i\,M(z)\,a')(z - i\,M(z)\,d') + M^2(z)b'^2}, \tag{A.10}$$

$$\Omega_{22}(k,z) = \frac{z - i\,M(z)\,d'}{(z - i\,M(z)\,a')(z - i\,M(z)\,d') + M^2(z)b'^2}. \tag{A.11}$$

The Laplace transform for the memory kernel, Eq. (53), is given by

$$M(z) = \frac{\Lambda^2}{2}\frac{1}{z + \Lambda^2}. \tag{A.12}$$

So, inserting this result in Eqs. (A.9), (A.10) and (A.11), we get:

$$\Omega_{11}(k,z) = \frac{P(z,t_a)}{Q(z)}, \tag{A.13}$$

$$\Omega_{12}(k,z) = \Omega_{21}(z) = \frac{-(t_b\,z + t_b\,\Lambda^2)}{Q(z)}, \tag{A.14}$$

$$\Omega_{22}(k,z) = \frac{P(z,t_d)}{Q(z)}, \tag{A.15}$$

where $t_j = i\,\frac{j'\Lambda^2}{2}$, $j = a, b, d$, and

$$P(z,t_j) = z^3 + 2\,\Lambda^2\,z^2 + (\Lambda^4 - t_j)\,z - t_j\,\Lambda^2, \tag{A.16}$$

$$Q(z) = z^4 + 2\,\Lambda^2\,z^3 + (\Lambda^4 - u)z^2 - u\,\Lambda^2\,z + v, \tag{A.17}$$

with $u = \frac{i(a'+d')\Lambda^2}{2}$ and $v = \frac{(b'^2 - a'd')\Lambda^4}{4}$. From Eqs. (A.2), (A.3) and (A.4), we have that:

$$u = \frac{\Lambda^2\,\epsilon(1 + 2\sinh^2\theta)((k^2 - m^2)^2 + \epsilon^2)}{-(k^2 - m^2)^2 - \epsilon^2(\cosh^4\theta + \sinh^4\theta) + \frac{\epsilon^2}{2}\sinh^2 2\theta}, \tag{A.18}$$

and

$$v = -\frac{\Lambda^4}{4}\frac{((k^2 - m^2)^2 + \epsilon^2)}{-(k^2 - m^2)^2 - \epsilon^2(\cosh^4\theta + \sinh^4\theta) + \frac{\epsilon^2}{2}\sinh^2 2\theta}, \tag{A.19}$$

so $u < 0$ and $v > 0$. In order to get the inverse Laplace transform of each component of the Ω-matrix, we must seek for the solutions of the quartic equation $Q(z) = 0$. As it is well known, a general quartic equation is a fourth-order polynomial equation of the form

$$z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0. \tag{A.20}$$

Using the familiar algebraic technique [42], it is easy to show that the roots of Eq.(A.20) are given by:

$$z_1 = -\frac{1}{4}a_3 + \frac{1}{2}R + \frac{1}{2}D, \tag{A.21}$$

$$z_2 = -\frac{1}{4}a_3 + \frac{1}{2}R - \frac{1}{2}D, \tag{A.22}$$

$$z_3 = -\frac{1}{4}a_3 - \frac{1}{2}R + \frac{1}{2}E, \tag{A.23}$$

$$z_4 = -\frac{1}{4}a_3 - \frac{1}{2}R - \frac{1}{2}E, \tag{A.24}$$

where:

$$R \equiv \left(\frac{1}{4}a_3^2 - a_2 + y_1\right)^{1/2}, \tag{A.25}$$

$$D \equiv \begin{cases} \Big(F(R) + G\Big)^{1/2} & \text{for } R \neq 0 \\ \Big(F(0) + H\Big)^{1/2} & \text{for } R = 0, \end{cases} \tag{A.26}$$

$$E \equiv \begin{cases} \Big(F(R) - G\Big)^{1/2} & \text{for } R \neq 0 \\ \Big(F(0) - H\Big)^{1/2} & \text{for } R = 0, \end{cases} \tag{A.27}$$

$$F(R) \equiv \frac{3}{4}a_3^2 - R^2 - 2a_2, \tag{A.28}$$

$$H \equiv 2\Big(y_1^2 - 4a_0\Big)^{1/2}, \tag{A.29}$$

$$G \equiv \frac{1}{4}(4a_3a_2 - 8a_1 - a_3^3)R^{-1}, \tag{A.30}$$

and y_1 is a real root of the following cubic equation:

$$y^3 - a_2y^2 + (a_1a_3 - 4a_0)y + (4a_2a_0 - a_1^2 - a_3^2a_0) = 0. \tag{A.31}$$

For convenience, let us assume that R, defined by Eq.(A.25), does not vanish. Comparing Eqs. (A.17) and (A.20), we easily see that $a_3 = 2\,\Lambda^2$, $a_2 = \Lambda^4 - u$, $a_1 = -u\,\Lambda^2$ and $a_0 = v$. Therefore, the inverse Laplace transform of Ω_{ab} is given by:

$$\begin{aligned} \Omega_{11}(k,\tau) \;=\; & \frac{P(z_1,t_a)}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}e^{z_1\tau} + \\ & + \frac{P(z_2,t_a)}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}e^{z_2\tau} + \\ & + \frac{P(z_3,t_a)}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}e^{z_3\tau} + \\ & + \frac{P(z_4,t_a)}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}e^{z_4\tau}, \end{aligned} \tag{A.32}$$

$$\Omega_{12}(k,\tau)=\Omega_{21}(k,\tau) \;=\; -\Bigg(\frac{t_b\,z_1+t_b\,\Lambda^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}e^{z_1\tau}+ +\frac{t_b\,z_2+t_b\,\Lambda^2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}e^{z_2\tau}+ +\frac{t_b\,z_3+t_b\,\Lambda^2}{(z_3-z_1)(z_3-z_2)(z_3-z_4)}e^{z_3\tau}+ +\frac{t_b\,z_4+t_b\,\Lambda^2}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}e^{z_4\tau}\Bigg), \tag{A.33}$$

and, finally, $\Omega_{22}(k,\tau)=\Omega_{11}(k,\tau;t_a\to t_d)$. The roots z_i are given by:

$$z_1=-\frac{\Lambda^2}{2}+\frac{1}{2}i\sigma+\frac{1}{2}i\gamma, \tag{A.34}$$

$$z_2=-\frac{\Lambda^2}{2}+\frac{1}{2}i\sigma-\frac{1}{2}i\gamma, \tag{A.35}$$

$$z_3=-\frac{\Lambda^2}{2}-\frac{1}{2}i\sigma+\frac{1}{2}i\gamma, \tag{A.36}$$

$$z_4=-\frac{\Lambda^2}{2}-\frac{1}{2}i\sigma-\frac{1}{2}i\gamma, \tag{A.37}$$

with $\sigma=(|u|-\,y_1)^{1/2}$ and $\gamma=(-\Lambda^4+|u|+y_1)^{1/2}$ being real quantities. The Eqs. (A.32) and (A.33) can be rewritten in a simpler form as:

$$\Omega_{11}(k,\tau) \;=\; -\Bigg(\Bigg(\cos\Big(\frac{(\sigma+\gamma)}{2}\tau\Big)+\frac{\Lambda^2}{(\sigma+\gamma)}\sin\Big(\frac{(\sigma+\gamma)}{2}\tau\Big)\Bigg)h_1+ +\Bigg(\cos\Big(\frac{(\sigma-\gamma)}{2}\tau\Big)+\frac{\Lambda^2}{(\sigma-\gamma)}\sin\Big(\frac{(\sigma-\gamma)}{2}\tau\Big)\Bigg)h_2+ +8\,t_a\,\sin\Big(\frac{\sigma\,\tau}{2}\Big)\sin\Big(\frac{\gamma\,\tau}{2}\Big)+ +4\,t_a\,\Lambda^2\Bigg(g_1\sin\Big(\frac{(\sigma+\gamma)}{2}\tau\Big)+g_2\sin\Big(\frac{(\sigma-\gamma)}{2}\tau\Big)\Bigg)\Bigg)\frac{e^{\frac{-\Lambda^2}{2}\tau}}{8\,\sigma\,\gamma}, \tag{A.38}$$

$$\Omega_{12}(k,\tau)=\Omega_{21}(k,\tau) \;=\; \frac{t_b}{2\,\sigma\,\gamma}\Bigg(\frac{\Lambda^2}{(\sigma+\gamma)}\sin\Big(\frac{(\sigma+\gamma)}{2}\tau\Big)-\frac{\Lambda^2}{(\sigma-\gamma)}\sin\Big(\frac{(\sigma-\gamma)}{2}\tau\Big)+ -2\,\sin\Big(\frac{\sigma\,\tau}{2}\Big)\sin\Big(\frac{\gamma\,\tau}{2}\Big)\Bigg)e^{\frac{-\Lambda^2}{2}\tau}, \tag{A.39}$$

where

$$h_1 \;=\; -(\sigma+\,\gamma)^2-\Lambda^4, \tag{A.40}$$

$$h_2 \;=\; (\sigma-\,\gamma)^2+\Lambda^4, \tag{A.41}$$

$$g_1 \;=\; i-\frac{2\Lambda^2}{(\sigma+\,\gamma)}, \tag{A.42}$$

$$g_2 \;=\; i-\frac{2\Lambda^2}{(\sigma-\,\gamma)}, \tag{A.43}$$

and, as before, $\Omega_{22}(k,\tau) = \Omega_{11}(k,\tau;t_a \to t_d)$. Let us consider the convergence of the stochastic process. In order for our stochastic process to converge, the retarded Green function for the diffusion problem should obey $G_{ab}(k,\tau)|_{\tau\to\infty} \to 0$. In other words, we must have $\Omega_{ab}(k,\tau)|_{\tau\to\infty} \to 0$. From these last expressions, it is easy to see that the stochastic process will converge, if the quantities σ and γ are real, as imposed before. This lead us to the following conditions: $|u|-y_1 > 0$ and $|u|+y_1-\Lambda^4 > 0$, or, combining those requirements, $|u| > \frac{\Lambda^4}{2}$. Remembering Eq. (A.18), we will have the following convergence criterium:

$$\frac{\epsilon(1+2\sinh^2\theta)((k^2-m^2)^2+\epsilon^2)}{(k^2-m^2)^2+\epsilon^2(\cosh^4\theta+\sinh^4\theta)-\frac{\epsilon^2}{2}\sinh^2 2\theta} > \frac{\Lambda^2}{2}, \tag{A.44}$$

Now let us present the quantity y_1. As was stated before, y_1 is a real root of a cubic equation:

$$z^3 + b_2\, z^2 + b_1\, z + b_0 = 0. \tag{A.45}$$

Comparing Eqs. (A.17), (A.20), (A.31) and (A.45), we have the following identifications: $b_2 = u - \Lambda^4$, $b_1 = -2\,\Lambda^4\, u - \,4\, v$ and $b_0 = -4\, u\, v - \, u^2\, \Lambda^4$. If we let:

$$q = \frac{1}{3}b_1 - \frac{1}{9}b_2^2, \tag{A.46}$$

$$r = \frac{1}{6}(b_1 b_2 - 3b_0) - \frac{1}{27}b_2^3, \tag{A.47}$$

we will have that

$$q = -\frac{4}{9}\Lambda^4\, u - \frac{\Lambda^8}{9} - \frac{4}{3}v - \frac{u^2}{9}, \tag{A.48}$$

$$r = \frac{4}{9}\Lambda^8\, u + \frac{2}{3}\Lambda^4\, v + \frac{1}{18}\Lambda^4\, u^2 + \frac{4}{3}uv - \frac{1}{27}\Lambda^{12} + \frac{1}{27}u^3. \tag{A.49}$$

So, writing $s_1 = (r + \sqrt{q^3+r^2})^{1/2}$ and $s_2 = (r - \sqrt{q^3+r^2})^{1/2}$, we have that:

$$y_1 = (s_1 + s_2) + \frac{\Lambda^4 - \, u}{3}. \tag{A.50}$$

As one can see, $y_1 > 0$. Also, in the limit $\epsilon \to 0$, y_1 becomes a polynomial of Λ.

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