Classification of irreps of the N-extended Supersymmetric Quantum Mechanics.*

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Abstract

We present the *Z. Kuznetsova, M. Rojas, F. Toppan, JHEP 03 (2006) 098* classification of irreps of the N-extended $1D$ supersymmetry algebra, linearly realized on a finite number of fields. Based on the 1-to-1 correspondence between Weyl-type Clifford algebras and classes of irreps of the N-extended $1D$ supersymmetry (ref. *A. Pashnev, F. Toppan, JMP 42 (2001) 5257*), the fields entering an irrep are accommodated in l different states. The admissible multiplets are given by a regular "dressing operator". The complete list of irreps is explicitly presented for $N \leq 10$.

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1 Introduction

The Supersymmetric Quantum Mechanics [1] has important applications in mathematical physics in connection with, e.g., the Morse theory and the Atiyah-Singer index. The large N extended supersymmetry (up to $N = 32$) plays a fundamental role as a dimensional-reduced quantum mechanical system derived from the maximal eleven-dimensional supergravity or its M-theory extension. In condensed matter, applications of supersymmetric quantum mechanics in the $N \to \infty$ limit have been investigated to describe the BCS model, see e.g. [2].

In this paper we present the classification of the irreducible representations of the algebra of the N-extended supersymmetric quantum mechanics, obtained in [3].

The N-extended one-dimensional supersymmetry algebra is a $\mathbf{Z}_2$ graded superalgebra with N odd generators Q_i, $i = 1, \ldots, N$ and a single even central charge H. The algebra is defined by

$$\begin{array}{rcl} \{Q_i, Q_j\} &=& \delta_{ij} H, \\ \left[Q_i, H\right] &=& 0. \end{array} \tag{1.1}$$

The irreducible representations linearly realized on an equal (finite) number n of bosonic and fermionic fields, depending on a parameter $t \in \mathbb{R}$ (the time coordinate) are labeled by the multiplets (see [4])

$$(n_1, n_2, \ldots, n_k) \tag{1.2}$$

where the n_i's are non-negative integers satisfying the condition

$$n_1 + n_3 + \ldots = n_2 + n_4 + \ldots = n \tag{1.3}$$

The odd (even)-indiced n_is denote the number of bosonic (fermionic) fields in the given irrep of degree $\frac{i-1}{2}$.

The problem solved in [3] consists in finding the allowed (1.2) multiplets for any given value N. This was an open problem in the literature.

2 Irreps of the N-extended $d = 1$ supersymmetry and Clifford algebras: the connection revisited

In this section we review the main results of ref. [4] concerning the classification of irreps of the N-extended one-dimensional supersymmetry algebra.

The N extended $D = 1$ supersymmetry algebra is given by

$$\{Q_i, Q_j\} = \eta_{ij} H \tag{2.1}$$

where the Q_i's are the supersymmetry generators (for $i, j = 1, \ldots, N$) and $H \equiv -i\frac{\partial}{\partial t}$ is a hamiltonian operator (t is the time coordinate). If the diagonal matrix η_{ij} is pseudo-Euclidean (with signature (p,q), $N=p+q$) we can speak of generalized supersymmetries. The analysis of [4] was done for this general case. For convenience in the present paper (despite the fact that our results can be straightforwardly generalized to pseudo-Euclidean supersymmetries, having applicability, e.g., to supersymmetric spinning particles moving in pseudo-Euclidean manifolds) we work exclusively with ordinary N-extended supersymmetries. Therefore for our purposes here $\eta_{ij}\equiv\delta_{ij}$.

The (D-modules) representations of the (2.1) supersymmetry algebra realized in terms of *linear* transformations acting on *finite* multiplets of fields satisfy the following properties. The total number of bosonic fields equal the total number of fermionic fields. For irreps of the N-extended supersymmetry the number of bosonic (fermionic) fields is given by d, with N and d linked through

$$\begin{aligned} N &= 8l+n, \\ d &= 2^{4l}G(n), \end{aligned} \tag{2.2}$$

where $l=0,1,2,\ldots$ and $n=1,2,3,4,5,6,7,8$. $G(n)$ appearing in (2.2) is the Radon-Hurwitz function [4]

n	1	2	3	4	5	6	7	8
$G(n)$	1	2	4	4	8	8	8	8

(2.3)

The modulo 8 property of the irreps of the N-extended supersymmetry is in consequence of the famous modulo 8 property of Clifford algebras. The connection between supersymmetry irreps and Clifford algebras is specified later.

The bosonic (fermionic) fields entering an irreducible multiplet can be grouped together according to their dimensionality. Throughout this paper we use, interchangeably, the words "dimension" or "spin" to refer to the dimensionality of the component fields. It is in fact useful, especially when discussing the $D=1$ dimensional reduction of higher-dimensional supersymmetric theories, to refer at the dimensionality of the $D=1$ fields as their "spin". The number (equal to l) of different dimensions (i.e. the number of different spin states) of a given irrep, will be referred to as the *length* l of the irrep. Since there are at least two different spin states (one for bosons, the other for fermions), obtained when all bosons (fermions) are grouped together within the same spin, the minimal length of an irrep is $l=2$.

A general property of (linear) supersymmetry in any dimension is the fact that the states of highest spin in a given multiplet are auxiliary fields, whose supersymmetry transformations are given by total derivatives. Just for $D=1$ total derivatives coincide with the (unique) time derivative. Using this specific property of the one-dimensional supersymmetry it was proven in [4] that all finite linear irreps of the (2.1) supersymmetry algebra fall into classes of equivalence, each class of equivalence being singled out by an associated minimal length ($l=2$) irreducible multiplet. It was further proven that the minimal length irreducible multiplets are in 1-to-1 correspondence with a subclass of Clifford algebras (the ones which satisfy a Weyl property). The connection goes as follows. The supersymmetry generators acting on a length-2 irreducible multiplet can be expressed as

$$Q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{pmatrix} \quad (2.4)$$

where the σ_i and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of the Clifford algebra relation

$$\Gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{pmatrix} \quad , \quad \{\Gamma_i, \Gamma_j\} = 2\eta_{ij} \quad (2.5)$$

The Q_i's in (2.4) are supermatrices with vanishing bosonic and non-vanishing fermionic blocks, acting on an irreducible multiplet m (thought as a column vector) which can be either bosonic or fermionic. The connection between Clifford algebra irreps of Weyl type and minimal length irreps of the N-extended one-dimensional supersymmetry is such that D, the dimensionality of the (Euclidean, in the present case) space-time of the Clifford algebra (2.5) coincides with the number N of the extended supersymmetries, according to

$\sharp$ of space-time dim. (Weyl-Clifford)	$\Leftrightarrow$	$\sharp$ of extended su.sies (in 1-dim.)
D	$=$	N

(2.6)

The matrix size of the associated Clifford algebra (equal to $2d$, with d given in (2.2)) corresponds to the number of (bosonic plus fermionic) fields entering the one-dimensional N-extended supersymmetry irrep.

The classification of Weyl-type Clifford irreps, furnished in [4], can be easily recovered from the well-known classification of Clifford irreps, given in [5] (see also [6] and [7]).

The (2.4) Q_i's matrices realizing the N-extended supersymmetry algebra (2.1) on length-2 irreps have entries which are either c-numbers or are proportional to the hamiltonian H. Irreducible representations of higher length ($l \geq 3$) are systematically produced [4] through repeated applications of the dressing transformations

$$Q_i \mapsto \widehat{Q}_i^{(k)} = S^{(k)} Q_i {S^{(k)}}^{-1} \quad (2.7)$$

realized by diagonal matrices $S^{(k)}$'s ($k = 1, \ldots, 2d$) with entries ${s^{(k)}}_{ij}$ given by

$${s^{(k)}}_{ij} = \delta_{ij}(1 - \delta_{jk} + \delta_{jk} H) \quad (2.8)$$

Some remarks are in order [4]

i) the dressed supersymmetry operators ${Q_i}'$ (for a given set of dressing transformations) have entries which are integral powers of H. A subclass of the ${Q_i}'$ s dressed operators is given by the local dressed operators, whose entries are *non-negative* integral powers of H (their entries have no $\frac{1}{H}$ poles). A local representation (irreps fall into this class) of an extended supersymmetry is realized by local dressed operators. The number of the extension, given by N' ($N' \leq N$), corresponds to the number of local dressed operators.

ii) The local dressed representation is not necessarily an irrep. Since the total number of fields (d bosons and d fermions) is unchanged under dressing, the local dressed representation is an irrep iff d and N' satisfy the (2.2) requirement (with N' in place of N).

iii) The dressing changes the dimension (spin) of the fields of the original multiplet m. Under the $S^{(k)}$ dressing transformation (2.7), $m \mapsto S^{(k)}m$, all fields entering m are unchanged apart the k-th one (denoted, e.g., as φ_k and mapped to $\dot{\varphi}_k$). Its dimension is changed from $[k] \mapsto [k]+1$. This is why the dressing changes the length of a multiplet. As an example, if the original length-2 multiplet m is a bosonic multiplet with d spin-0 bosonic fields and d spin-$\frac{1}{2}$ fermionic fields (in the following such a multiplet will be denoted as $(x_i;\psi_j) \equiv (d,d)_{s=0}$, for $i,j=1,\ldots,d$), then $S^{(k)}m$, for $k \leq d$, corresponds to a length-3 multiplet with $d-1$ bosonic spin-0 fields, d spin-$\frac{1}{2}$ fermionic fields and a single spin-1 bosonic field (in the following we employ the notation $(d-1,d,1)_{s=0}$ for such a multiplet).

Let us fix now the overall conventions. The most general multiplet is of the form $(d_1, d_2, \ldots, d_l)$, where d_i for $i = 1, 2, \ldots, l$ specify the number of fields of a given spin $s + \frac{i-1}{2}$. The spin s, i.e. the spin of the lowest component fields in the multiplet, will also be referred to as the "spin of the multiplet". When looking purely at the representation properties of a given multiplet the assignment of an overall spin s is arbitrary, since the supersymmetry transformations of the fields are not affected by s. Introducing a spin is useful for tensoring multiplets and becomes essential for physical applications, e.g. in the construction of supersymmetric invariant terms entering an action.

In the above multiplet l denotes its length, d_l the number of auxiliary fields of highest spins transforming as time-derivatives. The total number of odd-indiced equal the total number of even-indiced fields, i.e. $d_1+d_3+\ldots = d_2+d_4+\ldots = d$. The multiplet is bosonic if the odd-indiced fields are bosonic and the even-indiced are fermionic (the multiplet is fermionic in the converse case). For a bosonic multiplet the auxiliary fields are bosonic (fermionic) if the length l is an odd (even) number.

Just like the overall spin assignment, the assignment of a bosonic (fermionic) character to a multiplet is arbitrary since the mutual transformation properties of the fields inside a multiplet are not affected by its statistics. Therefore, multiplets always appear in dually related pairs s.t. to any bosonic multiplet there exists its fermionic counterpart with the same transformation properties (see also [8]).

In [4] it was shown that all dressed supersymmetry operators producing any length-3 multiplet (of the form $(d-p,d,p)$ for $p=1,\ldots,d-1$) are of local type. Therefore, for length-3 multiplets, we have $N'=N$.

3 Classification of the irreps

In this section we present a systematic procedure to produce and classify length $l>3$ irreps of the (2.1) supersymmetry algebra for arbitrary values of N. We apply it to fully classify all irreps up to $N \leq 10$ and, for the next cases of the oxidized $N = 11^{(*)}$

and $N=12$ supersymmetries, the length-4 irreps.

Our approach is based on the following points:

i) the (2.4) and (2.5) connection between oxidized Clifford irreps and (oxidized and reduced) length-2 irreps of the (2.1) supersymmetry algebra,

ii) the (2.7) dressing transformation of length-2 irreps, producing length $l>2$ local type representations of the (2.1) supersymmetry algebra,

iii) the matching condition (2.2) between the number of the extended supersymmetries and the dimension of the representation. It is satisfied if and only if the representation is irreducible and, finally,

iv) the algorithmic properties of the real Clifford irreps discussed at the end of Section **3**.

As explained in Section **2**, the dressing can produce $\frac{1}{H}$ poles in the dressed supersymmetry operators. An $S^{(k)}$ dressing (2.7, 2.8) of a given supersymmetry operator Q has the total effect of multiplying by $\frac{1}{H}$ all Q's entries belonging to the k-th column and by H all Q's entries belonging to the k-th row, leaving unchanged all remaining entries. In order to count (and remove) dressed operators with $\frac{1}{H}$ poles one has to determine how non-vanishing entries are distributed in the whole set of supersymmetry operators (since the Q's are 2×2 block-antidiagonal matrices, we can focus on the upper-right block, the lower-left block presenting the same structure). Up to $N\leq 8$, all non-vanishing entries of an oxidized supersymmetry fill the whole upper-right block (for $N=8$, e.g., we have eight supersymmetry operators with 8 non-overlapping non-vanishing entries each, s.t. $8\times 8=64$, filling the 8×8 upper block chessboard of the $N=8$ supersymmetry). Starting from $N\geq 9$ this is no longer the case. The 16×16 right upper block "chessboard" of the $N=9$ supersymmetry is filled with a total number of $9\times 16=144<16^2$ non-overlapping non-vanishing entries.

In the $N=9$ example each column and each row of the upper-right (bottom-left) block intercepts the same amount of 9 non-vanishing entries belonging to the whole set of 9 gamma matrices; the remaining $16-9=7$ entries are zero.

Not only the total number, but also the distribution of the non vanishing-entries inside the block matrices matters when computing the locality condition of the dressed supersymmetry operators. The structure of the non-vanishing entries filling the large-N oxidized supersymmetries can be recovered from the algorithmic construction of the Clifford irreps. For $N\geq 8$, the filling of the upper-right block can be symbolically presented (the block-symbol diagrams below) in terms of the three fundamental fillings of an 8×8 matrix. The three fundamental fillings, denoted as O, I, X, represent, respectively,

i) O $\equiv$ only vanishing entries,

ii) I $\equiv$ non-vanishing entries filling the diagonal,

iii) X $\equiv$ non-vanishing entries filling the whole 8×8 matrix.

The block-symbols, explicitly presented here for the oxidized supersymmetries with

$8 \leq N \leq 12$, are given by

$$N=8 \quad : \quad \begin{pmatrix} \mathrm{X} \end{pmatrix}$$

$$N=9 \quad : \quad \begin{pmatrix} \mathrm{I} & \mathrm{X} \\ \mathrm{X} & \mathrm{I} \end{pmatrix}$$

$$N=10 \quad : \quad \begin{pmatrix} \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} \\ \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} \\ \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} \\ \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} \end{pmatrix}$$

$$N=11^* \quad : \quad \begin{pmatrix} \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} \\ \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} \\ \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{O} \\ \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{O} \\ \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} \end{pmatrix} \tag{3.1}$$

$$N=12 \quad : \quad \begin{pmatrix} \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} \\ \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} \\ \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} \end{pmatrix}$$

Block-symbols can be straightforwardly computed for arbitrary large-N values of the oxidized supersymmetries.

For reduced supersymmetries extra holes appear in the block-symbols, corresponding to the non-vanishing entries belonging to the $N-N'$ supersymmetry operators that have been "removed" from the whole set of oxidized operators in order to produce the reduced N'-extended supersymmetry.

Concerning multiplets, it is worth reminding that the diagonal dressing operator

$$S = \begin{pmatrix} H \cdot \mathbf{1}_d & 0 \\ 0 & \mathbf{1}_d \end{pmatrix} \tag{3.2}$$

applied on a (d,d) length-2 multiplet reverses its statistics (the same transformation reverses the statistics of fields in any given multiplet).

Length-3 multiplets are obtained by applying, on a (d,d) length-2 multiplet, diagonal dressing operators S with a total number of k (with $1 \leq k \leq d-1$) single powers of H in the first d diagonal entries, while the $2d-k$ remaining diagonal entries are 1.

Length-4 multiplets require dressing operators with $\tilde{k}$ (for $1 \leq \tilde{k} \leq d-1$) single powers of H diagonal entries in the positions $d+1, \ldots, 2d$.

Length-5 (length-6) multiplets require a dressing operator S with at least one H^2 second power diagonal entry in the position $1, \ldots, d$ (and, respectively, $d+1, \ldots, 2d$).

Length-7 and length-8 multiplets require dressing operators with at least a third power, H^3, diagonal entry and so on.

We are now in the position to compute the length $l \geq 4$ irreducible representations of the oxidized supersymmetries. Let us illustrate at first an $N = 9$ example. An $N = 9$ length-3 irrep with 15 auxiliary fields (i.e. $(1, 16, 15)$) is such that the original $(16, 16)$ upper-right block $\mathcal{B}$ is mapped into a new block, $\mathcal{B} \mapsto \mathcal{B}'$, by multiplying 15 columns by H, while leaving the remaining column unchanged. The lengthening $3 \mapsto 4$, obtained by leaving unchanged the number of fields, 15, in the third position, produces a block-mapping $\mathcal{B}' \mapsto \mathcal{B}''$, where the new block is obtained from $\mathcal{B}'$ by multiplying a certain number of rows by $\frac{1}{H}$, while the remaining ones are left unchanged. The condition that no $\frac{1}{H}$ poles appear in $\mathcal{B}''$ implies that, at most, seven rows can be picked up. They have to be chosen among the ones corresponding to the zeroes of the single, unchanged, column of $\mathcal{B}'$. It turns out that $N = 9$ admits seven inequivalent length-4 irreps of the type $(1, 16-k, 15, k)$, for $k = 1, 2, \ldots, 7$.

The same strategy can be applied starting from $(2, 16, 14)$, $(3, 16, 13)$ and so on. At the end we produce the complete list of length-4 irreps of $N = 9$. This procedure straightforwardly works for computing length-4 irreps of any oxidized value of N, once the corresponding block-symbols are known.

For what concerns $l > 4$, let us illustrate the $N = 10$ length-5 example, since 10 is the least value of an extended supersymmetry admitting irreps with $l > 4$. Let us check, at first, whether we can produce a single auxiliary field in the fifth position. This amounts to multiply by $\frac{1}{H^2}$ a single row of the original $(32, 32)$ bottom-left block. Since all its entries, see (2.4), are already multiplied by H, this implies that the new bottom-left block admits a single $\frac{1}{H}$ pole in correspondence with the non-vanishing entries of the transformed row, while it is regular anywhere else. We get on the transformed row ten poles. In order to kill them we need to multiply (at least) the 10 corresponding columns of the bottom-left block by H. This multiplication corresponds to the transformation which maps (at least) 10 fields from the second to the fourth position. This transformation acts on the upper-right block by multiplying the corresponding rows by $\frac{1}{H}$. In its turn, these extra-poles have to be cancelled by multiplying a convenient number of columns by H (in correspondence with the transformation mapping fields from the first to the third position). The extra $\frac{1}{H}$ poles produced by this new compensating transformation on the corresponding rows of the bottom-left block do not produce any further singularity, due to the presence of the overall H factor mentioned above.

The same procedure can be later applied to verify whether there is enough room to have two, three or more fields in the fifth position.

Length $l \geq 6$ irreps can be analyzed along the same lines.

For what concerns the reduced extended supersymmetries, the computation of their irreps can be carried on just like the oxidized supersymmetries, but taking into account that their block-symbols admit extra holes. We concentrate on $N = 8$ reductions. The eight gamma matrices generating $N = 8$ under the (2.6) correspondence are all on equal footing. We can single out any one of them (let's say the one with a diagonal upper-right block) in order that the remaining ones generate $N = 7$. The diagonal holes in the $N = 7$ block-symbol imply that, just like the first $N = 9$ example discussed above, we can lengthen the $N = 8$ $(1, 8, 7)$ irrep into an $N = 7$ $(1, 7, 7, 1)$ irrep. The analysis of the $N = 5^{(*)}, 6$ (and $N = 3^{(**)}$ derived from $N = 4$) cases is done in the same way.

Let us now make some necessary remarks on the irreducible representations. Two types of dualities act on them. We have at first the *fermion* $\Leftrightarrow$ *boson* duality, obtained by exchanging, via the (3.2) dressing, the statistics of the component fields in the multiplet. A second type of duality can be referred to as the *high* $\Leftrightarrow$ *low* spin duality. This new duality involves the mapping of a $(d_1, d_2, \ldots, d_l)$ irreducible multiplet into its irreducible dual multiplet

$$(d_1, d_2, \ldots, d_l) \quad \Leftrightarrow \quad (d_l, d_{l-1}, \ldots, d_1) \tag{3.3}$$

obtained by turning the highest-spin fields into the lowest spin fields. Therefore this duality relates two opposite statistics multiplets if l is even and two multiplets with the same statistics if l is odd.

Let us denote with $(\,{}^1x_{j_1};\ {}^2x_{j_2}; \ldots ;\ {}^lx_{j_l})$ the set of fields entering $(d_1, d_2, \ldots, d_l)$ (here $j_i = 1, \ldots, d_i$). The dual irreducible $(d_l, d_{l-1}, \ldots, d_1)$ multiplet can be realized with the fields $(\,{}^lx_{j_l};\ {}^{l-1}\dot{x}_{j_{l-1}}; \ldots ;\ {}^1{x_{j_1}}^{(l-1)})$, where $x^{(k)}$ here denotes the application of the time derivative k-times. Applying the same transformation on the latter multiplet we obtain a new multiplet, $(\,{}^1{x_{j_1}}^{(l-1)};\ {}^2x_{j_2}^{(l-1)}; \ldots ;\ {}^lx_{j_l}^{(l-1)})$, whose supersymmetry transformations are nevertheless the same as the original ones. As a corollary, the class of the irreducible representations is closed under the (3.3) *high* $\Leftrightarrow$ *low* spin duality.

The *high* $\Leftrightarrow$ *low spin* duality (3.3) concides with the *fermion* $\Leftrightarrow$ *boson* (3.2) duality only when applied to self-dual (under (3.3)) multiplets of even length. It is a distinct duality transformation in the remaining cases.

For what concerns the total number $\overline{\kappa}$ of inequivalent irreps of the N-extended supersymmetry, it is given by the sum of the $\overline{\kappa}_l$ inequivalent irreps of length-l, namely,

$$\overline{\kappa} \;=\; \sum_{l=2}^{L} \overline{\kappa}_l \tag{3.4}$$

where L is the maximal length for an N-extended supersymmetry irrep.

$\overline{k}$ is the counting of inequivalent irreps irrispectively of the overall statistics of the multiplets. A factor 2 can be introduced if we want to discriminate the statistics of the multiplets (bosonic or fermionic). In this case the number of inequivalent irreps is κ, with

$$\kappa \;=\; 2\overline{\kappa} \tag{3.5}$$

Let us present now a series of results concerning the irreducible irreps.

Up to $N \leq 8$, length-4 irreps are present only for reduced supersymmetries. The complete list of length-4 irreps up to $N = 8$ is given by

$N = 1$	NO
$N = 2$	NO
$N = 3$	$(1, 3, 3, 1)$
$N = 4$	NO
$N = 5$	$(1, 5, 7, 3)$, $(3, 7, 5, 1)$, $(1, 6, 7, 2)$, $(2, 7, 6, 1)$, $(2, 6, 6, 2)$, $(1, 7, 7, 1)$
$N = 6$	$(1, 6, 7, 2)$, $(2, 7, 6, 1)$, $(2, 6, 6, 2)$, $(1, 7, 7, 1)$
$N = 7$	$(1, 7, 7, 1)$
$N = 8$	NO

(3.6)

Since there are no length-l irreps with $l \geq 5$ for $N \leq 9$, the above list, together with the already known length-2 and length-3 irreps, provides the complete classification of inequivalent irreps for $N \leq 8$.

Please notice that the length-4 irrep of $N = 3$, $(1, 3, 3, 1)$, is self-dual under the (3.3) *high* $\Leftrightarrow$ *low* spin duality, while two of the inequivalent length-4 $N = 5$ irreps are self-dual, $(2, 6, 6, 2)$ and $(1, 7, 7, 1)$. The remaining ones are pair-wise dually related ($(1, 5, 7, 3) \Leftrightarrow (3, 7, 5, 1)$ and $(1, 6, 7, 2) \Leftrightarrow (2, 7, 6, 1)$).

The list of inequivalent length-4 irreps is the same for both derivations (real and quaternionic) of the $N = 3$ and $N = 5$ extended supersymmetries. It is however convenient to distinguish between real and quaternionic derivations of the $N = 3, 5 \quad mod \quad 8$ extended supersymmetries, due to their different properties. As an example, the $(1, 3, 3, 1)$ length-4 irrep of the $N = 3^{(*)}$ supersymmetry can be oxidized to a length-4 irrep of the $(3, 3)$ pseudosupersymmetry (2.1), while the corresponding quaternionic $(1, 3, 3, 1)$ $N = 3^{(**)}$ irrep cannot be oxidized to a pseudosupersymmetry. Similarly, the quaternionically derived length-4 irreps of the $N = 5^{(**)}$ supersymmetry are oxidized to length-4 irreps of the $(5, 1)$ extended pseudosupersymmetry. For what concerns the real length-4 irreps of the $N = 5^{(*)}$ supersymmetry the picture is the following. Due to the reduction chain from the $N = 8$ oxidized supersimmetry

$$N = 8 \rightarrow N = 7 \rightarrow N = 6 \rightarrow N = 5^{(*)} \tag{3.7}$$

it turns out that the $(1, 7, 7, 1)$ irrep of $N = 5^{(*)}$ can be *oxidized* as an $N = 6$ and $N = 7$ irrep. The $(1, 6, 7, 2) \Leftrightarrow (2, 7, 6, 1)$ and $(2, 6, 6, 2)$ multiplets, thought as $N = 5^{(*)}$ irreps, can be oxidized and promoted to be $N = 6$ irreps.

In the Appendix **B** the complete classification of inequivalent irreps for $N = 9, 10$ is presented. Therefore, we are able to produce here another table, expressing the maximal length L and the total number $\overline{\kappa}$ of inequivalent irreps for the N-extended supersymmetries with $N \leq 10$. We have

su.sies	L	$\overline{\kappa}_2 + \ldots + \overline{\kappa}_L = \overline{\kappa}$
$N = 1$	2	1
$N = 2$	3	$1 + 1 = 2$
$N = 3$	4	$1 + 3 + 1 = 5$
$N = 4$	3	$1 + 3 = 4$
$N = 5$	4	$1 + 7 + 6 = 14$
$N = 6$	4	$1 + 7 + 4 = 12$
$N = 7$	4	$1 + 7 + 1 = 9$
$N = 8$	3	$1 + 7 = 8$
$N = 9$	4	$1 + 15 + 28 = 44$
$N = 10$	5	$1 + 31 + 176 + 140 = 348$

(3.8)

We conclude this section pointing out that the procedure here outlined can be systematically carried on to fully classify inequivalent irreps for arbitrarily large values of N; the limitations are only due to the increasing of the required computational work.

4 Some results

Classification of the $N = 9, 10$ irreps and length-4 $N = 11^{(*)}, 12$ irreps

i) **Classification of the $N = 9$ irreps**

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 16 - h$, $d_2 = 16 - k$.

$N = 9$ presents 4 length-4 irreducible self-dual (under (3.3)) multiplets for

$$h = k = 1, 2, 3, 4 \tag{4.1}$$

and $2 \times (6 + 4 + 2) = 24$ non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{array}{lll} h = 1 & \& & k = 2, \ldots, 7 \\ h = 2 & \& & k = 3, \ldots, 6 \\ h = 3 & \& & k = 4, 5 \end{array} \tag{4.2}$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

The previous results can be summarized as follows. Inequivalent length-4 irreps are in 1-to-1 correspondence with the ordered pair of positive integers h, k satisfying the constraint

$$h + k \quad \leq \quad 8. \tag{4.3}$$

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (3.4), the statistics of the multiplets) is

$$\overline{k}_4 = 28 \qquad (4.4)$$

ii) **Classification of the $N = 10$ irreps**

$N = 10$ admits irreps up to length $l = 5$. We have

iia) *The length-4 classification.*

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 32 - h$, $d_2 = 32 - k$.

$N = 10$ presents 8 length-4 irreducible self-dual (under (3.3)) multiplets for

$$h = k = 1, 2, \ldots, 8 \qquad (4.5)$$

and a set of $2 \times 3(\sum_{j=1}^{7} j) = 168$ non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{array}{lll} h = 1 & \& & k = 2, \ldots, 22 \\ h = 2 & \& & k = 3, \ldots, 20 \\ h = 3 & \& & k = 4, \ldots, 18 \\ h = 4 & \& & k = 5, \ldots, 16 \\ h = 5 & \& & k = 6, \ldots, 14 \\ h = 6 & \& & k = 7, \ldots, 12 \\ h = 7 & \& & k = 8, 9, 10 \end{array} \qquad (4.6)$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

If we set

$$r = min(h, k) \qquad (4.7)$$

the previous results can be summarized as follows. Inequivalent length-4 irreps are in 1-to-1 correspondence with the ordered pair of positive integers h, k satisfying the constraint

$$h + k + r \leq 24. \qquad (4.8)$$

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (3.4), the statistics of the multiplets) is

$$\overline{k}_4 = 176 \qquad (4.9)$$

iib) *The length-5 classification*

A length-5 multiplet $(d_1, d_2, d_3, d_4, d_5)$ is characterized by three independent positive integers, let's say d_1, d_2, d_5, since $d_4 = 32 - d_2$ and $d_3 = 32 - d_1 - d_5$. The full list of length-5 irreps of the $N = 10$ supersymmetry can be listed according to the number d_5 of highest-spin auxiliary fields. The maximal number of auxiliary fields is 7. At any fixed $d_5 = 1, \ldots, 7$ the number of inequivalent irreps is $(8 - d_5)^2$. Therefore, the total number $\overline{k}_5$ of length-5 inequivalent irreps is given by

$$\overline{k}_5 = 1^2 + 2^2 + \ldots + 7^2 = 140 \tag{4.10}$$

The full list of irreps is here produced in terms, at any fixed d_5, of the ordered $\underline{d_1, d_2}$ pairs. We have

$$\begin{array}{lcl}
d_5 = 7 & : & \underline{1,10}. \\
d_5 = 6 & : & \underline{1,10}, \underline{1,11}, \underline{1,12}/\underline{2,12}. \\
d_5 = 5 & : & \underline{1,10}, \ldots, \underline{1,14}/\underline{2,12}, \ldots, \underline{2,14}/\underline{3,14}. \\
d_5 = 4 & : & \underline{1,10}, \ldots, \underline{1,16}/\underline{2,12}, \ldots, \underline{2,16}/\underline{3,14}, \ldots, \underline{3,16}/\underline{4,16}. \\
d_5 = 3 & : & \underline{1,10}, \ldots, \underline{1,18}/\underline{2,12}, \ldots, \underline{2,18}/\underline{3,14}, \ldots, \underline{3,18}/\underline{4,16}, \ldots, \underline{4,18}/\underline{5,18}. \\
d_5 = 2 & : & \underline{1,10}, \ldots, \underline{1,20}/\underline{2,12}, \ldots, \underline{2,20}/\underline{3,14}, \ldots, \underline{3,20}/\underline{4,16}, \ldots, \underline{4,20}/ \\
 & & \underline{5,18}, \ldots, \underline{5,20}/\underline{6,20}. \\
d_5 = 1 & : & \underline{1,10}, \ldots, \underline{1,22}/\underline{2,12}, \ldots, \underline{2,22}/\underline{3,14}, \ldots, \underline{3,22}/\underline{4,16}, \ldots, \underline{4,22}/ \\
 & & \underline{5,18}, \ldots, \underline{5,22}/\underline{6,20}, \ldots, \underline{6,22}/\underline{7,22}.
\end{array} \tag{4.11}$$

One can check that the above set of irreducible multiplets is indeed closed under the (3.3) *high* $\Leftrightarrow$ *low spin* duality transformations.

iii) **Classification of the length-4 $N = 11^{(*)}$ irreps**

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 64 - h$, $d_2 = 64 - k$.

$N = 11^{(*)}$ presents 16 length-4 irreducible self-dual (under (3.3)) multiplets for

$$h = k = 1, 2, \ldots, 16 \tag{4.12}$$

and 776 non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{array}{ll}
h = 1 \ \& \ k = 2, \ldots, 53 & h = 9 \ \& \ k = 10, \ldots, 30 \\
h = 2 \ \& \ k = 3, \ldots, 50 & h = 10 \ \& \ k = 11, \ldots, 28 \\
h = 3 \ \& \ k = 4, \ldots, 47 & h = 11 \ \& \ k = 12, \ldots, 26 \\
h = 4 \ \& \ k = 5, \ldots, 44 & h = 12 \ \& \ k = 13, \ldots, 24 \\
h = 5 \ \& \ k = 6, \ldots, 41 & h = 13 \ \& \ k = 14, \ldots, 22 \\
h = 6 \ \& \ k = 7, \ldots, 38 & h = 14 \ \& \ k = 15, \ldots, 20 \\
h = 7 \ \& \ k = 8, \ldots, 35 & h = 15 \ \& \ k = 16, \ldots, 18 \\
h = 8 \ \& \ k = 9, \ldots, 32 &
\end{array} \tag{4.13}$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

The previous results can be summarized as follows. Let us set, as before (4.7), $r = min(h,k)$ and introduce the $s(r)$ function defined through

$$s(r) \quad = \quad \left\{ \begin{array}{cl} 8-r & \text{for } r = 1, \ldots, 7 \\ 0 & \text{otherwise} \end{array} \right\} \tag{4.14}$$

Inequivalent length-4 irreps are in 1-to-1 correspondence with the ordered pair of positive integers h, k satisfying the constraint

$$h + k + r - s(r) \quad \leq \quad 48. \tag{4.15}$$

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (3.4), the statistics of the multiplets) is

$$\overline{k}_4 \quad = \quad 792. \tag{4.16}$$

iii) **Classification of the length-4 $N = 12$ irreps**

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 64 - h$, $d_2 = 64 - k$.

$N = 12$ presents 12 length-4 irreducible self-dual (under (3.3)) multiplets for

$$h = k = 1, 2, \ldots, 12 \tag{4.17}$$

and 584 non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{array}{ll} h = 1 \ \& \ k = 2, \ldots, 52 & h = 7 \ \& \ k = 8, \ldots, 28 \\ h = 2 \ \& \ k = 3, \ldots, 48 & h = 8 \ \& \ k = 9, \ldots, 24 \\ h = 3 \ \& \ k = 4, \ldots, 44 & h = 9 \ \& \ k = 10, \ldots, 21 \\ h = 4 \ \& \ k = 5, \ldots, 40 & h = 10 \ \& \ k = 11, \ldots, 18 \\ h = 5 \ \& \ k = 6, \ldots, 36 & h = 11 \ \& \ k = 12, \ldots, 15 \\ h = 6 \ \& \ k = 7, \ldots, 32 & \end{array} \tag{4.18}$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (3.4), the statistics of the multiplets) is

$$\overline{k}_4 \quad = \quad 596. \tag{4.19}$$

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