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GRAVITATIONAL COUPLING OF NEUTRINOS TO MATTER VORTICITY II: MICROSCOPIC ASYMMETRIES IN ANGULAR-MOMENTUM MODES(*)

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(*) Presented at the III Escola da Cosmologia e Gravitação - February, 1982

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ABSTRACT

We examine the gravitational coupling of neutrinos to matter vorticity; in the context of the Einstein's theory of gravitation and for technical simplicity we have considered the Gödel model as the gravitational background, whose matter content has a non-null vorticity. Dirac's equation is solved by separation of the neutrino amplitudes into invariant angular-momentum and energy modes. These modes provide two distinct representation bases for the algebra of the total angular momentum of the system (one finite-dimensional and the other infinite-dimensional). The presence of a vorticity field of matter generates, via gravitation, microscopic asymmetries in neutrino physics. The angular momentum space appears to be polarized along the direction determined by the local vorticity field $\vec{\Omega}$, At the microscopic level, currents are asymmetric along the direction determined by the vorticity field: Neutrino (anti neutrino) currents are larger along the direction antiparallel (parallel) to the vorticity field. In the case of production of pairs under CP violation a net number asymmetry may be generated between neutrinos and antineutrinos.

1. INTRODUCTION

Our purpose is to describe the effect of matter vorticity in the physics of neutrinos, the coupling of neutrinos to the vorticity field being realized through gravitation. This problem is not purely academic because the observed anisotropy of the microwave background radiation can possibly be due to a large scale primordial vorticity of the universe [1,2]. This fact and the present observed rotation of galaxies and nebulae could be an indication that the rotation of matter was a remarkable feature of earlier eras, playing an important role in the dynamics of the primordial universe.

The present paper continues a program initiated in ref. (3), in which we have examined microscopic asymmetries in neutrino physics (generated by matter vorticity), the amplitudes for neutrinos/anti-neutrinos being described by quasi-Cartesian invariant excitation modes of the neutrino field. Here we discuss this problem in terms of hyperbolic excitation modes of neutrino field, which correspond to a new coordinatization of the group manifold of the model. The advantage of these hyperbolic modes over the quasi-Cartesian modes is because they define a complete basis of total angular momentum eigenstates for the coupled neutrino field. Also in these modes we were able to separate Dirac equation for mass $\mu \neq 0$, which shall be the subject of another publication [4].

The gravitational field is considered here as described by the Theory of General Relativity (Einstein's theory of Gravitation) and for technical simplicity we take Gödel universe [5] as the gravitational background. It is the simplest known solution of Einstein field equations with rotating incoherent matter. The vorticity field of matter is connected to the property that matter rotates with nonzero angular velocity, in the local inertial frames of its comoving observers. The model admits a global time-like Killing vector, a fact that is crucial for constructing invariant energy modes of the neutrino field. Neutrinos are introduced as test fields over the background gravitational field, and are described by spinorial fields which satisfy Dirac's equation on the curved space-time.

In section II we characterize the Gödel universe as the Lie group $H^3 \times R$ with a left-invariant metric defined on it. This garantees that all vector fields over $H^3 \times R$ exist globally, and that the hyperbolic excitation modes - in which we decompose the neutrino field - are invariantly and globally defined over the manifold. In section III, the local dynamics of neutrinos is discussed, with its basis in Dirac's equation, obtaining as a result the local precession of the spin of the neutrino and the conservation of helicity. A complete basis of neutrino solutions is obtained, which are eigenstates of energy, helicity, total angular momentum and of the projection of the angular momentum along the axis determined locally by the vorticity field. They satisfy boundary conditions related to the test field character of neutrinos. In sections IV, V we construct the Fourier space associated to the above basis and discuss the local microscopic asymmetry of neutrino emission which appears in the presence of a vorticity field; we also discuss the asymetry between neutrino and anti neutrino amplitudes which could appear due to CP violation and could produce a net asymmetry between the number of neutrinos and anti neutrinos.

2. THE STRUCTURE OF GÖDEL UNIVERSE AND THE HYPERBOLIC EXCITATION OF NEUTRINO FIELDS

Gödel's universe is shown here to have the structure of the simply connected Lie group H^3xR,modulo identification of points, with a left-invariant metric introduced on $H^3x R$ and which is a solution of Einstein field equations for a perfect fluid.This provides a global characterization of the complete basis of solutions in which we expand neutrino fields, because the vector fields and forms used to construc the invariant excitation modes are globally defined over the group manifold. The methods used on this section are borrowed from Ozsvath and Schücking [6].

Let E_4 be the four-dimensional Euclidean space with Cartesian coordinates $q^\mu=(q^0,q^1,q^2,q^5)$, and the unit vectors along the Cartesial axes denotes by $\vec{e}_\mu$. With a multiplication law defined by

1st factor \ 2nd factor	$\vec{e}_0$	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\vec{e}_0$	$\vec{e}_0$	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\vec{e}_1$	$\vec{e}_1$	$-\vec{e}_0$	$\vec{e}_3$	$-\vec{e}_2$
$\vec{e}_2$	$\vec{e}_2$	$-\vec{e}_3$	$\vec{e}_0$	$-\vec{e}_1$
$\vec{e}_3$	$\vec{e}_3$	$\vec{e}_2$	$\vec{e}_1$	$\vec{e}_0$

(2.1)

E_4 becomes an algebra, the quaternion algebra, and the vectors

$$\vec{q} = q^{\mu}\ \vec{e}_{\mu} = q^{0}\ \vec{e}_{0} + \sum_{i} q^{i}\ \vec{e}_{i} \qquad (2.2)$$

are called Gödel quaternions. The algebra multiplication of quaternions is non-commutative and satisfies the properties of associativity and distributivity. From (2.1) we have that $\vec{e}_0$ is the identity of the algebra, with the quaternions of the type $\vec{q} = q_0 \vec{e}_0$ isomorphic to the field of real numbers, and we hence identify $q_0\ \vec{e}_0 \sim q_0$.

For a quaternion (2.2) we define its conjugate quaternion by

$$\vec{q}^{*} = q_0\ \vec{e}_0 - \sum_{i=1}^{3} q^{i}\ \vec{e}_{i} \qquad (2.3)$$

We then have $\vec{q}\ \vec{q}^{*} = \vec{q}^{*}\ \vec{q} = (q^0)^2 + (q^1)^2 - (q^2)^2 - (q^3)^2$. Denoting $N(\vec{q}) = (q^0)^2 + (q^1)^2 - (q^2)^2 - (q^3)^2$, every quaternion $\vec{q}$ such that $N(\vec{q}) \neq 0$ has an inverse $\vec{q}^{-1} = (N(q))^{-1}\vec{q}^{*}$, which obviously satisfies $\vec{q}\ \vec{q}^{-1} = \vec{q}^{-1}\ \vec{q} = 1$.

The equation of the 3-hyperboloid H^3 can be expressed

$$\vec{q}\ \vec{q}^{*} = (q^0)^2 + (q^1)^2 - (q^2)^2 - (q^3)^2 = 1 \qquad (2.4)$$

We now identify H^3 with the group of motions of H^3, with H^3 acting on itself by left multiplication. In fact, for any quaternion $\vec{V} \in H^3 (\vec{V}^{*}\vec{V} = 1)$, a left motion of H^3 on itself is expressed by

$$\vec{q}\,' = \vec{V}\ \vec{q} \qquad (2.5)$$

and we have, using that $(\vec{a}\vec{b})^* = \vec{b}^* \vec{a}^*$,

$$\vec{q}' \vec{q}'^* = 1 = \vec{q} \vec{q}^*$$

H^3 is a simply transitive group since for each $\vec{a} \in H^3$, there exists only one left translation $\vec{r}$ from $\vec{a}$ to a given $\vec{a}'$, namely $\vec{r} = \vec{a}' \vec{a}^*$.

H^3 acting on itself by left multiplications (2.3) is a group, and the independent left invariant [7] vector fields and /or forms over H^3 yield a representation of the algebra of H^3. To obtain these fields and forms we proceed as follows. Representing the unit Gödel quaternions $(\vec{e}_0, \vec{e}_i)$ by the matrices

$$\vec{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \vec{e}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\vec{e}_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \qquad \vec{e}_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.6}$$

we replace every quaternium $\vec{q} \in H^3$ by the matrix

$$A = \begin{pmatrix} q^0 - q^3 & q^1 - q^2 \\ -q^1 - q^2 & q^0 + q^3 \end{pmatrix} \tag{2.7}$$

with

$$\det A = 1 \tag{2.8}$$

The quaternion multiplication goes over to matrix multiplication. Introducing on H^3 the coordinates (t, r, ϕ) by the transformations

$$q^0 = \cosh r \ \cos \frac{\sqrt{2}}{2} t$$

$$q^1 = \cosh r \ \sin \frac{\sqrt{2}}{2} t$$

$$q^2 = -\sinh r \ \cos \left(\frac{\sqrt{2}}{2} t - \phi\right) \qquad (2.9)$$

$$q^3 = \sinh r \ \sin \left(\frac{\sqrt{2}}{2} t - \phi\right)$$

where $0 \leq \frac{\sqrt{2}}{2} t, \phi \leq 2\pi$, $0 \leq r < \infty$, the left-ivariant 1-forms ω^μ over H^3 are obtained by calculating [8]

$$\sigma = A^{-1}\, dA = \sigma^\mu \vec{e}_\mu \qquad (2.10)$$

and we have

$$\sigma = \left(\begin{array}{c|c} -\frac{\sqrt{2}}{2}(dt + \sqrt{2}\sinh^2 r\, d\phi) & -[\sin(\sqrt{2}t-\phi)+\cos(\sqrt{2}t-\phi)]\,dr + \\ & +[\cos(\sqrt{2}t-\phi)-\sin(\sqrt{2}t-\phi)]\sinh r\cosh r\, d\phi \\ \hline [\sin(\sqrt{2}t-\phi)-\cos(\sqrt{2}t-\phi)]\,dr & \frac{\sqrt{2}}{2}(dt+\sqrt{2}\sinh^2 r\, d\phi) \\ -[\cos(\sqrt{2}t-\phi)+\sin(\sqrt{2}t-\phi)]\sinh r\cosh r\, d\phi & \end{array}\right)$$

and the last equality (2.10) yields the three independent left-invariant 1-forms

$$\sigma^1 = -\sin(\sqrt{2}t - \phi)\,dr + \cos(\sqrt{2}t - \phi)\sinh r \ \cosh r \ d\phi$$

$$\sigma^2 = \cos(\sqrt{2}t - \phi)\,dr + \sin(\sqrt{2}t - \phi)\sinh r \ \cosh r \ d\phi \qquad (2.11)$$

$$\sigma^3 = \frac{\sqrt{2}}{2}(dt + \sqrt{2} \ \sinh^2 r \ d\phi)$$

Dual to (2.11) we have the corresponding left-invariant vector fields

$$X_3 = \sqrt{2}\ \partial/\partial t$$

$$X_1 = -\sqrt{2}\cos(\sqrt{2}t-\phi)\ \frac{\sinh r}{\cosh r}\ \frac{\partial}{\partial t} - \sin(\sqrt{2}t-\phi)\ \frac{\partial}{\partial r} + \frac{\cos(\sqrt{2}t-\phi)}{\sinh r\ \cosh r}\ \frac{\partial}{\partial \phi}$$

$$X_2 = -\sqrt{2}\sin(\sqrt{2}t-\phi)\ \frac{\sinh r}{\cosh r}\ \frac{\partial}{\partial t} + \cos(\sqrt{2}t-\phi)\ \frac{\partial}{\partial r} + \frac{\sin(\sqrt{2}t-\phi)}{\sinh r\ \cosh r}\ \frac{\partial}{\partial \phi}$$

(2.12)

The left-invariant vector fields and forms (2.11) e (2.12) satisfy the algebra of H^3,

$$[X_3, X_1] = -2X_2$$
$$[X_3, X_2] = 2X_1 \qquad (2.13)$$
$$[X_1, X_2] = 2X_3$$

and

$$d\sigma^1 = 2\quad \sigma^2 \wedge \sigma^3$$
$$d\sigma^2 = -2\quad \sigma^1 \wedge \sigma^3 \qquad (2.14)$$
$$d\sigma^3 = -2\quad \sigma^1 \wedge \sigma^2$$

We have the analogous picture for right motions of the Lie group H^3 into itself, namely (cf.(2.5))

$$\vec{q}\,' = \vec{q}\,\vec{v} \qquad (2.13)$$

The corresponding right-invariant vector fields and forms over H^3 are obtained (similarly to the above method) by calculating [8]

$$\rho = dA\ A^{-1} = \rho^\mu\ \vec{e}_\mu \qquad (2.14)$$

and we have

$$\rho = \begin{pmatrix} \sin\phi\ dr - \sqrt{2}\ \sinh r\ \cosh r\ \cos\phi\ dt & \cos\phi\ dr + \frac{\sqrt{2}}{2}(\sinh^2 r + \cosh^2 r + 2\sinh r\ \cosh r \\ + \cosh r\ \sinh r\ \cos\ \ d & \sin\phi)dt + (\sinh^2 r + \sinh r\ \cosh r\ \sin\phi)d\phi \\ \hline \cos\phi\ dr + \frac{\sqrt{2}}{2}(-\sinh^2 r - \cosh^2 r + & -\sin\phi\ dr + \sqrt{2}\ \sinh r\ \cosh r\ \cos\phi\ dt \\ 2\sinh r\ \cosh r\ \sin\phi)dt + & -\sinh r\ \cosh r\ \cos\phi\ d\phi \\ (\sinh^2 r - \sinh r\ \cosh r\ \sin\phi)d\phi & \end{pmatrix}$$

The last equality (2.14) gives the three independent right-invariant 1-forms

$$\rho^1 = \frac{\sqrt{2}}{2}\ (\sinh^2 r + \cosh^2 r)dt - \sinh^2 r\ d\phi$$
$$\rho^2 = -\cos\phi\ dr - \sqrt{2}\ \sinh r\ \cosh r\ \sin\phi\ dt + \sinh r\ \cosh r\ \sin\phi\ d\phi$$
$$\rho^3 = -\sin\phi\ dr + \sqrt{2}\ \sinh r\ \cosh r\ \cos\phi\ dt - \sinh r\ \cosh r\ \cos\phi\ d\phi$$

(2.15)

with the corresponding dual right-invariant vector fields

$$Y_1 = 2\left(\frac{\sqrt{2}}{2}\ \frac{\partial}{\partial t} + \frac{\partial}{\partial\phi}\right)$$

$$Y_2 = \sqrt{2}\ \sin\phi\ \frac{\sinh r}{\cosh r}\ \frac{\partial}{\partial t} - \cos\phi\ \frac{\partial}{\partial r} + \sin\phi\ \frac{\sinh^2 r + \cosh^2 r}{\sinh r\ \cosh r}\ \frac{\partial}{\partial\phi}$$

$$Y_3 = -\sqrt{2}\cos\phi\ \frac{\sinh r}{\cosh r}\ \frac{\partial}{\partial t} - \sin\phi\ \frac{\partial}{\partial r} - \cos\phi\ \frac{\sinh^2 r + \cosh^2 r}{\sinh r\ \cosh r}\ \frac{\partial}{\partial\phi}$$

(2.16)

which provide the representations of the algebra of H^3,

$$[Y_1, Y_2] = -2Y_3$$

$$[Y_3, Y_1] = -2Y_2 \qquad (2.16)$$
$$[Y_2, Y_3] = 2Y_1$$

and

$$d\rho^1 = -2 \quad \rho^2 \Lambda \rho^3$$
$$d\rho^2 = -2 \quad \rho^1 \Lambda \rho^3 \qquad (2.17)$$
$$d\rho^3 = 2 \quad \rho^1 \Lambda \rho^2$$

We obviously have

$$[X_i, Y_j] = 0 \quad , \quad i,j = 1,2,3 \qquad (2.18)$$

Taking on the one-dimensional manifold R the coordinate z, with vector field $X_4 = \partial/\partial z$ and dual 1-form $\sigma^4 = dz$, the group H^3xR can be characterized by the left-invariant 1-forms ($\sigma^1, \sigma^2, \sigma^3, \sigma^4$) which provide a representation of the algebra of H^3 x R, namely satisfy (2.14) and $d\sigma^3 = 0$, and which are a basis for the 1-forms on H^3 x R. Correspondingly the left-invariant dual vector fields (X_1, X_2, X_3, X_4) satisfy (2.13) and

$$[X_i, X_4] = 0 \quad , \ i=1,2,3 \ , \qquad (2.19)$$

and provide a basis for the vector fields on H^3 x R. The manifold H^3 x R is the covering group of the algebra (2.13) and (2.19).

We obtain the Gödel universe by introducing on H^3 x R the left-invariant metric

$$ds^2 = \frac{4}{\omega^2}\left[(\sqrt{2}\ \sigma^3)^2-(\sigma^1)^2-(\sigma^2)^2-(\sigma^4)^2\right] \qquad (2.20)$$

where ω is a positive constant. The metric (2.20) is a solution of Einstein equations [9] with cosmological constant Λ and incoherent matter whose density ρ must satisfy

$$k\rho = \omega^2 = -2\Lambda \qquad (2.21)$$

he four-velocity of matter is $\partial/\partial t$. The model is stationary because (2.20) admits a time like Killing vector. The velocity field of matter has zero expansion and shear but has a non-null vorticity

$$\Omega = \sqrt{2}\ \omega\ \partial/\partial z \qquad (2.22)$$

We remark that the Gödel universe is locally isometric to (2.20), but concerning connectivity-in-the-large the above model is obtained from the Gödel model by identification of the points $(\frac{\sqrt{2}}{2}t + 2n\pi,\ r,\ \psi,\ z)$, n=integer. In the Gödel universe any geodesic of the congruence determined by $\partial/\partial t$ is time-like and open.

From (2.18) and (2.20) we have that Gödel's geometry admits the five Killing vectors

$$(Y_1, Y_2, Y_3, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) \qquad (2.23)$$

All these vector fields are globally defined on the group manifold [10]. We then select the Killing vector fields

$$\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi} = \frac{1}{2} Y_1 - \frac{\sqrt{2}}{2} \frac{\partial}{\partial t}\right) \qquad (2.24)$$

to construct the global invariant modes ϕ defined by [11]

$$\mathcal{L}_{\partial/\partial z} \phi_{(3)} = i\, k_3\, \phi_{(3)}\ , \qquad \mathcal{L}_{\partial/\partial \phi} \phi_{(2)} = -i\, m\, \phi_{(2)} \qquad (2.25)$$

$$\mathcal{L}_{\partial/\partial t} \phi_{(o)} = -i\, \varepsilon\, \phi_{(o)} \qquad (2.20)$$

with respective solutions $\phi_{(3)} \sim e^{ik_3 z}$, $\phi_{(2)} \sim e^{-im\phi}$ and $\phi_{(o)} \sim e^{-i\varepsilon t}$. $\partial/\partial t$ is a globally defined time-like Killing vector generating time translations and we interpret (2.20) as the definition of invariant energy modes; $\partial/\partial t$ actually defines the Hamiltonian operator which describes the local dynamics of neutrinos. We use the invariant modes $\phi_{(i)}$ to separate neutrino amplitudes in the modes $(\varepsilon.m,k_3)$ and which are globally defined.

3. THE LOCAL DYNAMICS OF NEUTRINOS AND THE SOLUTIONS OF DIRAC'S EQUATION

Neutrinos in interaction with gravitation is describrd by spinorial fields in the curved space-time. For a general review of spinors on a curved space-time see Ref. [12]. Here we use four-component spinors from the point of view of the tetrad formalism. We choose a tetrad $e^{(A)}_{\alpha}(x)$ such that the line element is expressed [13] as

$$ds^2 = \eta_{AB}\, \theta^A\, \theta^B \qquad (3.1)$$

where $\theta^A = e_\alpha^{(A)}\, dx^\alpha$. The definition of the neutrino wave function in a curved space-time involves two group structures. Its spinor character is defined with respect to the local Lorentz structure (3.1), that is, it provides a spinoral representation of the local Lorentz group

$$\theta'^A = L^A{}_B\,(x)\,\theta^B \tag{3.2}$$

with

$$L^A{}_D\,(x)\;\eta_{AB}\;L^B{}_C\,(x) = \eta_{DC} \tag{3.3}$$

These transformations, which can be made independently at each space-time point, have (3.1) invariant. Under (3.2) and (3.3) the spinors ψ transform as

$$\psi'(x) = S(x)\psi(x) \tag{3.4}$$

where the 4x4 matrix S(x) must satisfy [14]

$$(L^{-1})^A{}_B(x)\,\gamma^B = S(x)\,\gamma^A\,S^{-1}(x) \tag{3.5}$$

On the other hand, spinors ψ transform as scalar functions with respect to general coordinate transformations of the space-time, and thus provide a scalar representation of the isometry group of the space-time.

The Lagrangian for neutrinos is

$$i\sqrt{-g}\;(\overline{\psi}\,\gamma^A\,\nabla_A\,\psi - \nabla_A\,\overline{\psi}\,\gamma^A\,\psi) \tag{3.6}$$

In the above formalism $\overline{\psi} = \psi^{+} \gamma^{0}$, where γ^{0} is the constant Dirac matrix. The spinor covariant derivatives are given by

$$\nabla_A \psi = e^{\alpha}_{(A)} \partial_\alpha \psi - \Gamma_A \psi$$
$$\nabla_A \overline{\psi} = e^{\alpha}_{(A)} \partial_\alpha \overline{\psi} + \overline{\psi} \Gamma_A \qquad (3.7)$$

where the Fock-Ivanenko coefficients Γ_A have the form

$$\Gamma_A = -\frac{1}{4} \gamma_{BCA} \gamma^B \gamma^C \qquad (3.8)$$

The Ricci rotation coefficients γ_{ABC} are defined by

$$\gamma_{ABC} = -e^{\alpha}_{(A)||\beta} e_{\alpha(B)} e^{\beta}_{(C)} \qquad (3.9)$$

and Dirac equation for neutrinos coupled to gravitation is expressed as

$$\gamma^A \nabla_A \psi = \gamma^A (e^{\alpha}_{(A)} \partial_\alpha - \Gamma_A) \psi = 0 \qquad (3.10)$$

For (2.20) we take

$$\theta^0 = a(dt + \sqrt{2} \sinh^2 r \, d\phi)$$
$$\theta^1 = a \, dr$$
$$\theta^2 = a \sinh r \cosh r \, d\phi \qquad (3.11)$$
$$\theta^3 = a \, dz$$

where $a = 2/\omega$. With this choice the Fock-Ivanenko coefficients (3.8) have the expression

$$\Gamma_0 = \frac{\sqrt{2}}{2a}\gamma^1\gamma^2$$

$$\Gamma_1 = \frac{\sqrt{2}}{2a}\gamma^0\gamma^2 \tag{3.12}$$

$$\Gamma_2 = -\frac{\sqrt{2}}{2a}\gamma^0\gamma^1 + \frac{1}{2a}\frac{\cosh^2 r + \sinh^2 r}{\cosh r \sinh r}\gamma^2\gamma^1$$

$$\Gamma_3 = 0$$

For a neutrino field in invariant energy excitation modes (2.20), and eigenstates of γ^5,

$$\gamma^5\psi = L\psi, \quad L^2 = 1 \ .$$

We have in the representation used [13]

$$\psi = \begin{pmatrix} \phi(\vec{x}) \\ L\phi(\vec{x}) \end{pmatrix} e^{-i\varepsilon t} \tag{3.13}$$

and using (3.11) and (3.12) Dirac equation (3.10) yields

$$\varepsilon L\, \psi = \vec{\Sigma}\cdot\vec{\pi}\,\psi \tag{3.14}$$

Here $\vec{\Sigma}$ is the spin matrix $\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ and $\vec{\pi}$ is the generalized <u>local</u> momentum operator

$$\vec{\pi} = i\,a\,\vec{e}^{\,\alpha}\,\partial_\alpha - i\,\vec{n}_1 + \gamma^5\vec{\Omega} \tag{3.15}$$

where $\vec{n}_1 = (\frac{1}{2}\ \frac{\cosh^2 r + \sinh^2 r}{\cosh r \ \sinh r}\ ,\ 0,\ 0)$ and

$$\omega\ \vec{\Omega} = (0,\ 0,\ \frac{\sqrt{2}\ \omega}{2}) \qquad (3.16)$$

is the vorticity of matter in the local frame (3.11). We use the notation $\vec{A}\cdot\vec{B} = \sum_{k=1}^{3} A^k B^k$. From (3.14) we have that the operator $L\ \vec{\Sigma}\cdot\vec{\pi}$ is the Hamiltonian of the system (expressed in terms of objects defined in the local frame determined by (3.11), in the sense that the time development of any operator acting on the space of neutrino wave functions is proportional to the commutator of the operator and $L\vec{\Sigma}\cdot\vec{\pi}$. With respect to this Hamiltonian $\vec{\Sigma}\cdot\vec{\pi}$ is conserved, that is, the projection of the spin $\vec{\Sigma}$ on the direction of the local momentum $\vec{\pi}$ is conserved. In this sense $L = \vec{\Sigma}\cdot\vec{\pi}/\varepsilon$ has a precise meaning as the helicity of neutrino, in the local Lorentz frames determined by (3.11). The wave functions (3.13) are <u>energy</u> and <u>helicity</u> eigenstates for neutrinos. Later we shall characterize neutrino amplitudes by $L = -1$, $\varepsilon > 0$ and antineutrino amplitudes by $L = +1$, $\varepsilon > 0$.

The motion of the local momentum $\vec{\pi}$ is calculated $\dot{\vec{\pi}} = i[\vec{\pi}, L\vec{\Sigma}\cdot\vec{\pi}]$ and we have

$$\dot{\vec{\pi}} = \sqrt{2}\ \varepsilon\ L\ \vec{\Sigma}\Lambda\vec{\Omega} \qquad (3.17)$$

Since the projection $\vec{\Sigma}\cdot\vec{\pi}$ is conserved, that is, the helicity L of neutrinos is conserved, we have from (3.17) that, for a given sign of ε, the spin $\vec{\Sigma}$ precesses locally about the direction determined by $\vec{\Omega}$, with angular velocity proportional to $\sqrt{2}\,\varepsilon\,\vec{\Omega}$ and independent of the sign of L, that is, independent of being neutrino or antineutrino.

To separate Dirac equation for neutrino in Gödel's background we consider neutrino wave functions which belongs to the complete set of modes $(\varepsilon, k_3, m, m')$ described by

$$\psi = \begin{pmatrix} \phi(r,\phi) \\ L\phi(r,\phi) \end{pmatrix} e^{-ik_3 z - i\varepsilon t} \qquad (3.18)$$

where

$$\phi(r,\phi) = \begin{pmatrix} \alpha(r)\ e^{-im\phi} \\ \beta(r)\ e^{-im'\phi} \end{pmatrix} \qquad (3.19)$$

which are invariantly and globally defined, as we have discussed in sec. 2. Using (3.18)/(3.19) and the explicit expressions of $e^{\alpha}_{(A)}$ from (3.11), Dirac equation (3.14) reduces to

$$\frac{d\beta}{dr} + \frac{m'}{\sinh r \cosh r}\beta - \sqrt{2}\,\varepsilon\,\frac{\sinh r}{\cosh r}\beta + \frac{1}{2}\left(\frac{\cosh^2 r + \sin^2 r}{\cosh r \sinh r}\right)\beta =$$

$$= -iE_1\alpha \qquad (3.20a)$$

$$\frac{d\alpha}{dr} - \frac{m}{\sinh r \cosh r}\alpha + \sqrt{2}\,\varepsilon\,\frac{\sinh r}{\cosh r}\alpha + \frac{1}{2}\left(\frac{\cosh^2 r + \sinh^2 r}{}\right)\alpha =$$

$$= -iE_2\beta \qquad (3.20b)$$

where we have introduced the notation

$$E_1 = L\left(-\varepsilon + \frac{\sqrt{2}}{2} - Lk_3\right)$$

$$E_2 = L\left(-\varepsilon - \frac{\sqrt{2}}{2} + Lk_3\right) \qquad (3.21)$$

Introducing the variable $x = \cosh r\ 2r$, the second-order equations resulting from (3.20) are

$$(x^2-1)\frac{d^2\alpha}{dx^2}+\left[2x+(m'-1/2)-(m-1/2)\right]\frac{d\alpha}{dx}+\left[\frac{Q}{x+1}+k-\frac{(m'-1/2)(m-1/2)}{x^2-1}\right]\alpha=0 \qquad (3.22a)$$

where $Q=\varepsilon^2+\frac{\sqrt{2}}{2}\varepsilon(m'+m+1)+\frac{m+m'}{2}$, and

$$(x^2-1)\frac{d^2\beta}{dx^2}+\left[2x+(m'+1/2)-(m+1/2)\right]\frac{d\beta}{dx}+\left[\frac{Q'}{x+1}+k-\frac{(m'+1/2)(m+1/2)}{x^2-1}\right]\beta=0 \qquad (3.22b)$$

where $Q'=\varepsilon^2+\frac{\sqrt{2}}{2}\varepsilon(m+m'-1)-\frac{m+m'}{2}$. For both cases

$$4k=(E_1E_2-2\varepsilon^2+1)=-\varepsilon^2-(k_3-\frac{\sqrt{2}}{2}L)^2+1 \qquad (3.23)$$

For consistency, if we take a given solution α of (3.22a) the corresponding solution β is obtained by using (3.20b); similarly for a given solution β of (3.22b), the corresponding solution α is obtained from (3.20a).

We distinguish the set of solutions [15]

$$\psi(m,m',k_3,L,\varepsilon)=\begin{pmatrix}\phi(m,m',k_3,L,\varepsilon)\\ L\phi(m,m',k_3,L,\varepsilon)\end{pmatrix}e^{-ik_3z}\,e^{-i\varepsilon t} \qquad (3.24)$$

where

$$\phi(m,m',k_3,L,\varepsilon)=\begin{pmatrix}(m+m'+1)\left(\frac{x+1}{x-1}\right)^{\frac{m'-m+1}{4}}(x^2-1)^{\frac{m+m'}{4}}(x+1)^{\frac{\sqrt{2}}{2}\varepsilon}F(a,b,c;\frac{1-x}{2})e^{-im\phi}\\ iL(\varepsilon+Lk-\frac{\sqrt{2}}{2})\left(\frac{x+1}{x-1}\right)^{\frac{m'-m-1}{4}}(x^2-1)^{\frac{m+m'}{4}}(x+1)^{\frac{\sqrt{2}}{2}\varepsilon}F(a,b,c+1;\frac{1-x}{2})e^{-im'\phi}\end{pmatrix} \qquad (3.25)$$

$F(a,b,c;\frac{1-x}{2})$ is the hypergeometric function [16] with argument $\frac{1-x}{2}$ and parameters

$$a = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} \pm \frac{n}{2}$$

$$b = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} \mp \frac{n}{2} \qquad (3.26)$$

$$c = \frac{m+m'+1}{2}$$

where

$$n = \sqrt{\varepsilon^2 + (k_3 - \frac{\sqrt{2}}{2}L)^2} \qquad (3.27)$$

On the space of solutions (3.25) we now define the operators

$$J_{(-)} = \begin{pmatrix} J^{(1)}_{(-)} & 0 \\ 0 & J^{(2)}_{(-)} \end{pmatrix} \qquad (3.28)$$

where

$$J^{(1)}_{(-)} = e^{i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} + i\frac{x}{(x^2-1)^{1/2}}\frac{\partial}{\partial\phi} + i\frac{\sqrt{2}}{2}(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t} + (\frac{\sigma-1}{2})\frac{1}{(x^2-1)^{1/2}} + \frac{\sigma}{2}\frac{x}{(x^2-1)^{1/2}}\} \qquad (3.29)$$

$$J^{(2)}_{(-)} = e^{i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} + i\frac{x}{(x^2-1)^{1/2}}\frac{\partial}{\partial\phi} + i\frac{\sqrt{2}}{2}(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t} + (\frac{\sigma+1}{2})\frac{1}{(x^2-1)^{1/2}} - \frac{\sigma}{2}\frac{x}{(x^2-1)^{1/2}}\} \qquad (3.30)$$

and

$$J_{(+)} = \begin{pmatrix} J^{(1)}_{(+)} & 0 \\ 0 & J^{(2)}_{(+)} \end{pmatrix} \tag{3.31}$$

where

$$J^{(1)}_{(+)} = e^{-i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} - i\frac{x}{(x^2-1)^{1/2}}\frac{\partial}{\partial\phi} - i\frac{\sqrt{2}}{2}(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t} + (\frac{\sigma+1}{2})\frac{1}{(x^2-1)^{1/2}}$$

$$-\frac{\sigma}{2}\frac{x}{(x\ \ -1)^{1/2}}\} \tag{3.32}$$

$$J^{(2)}_{(+)} = e^{-i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} - i\frac{x}{(x^2-1)^{1}{}_{2}}\frac{\partial}{\partial\phi} - i\frac{\sqrt{2}}{2}(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t} +$$

$$+ (\frac{\sigma-1}{2})\frac{1}{(x\ \ -1)^{1/2}} + \frac{\sigma}{2}\frac{x}{(x^2-1)^{1/2}}\} \tag{3.33}$$

We have denoted

$$m' = m+\sigma \tag{3.34}$$

We define $J_{(3)}$ by the relation $[J_{(+)}, J_{(-)}] = 2J_{(3)}$ and obtain

$$J_{(3)} = i(\frac{\partial}{\partial\phi} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial t}) + \frac{\sigma}{2}\sigma^3 \tag{3.35}$$

The effect of the operator (3.28), (3.31) and (3.35) on the set of solutions (3.25) is

$$J_{(-)}\phi(m,m') = 2(\frac{m+m'+1}{2})\ \phi(m-1,m'-1) \tag{3.36}$$

$$J_{(+)} \;\phi(m,m') = -\frac{ab}{(m+m'+1)}\,\phi(m+1,m'+1) \tag{3.37}$$

$$J_{(3)} \;\phi(m,m') = (m+\frac{\sigma}{2}+\frac{\sqrt{2}}{2}\,\varepsilon)\;\phi(m,m') \tag{3.38}$$

$$J^2\,\phi(m,m') = \{\frac{1}{2}(J_{(+)}J_{(-)}+J_{(-)}J_{(+)})+J^2_{(3)}\}\phi(m,m') = \frac{n^2-1}{4}\;\phi(m,m')$$

From the definition of $J_{(3)}$ and from the relations

$$\left[J_{(+)},J_{(3)}\right] = -J_{(+)} \tag{3.39}$$

$$\left[J_{(-)},J_{(3)}\right] = J_{(-)} \tag{3.40}$$

we see that the operator $J_{(3)}$, $J_{(+)}$, $J_{(-)}$ generate the algebra of angular-momentum. By using (3.39) we can show that if $\phi(m,m')$ is a solution - which is eigenstate of $J_{(3)}$ with eigenvalue $\frac{m+m'}{2} - \frac{\sqrt{2}}{2}\,\varepsilon$ - then $J_{(+)}\;\phi(m,m')$ is also a solution of the set (3.25) which is eigenstate of $J_{(3)}$ with eigenvalue $\frac{m+m'}{2} - \frac{\sqrt{2}}{2}\,\varepsilon+1$. Analogously from (3.40) $J_{(-)}\;\phi(m,m')$ is also a solution of the set (3.25), which is eigenstate of $J_{(3)}$ with eigenvalue $\frac{m+m'}{2} - \frac{\sqrt{2}}{2}\,\varepsilon - 1$. So given $\phi(m,m')$ it is possible to construct a sequence (in values of (m,m')) extending indefinitely in both directions or terminating if $J_{(+)}\phi$ and/or $J_{(-)}\;\phi$ vanishes for some value of $\frac{m+m'}{2}$.

Unfortunately in the present case it is not possible to use the same procedure as in the case of the spherical harmonics basis for setting bounds on the range of $m+\sigma/2$, because the operators $J_{(1)} = \frac{1}{2}(J_{(+)}+J_{(-)})$ and $J_{(2)} = \frac{1}{2i}(J_{(+)}-J_{(-)})$ lack any hermiticity property, with respect to the normalization scalar product defined in section 4 for the function (3.24)/(3.25). There occurs an exception for $\sigma=0$, in which case $J_{(1)}$ and $J_{(2)}$ are anti-hermitian. $J_{(3)}$ in all cases is obviosly hermitian [17]. To proceed we shall then make use of regularity and boundary conditions on the wave functions, and obtain two distinct sets of solutions, one infinite dimensional and the other finite dimensional representation basis of the algebra of angular-momentum.

On the set of solutions (3.24)/(3.25) we now impose boundary and regularity conditions, namely that neutrino fields (which are test fields and do not contribute to the curvature of the cosmological background) are finite perturbations at any space-time point. We impose [18]

$$\lim_{x\to 1} \psi^{+}\psi = \text{finite} \tag{3.41}$$

$$\lim_{x\to\infty} (x^2-1)^{1/2}\, \psi^{+}\psi = 0 \tag{3.42}$$

By using (3.25), the regularity condition (3.41) implies

$$m \geq \frac{1}{2} \tag{3.43}$$

So starting from a given solution $\psi(m,m')$ and sucessively applying $J_{(-)}$ we necessarily arrive at a solution which do not satisfy (3.43) unless $J_{(-)}\psi = 0$ for some value (m,m'). From (3.36) we have that the sequence finishes on the left for $\frac{m+m'}{2} = -1/2$, and we must then have

$$\frac{m+m'}{2} \geq -1/2 \tag{3.44}$$

that is, $\frac{m+m'}{2}$ takes half-integer values greater or equal to $-1/2$. In the right the sequence could in principle extend to infinite values of $\frac{m+m'}{2}$ by successive application of $J_{(+)}$. Condition (3.42) will nevertheless impose a bound on the values of $\frac{m+m'}{2}$ on the right.

From (3.42) two distinct possibilities arise [19]. Either

(I) $a =$ negative integer or zero (3.45)

or

(II) $c - b =$ negative integer or zero (3.46)

with

$$a = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} \tag{3.47}$$

$$b = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} - \frac{n}{2} \tag{3.48}$$

for both cases (I) and (II), and we obtain the two distinct set of solutions:

Type (I) solutions

We denote any negative integer or zero by $\frac{m+m'}{2} - j$, with $j =$ half-integer $\geq \frac{m+m'}{2}$, that is,

$$-1/2 \leq \frac{m+m'}{2} \leq j \tag{3.49}$$

From (3.45) and (3.47) we then have

$$j + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} = 0 \qquad (3.50)$$

which implies

$$\varepsilon = -\sqrt{2}\,(2j+1) - \sqrt{(2j+1)^2 + (k_3 - L\sqrt{2}/2)^2} \qquad (3.51)$$

The corresponding positive-energy solutions of type (I) are obtained from the symmetry $\psi \to i\gamma^2\psi^*$ of Dirac equation (3.10), where * denotes complex-conjugation. We remark, for example, that the eigenvalues of $J_{(3)}$ and J^2 for this case are given by $m + \frac{\sigma}{2} + \frac{\sqrt{2}}{2}\varepsilon$ and $(j - \frac{\sqrt{2}}{2}\varepsilon)(j - \frac{\sqrt{2}}{2}\varepsilon + 1)$, respectively.

Type (II) solution

We here denote any negative integer or zero by $-(j + 1/2)$, where

$$j = \text{half-integer} \geq -\frac{1}{2} \qquad (3.52)$$

From (3.46) and (3.48) we have

$$j - \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} = 0 \qquad (3.53)$$

which implies

$$\varepsilon = +\sqrt{2}\,(2j + 1) + \sqrt{(2j+1)^2 + (k_3 - L\sqrt{2}/2)^2} \qquad (3.54)$$

The corresponding negative-energy states of type (II) are obtained from the symmetry $\psi \to i\gamma^2\psi^*$ of Dirac equation (3.10).

We remark that for type (I) solutions the values of $\frac{m+m'}{2}$ are bounded for a given j (cf.(3.49)), and for type (II) solutions the range $\frac{m+m'}{2} \geq -\frac{1}{2}$ is completely independent of the value of j. In other words, for a given $j = $ half-integer $\geq -1/2$, type (I) solutions provide a finite dimensional ($\dim = j+3/2$) representation basis for the algebra of angular momentum, while type (II) solutions provide an infinite dimensional representation basis for the algebra of angular momentum. In the above discussion we have discarded normalizable solutions which could not constitute a basis of representation for the algebra of angular-momentum, although we should mention that some of these solutions have interesting features as zero energy and eigenvalue of $J_{(3)}$ equal to an integer.

The zero-energy modes in both types (I) and (II) solutions occur for $j = -1/2$ and $k_3 = L\sqrt{2}/2$. For these modes $J_{(3)} = \frac{\sigma-1}{2}$ (actually its eigenvalues), and the statistics (boson or fermion character) depends on the value of σ. Also the modes $j = -1/2$, $k_3 = 0$ with corresponding $|\varepsilon| = \sqrt{2}/2$ have the eigenvalues of the total angular-momentum projection $J_{(3)} = \pm 1 + \frac{\sigma}{2}$ as well $\Sigma^3 = \pm 1 + \frac{\sigma}{2}$, respectively for positive/negative energy solutions - in other words, due to the gravitational coupling to matter vorticity, these massless fermions for $\sigma=0$ are converted to bosons polarized along the direction $\vec{\Omega}$, with eigenvalues of projections $J_{(3)} = \Sigma^3 = \pm 1$, respectively for positive/negative energy.

4. COMPLETE SET OF SOLUTIONS, NORMALIZATION AND GENERALIZED FOURIER SPACE OF NETRINO AMPLITUDES

We restrict ourselves to the complete basis of type (I) solutions for two reasons. It is physically more satisfactory because it corresponds to a finite dimensional representation of the angular-momentum algebra of the system, that is, for a fixed energy ε and for a given value of the total angular momentum $(j-\frac{\sqrt{2}}{2}\varepsilon)(j-\frac{\sqrt{2}}{2}\varepsilon+1)$, where j = half-integer $\geq -1/2$ we have $j+3/2$ eigenstates of the angular-momentum projection on the local axis $\vec{\Omega}$; also for simplicity, because all following results are analogous to the ones obtained if we have also considered type (II) basis. Without loss of generality, in what follows we consider only the case $\sigma = 0$. We have the complete basis:

Positive-energy modes

$$\psi_{(+)}(j,m,k_3,L,\varepsilon) = \begin{pmatrix} \phi_{(+)}(j,m,k_3,L,\varepsilon) \\ \\ L\phi_{(+)}(j,m,k_3,L,\varepsilon) \end{pmatrix} e^{im\phi}\, e^{-ik_3 z}\, e^{-i|\varepsilon|t} \tag{4.1}$$

where

$$\phi_{(+)}(j,m,k_3,L,\varepsilon) = \begin{pmatrix} -i(-|\varepsilon|+Lk_3-\frac{\sqrt{2}}{2})(x^2-1)^{\frac{m+1/2}{2}}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|-\frac{1}{2}}F(a,b,c+1;\frac{1-x}{2}) \\ \\ L(2m+1)(x^2-1)^{\frac{m-1/2}{2}}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|+\frac{1}{2}}F(a,b,c;\frac{1-x}{2}) \end{pmatrix} \tag{4.2}$$

Negative-energy modes

$$\psi_{(-)}(j,m,k_3,L,\varepsilon) = \begin{pmatrix} \phi_{(-)}(j,m,k_3,L,\varepsilon) \\ \\ L\phi_{(-)}(j,m,k_3,L,\varepsilon) \end{pmatrix} e^{-im\phi}\, e^{-ik_3 z}\, e^{i|\varepsilon|t} \tag{4.3}$$

where

$$\phi_{(-)}(j,m,k_3,L,\varepsilon) = \begin{pmatrix} (2m+1)(x\ -1)^{\frac{m-1/2}{2}}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|+\frac{1}{2}} F(a,b,c;\ \frac{1-x}{2}) \\ iL(-|\varepsilon|+Lk_3-\frac{\sqrt{2}}{2})(x^2-1)^{\frac{m+1/2}{2}}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|-\frac{1}{2}} F(a,b,c+1;\ \frac{1-x}{2}) \end{pmatrix} \tag{4.4}$$

In the above, j = half-integer $\geq -\frac{1}{2}$,

$$-1/2 \leq m \leq j \tag{4.5}$$

and

$$\begin{aligned} a &= m-j \\ b &= m+j-\sqrt{2}|\varepsilon|+1 \\ c &= m+1/2 \end{aligned} \tag{4.6}$$

For all cases

$$|\varepsilon| = \sqrt{2}(2j+1) + \sqrt{(2j+1)^2 + (k_3-\frac{\sqrt{2}}{2}L)^2} \tag{4.7}$$

The lower bound $m = 1/2$ in (4.5) is not in contradiction with the regularity condition (3.43) because

$$\lim_{m \to -1/2} (2m+1)(x-1)^{\frac{m-1/2}{2}} (x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon| + \frac{1}{2}} F(a,b,m+1/2; \tfrac{1-x}{2}) = \text{finite}$$

for all x.

The positive-energy (4.1) and negative-eenrgy (4.3) set of solutions are related by

$$\psi_{(+)}(k_3,L) = -iL\gamma^5\gamma^2 \; \psi^*_{(-)}(-k_3,-L) \tag{4.8}$$

We now discuss the normalization of the complete set of modes (j,m,L,k_3,ε) defined in (4.1) - (4.7). Let us consider the local classical Dirac current

$$j^{(A)} = \overline{\psi}\; \gamma^A \psi = C_\alpha^{(A)}(x)\; \overline{\psi}\; \gamma^\alpha(x)\, \psi \tag{4.9}$$

The component $j^{(o)} = \psi^+\psi$ of (4.9) is the <u>local number density</u> of neutrinos. As expected $j^{(o)}$ transforms as the zeroth component of a Lorentz vector with respect to local Lorentz transformations (3.3) and it is a scalar function with respect to coordinate transformations (and/or point transformations) of the space-time. The local number $j^{(o)}\sqrt{-g}\; d^4x$ is thus a scalar and integrated over a given volume of the manifold

$$\int \sqrt{-g}\; j^{(o)} d^4x \tag{4.10}$$

yields a positive definite quantity which is coordinate invariant.

Neutrino amplitudes are thus normalized according to the integral (4.10), taken over the whole Gödel manifold for reasons extensively discussed in Ref. [3], and for the complete set (4.1) - (4.4) we have the δ normalization

$$<\psi_{(r)}(j',m',k_3',\varepsilon')|\psi_{(s)}(j,m,k_3,\varepsilon)> = (2\pi)^3 N^2 \delta_{rs}\,\delta_{jj'}\,\delta_{mm'}\,\delta(k_3-k_3')\cdot$$

$$\cdot\,\delta(|\varepsilon|-|\varepsilon'|) \qquad (4.11)$$

where r,s = +, - corresponding respectively to positive (4.1) and negative (4.3) energy solutions, and [10]

$$N^2 = \frac{4<\alpha>}{\omega^4}\,\frac{|\varepsilon|}{(|\varepsilon|+Lk_3-\sqrt{2}/2)} \qquad (4.12)$$

where

$$<\alpha> = \frac{2^{2m-\sqrt{2}|\varepsilon|+3}\,(j+1/2)((m+1/2)!)^2(j-m)!(\sqrt{2}|\varepsilon|-j-m-1)!}{(\sqrt{2}|\varepsilon|-2j-1)(j+1/2)!\;(\sqrt{2}|\varepsilon|-j-3/2)!} \qquad (4.13)$$

The factor $(2\pi)^2 N^2$ in the right-hand side of (4.11) can be interpreted as inversely proportional to the local number density of states (j,m,k_3,L), that is, the local number density in the Fourier space associated to the complete basis of solutions (4.1) - (4.4). It is clear from (4.12) that the local number density of states (j,m,k_3,L) depends strongly on the sign of Lk_3.

Since we have used the local number density $j^{(o)}$ to normalize the wave functions, the normalization depends on the orientation of the field of tetrad frames $e^{\alpha}_{(\ )}(x)$, with an arbitrariness due to local Lorentz transformations (3.2)/(3.3). The present orientation of the tetrad frame in which (4.11) and (4.12) were calculated in nevertheless a <u>preferred orientation</u> in the sense that (3.11) is based on the matter flow of the model - actually the zeroth vector of the tetrad frame is defined by the four-velocity field of matter $e^{\alpha}_{(o)} = \delta^{\alpha}_{o}$, and (4.11) and (4.12) are invariant under Lorentz transformations which preserve this condition, that is, $L^{o}{}_{A} = \delta_{A}{}^{o}$. The matter flow of the model singles out (4.11) and (4.12).

The Fourrier space associated to the complete basis (4.1) - (4.4) is constructed as follows. The kernel of the transformation is defined by [21]

$$K(j,m,k_3,\varepsilon;x) = K_{(+)}(j,m,k_3,|\varepsilon|;x) + K_{(-)}(j,m,k_3,|\varepsilon|;x) \qquad (4.14)$$

where

$$K_{(+)} = \mathrm{diag}\left(\frac{\beta}{<\beta>^{1/2}}, \frac{\alpha}{<\alpha>^{1/2}}, \frac{\beta}{<\beta>^{1/2}}, \frac{\alpha}{<\alpha>^{1/2}}\right) \exp(-im\phi + ik_3 z + i|\varepsilon|t) \qquad (4.15)$$

and

$$K_{(-)} = \mathrm{diag}\left(\frac{\alpha}{<\alpha>^{1/2}}, \frac{\beta}{<\beta>^{1/2}}, \frac{\alpha}{<\alpha>^{1/2}}, \frac{\beta}{<\beta>^{1/2}}\right) \exp(im\phi + ik_3 z - i|\varepsilon|t) \qquad (4.16)$$

where

$$\alpha = (2m+1)(x^2-1)^{\frac{m-1/2}{2}} (x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon| + \frac{1}{2}} F(a,b,c;\frac{1-x}{2})$$

$$\beta = (x^2-1)^{\frac{m+1/2}{2}} (x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon| - \frac{1}{2}} F(a,b,c+1; \tfrac{1-x}{2}) \tag{4.17}$$

and

$$\langle\alpha\rangle = \int_1^\infty \alpha^2(x)\,dx, \quad \langle\beta\rangle = \int_1^\infty \beta^2(x)\,dx = \frac{\langle\alpha\rangle}{4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)} \tag{4.18}$$

The parameters a, b and c are given in (4.6) and $\langle\alpha\rangle$ in (4.13). The Fourier transform of a neutrino field ψ has the expression

$$F[\psi] = (\psi_F\,'j',m',k_3',\varepsilon') = \int \sqrt{-g}\, d^4x\, K(j',m',k_3',\varepsilon';x)\, \psi(x) \tag{4.19}$$

where the integration is taken over the whole manifold.

For (4.14) we have the unitarity property

$$\int \sqrt{-g}\, d^4x\, K(j',m',k_3',\varepsilon';x)\, K^{\dagger}(j,m,k_3,\varepsilon;x) = 2(2\pi)^3\, \mathbf{1}\, \cdot$$

$$\cdot\, \delta_{mm'}\, \delta_{jj'}\, \delta(k_3-k_3')\, \delta(|\varepsilon|-|\varepsilon'|) \tag{4.20}$$

We remark that the first term $K_{(+)}$ of the kernel (4.14) can be considered as projector - with respect to the operation (4.19) - into positive-energy states since its action on negative-energy states (4.3)/(4.4) results zero; analogously the second term $K_{(-)}$ in (4.14) is a projector into negative-energy states since its action on positive-energy states (4.1)/(4.2) gives zero. Because the inverse of a projector is not a one-to-one map, the inverse Fourier transform is then defined separately for positive - and negative-energy amplitudes, with kernels $K_{(+)}$ and $K_{(-)}$ respectively, that is

$$F^{-1}\left[\psi_F(j,m,k_3,|\varepsilon|,+)\right] = \sum_{j=-1/2}^{\infty} \sum_{m=-1/2}^{j} \int_{\varepsilon>o} \frac{dk_3 d\varepsilon}{(2\pi)^3} K^+_{(\ )}(j,m,k_3,\varepsilon;x)\psi_F \tag{4.21}$$

for positive- and negative-energy states respectively. We have the unitary properties

$$\sum_{j=-1/2}^{\infty} \sum_{m=-1/2}^{j} \int_{\varepsilon>o} \frac{dk_3 d\varepsilon}{(2\pi)^3} K^+_{(+)}(j,m,k_3,\varepsilon;x)K_{(+)}(j,m,k_3,\varepsilon;x') = \frac{\delta^4(x-x')}{\sqrt{-g}} \mathbb{1} \tag{4.22}$$

$$\sum_{j=-1/2}^{\infty} \sum_{m=-1/2}^{j} \int_{\varepsilon>o} \frac{dk_3 d\varepsilon}{(2\pi)^3} K^+_{(-)}(j,m,k_3,\varepsilon;x)K_{(-)}(j,m,k_3,\varepsilon;x') = \frac{\delta^4(x-x')}{\sqrt{-g}} \mathbb{1} \tag{4.23}$$

which actually imply $FF^{-1} = F^{-1}F = 1$, as expected.

The Fourier transform of a positive-energy amplitude (4.1) /(4.2) is the four-spinor

$$\psi_F(j,m,k_3,|\varepsilon|,+) = (2\pi)^3 \begin{pmatrix} -i(Lk_3-|\varepsilon|-\frac{\sqrt{2}}{2})<\beta>^{1/2} \\ L <\alpha>^{1/2} \\ -iL(Lk_3-|\varepsilon|-\frac{\sqrt{2}}{2})<\beta>^{1/2} \\ <\alpha>^{1/2} \end{pmatrix} \cdot$$

$$\cdot\delta_{mm'}\ \delta_{jj'}\ \delta(k_3-k_3')\ \delta(|\varepsilon|-|\varepsilon'|) \tag{4.24}$$

The local Lorentz group (3.2), (3,3) - with respect to which the spinor structure is defined - induces on the Fourier space the group of transformations

$$\tilde{\psi}_F(j,m,k_3,\varepsilon,\pm) = \sum_{j'} \sum_{m'} \int \frac{dk_3'\, d\varepsilon'}{(2\pi)^3}\, S(j,m,k_3,\varepsilon;j',m',k_3',\varepsilon')\psi_F(j',m',k_3',\varepsilon',\pm) \tag{4.25}$$

where

$$S(jmk_3\varepsilon;j'm'k_3'\varepsilon') = \int \sqrt{-g}\, d^4x\, K_{(\pm)}(jmk_3\varepsilon;x)\; S(x)\; K^+_{(\pm)}(j'm'k_3'\varepsilon';x) \tag{4.26}$$

The Fourier space described above is actually a <u>momentum space</u> for neutrinos. In fact, expressing a positive-energy state (4.1), (4.2) as

$$\psi_{(+)}(L,k_3) = \sum_{j=-1/2}^{\infty} \sum_{m=-1/2}^{j} \int \frac{dk_3 d\varepsilon}{(2\pi)^3}\, K^+_{(+)}(j,m,k_3,\varepsilon;x)\;\psi_F(j,m,k_3,\varepsilon,+)$$

and using Dirac's equation $\gamma^A \nabla_A \psi = 0$ we obtain the transformed Dirac's equation

$$-i\, \Pi_A\, \gamma^A\, \psi_F = 0 \tag{4.27}$$

where Π_A is given by

$$\Pi_A = \left(|\varepsilon|,\; 0,\; -2\left[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)\right]^{1/2},\; -k_3 + \frac{\sqrt{2}}{2}L\right) \tag{4.28}$$

We have

$$\Pi_A\, \Pi^A = 0 \tag{4.29}$$

as expected for a massless particle, where $\Pi^A = \eta^{AB}\, \Pi_B$. The form of the component Π_3 (along the local direction of the vorticity field) shows that the "leptonic charge" L behaves like the coupling constant in the coupling of the spinor structure of neutrino to the vorticity field. For a negative-energy solution (4.3)/(4.4)

$$\psi_{(-)}(L,k_3) = \sum_{j=-1/2}^{\infty} \sum_{m=-1/2}^{j} \int \frac{dk_3 d\varepsilon}{(2\Pi)^3} K^{+}_{(-)}(j,m,k_3,\varepsilon;x)\ \psi_F(j,m,k_3,\varepsilon,-) \tag{4.30}$$

we analogously obtain (4.27) where Π_A is given now by

$$\Pi_A = (-|\varepsilon|,\ 0,\ 2\left[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)\right]^{1/2},\ -k_3 + \frac{\sqrt{2}}{2} L) \tag{4.31}$$

with $\Pi_A\ \Pi^A = 0$. The same results (4.27)-(4.31) are obtained if we have instead used the infinite dimensional representation basis which was discussed in Section 3, the only difference being that the parameters j and m are completely independent, with range $-1/2 \leq j < \infty$, $-1/2 \leq m < \infty$. We remark that Π_3 has the same sign in (4.28) and (4.31) due to our definition of (4.16) - in fact, if in (4.16) we change $k_3 \to -k_3$ and $L \to -L$ we have in (4.31) that $\Pi_3 \to -\Pi_3$ without altering other components. It follows that the corresponding Π_A for negative-energy solutions has the opposite sign of Π_A for the positive-energy solutions, a behaviour characteristic of "plane-wave type" positive- and negative-energy amplitudes related through the property (4.8). This fact is important when we consider symmetry transformations between particle and antiparticle amplitudes.

We now calculate the component (along the local vorticity field $\vec{\Omega}$) $j_F^{(3)}$ of the local four-current

$$j_F^{(A)} = \overline{\psi}_F \, \gamma^A \, \psi_F = (\psi_F^+ \, \psi_F, \, \overline{\psi}_F \, \vec{\gamma} \, \psi_F) \tag{4.32}$$

and we obtain, using (4,24),

$$j_F^{(3)} = 4(2\pi)^3 \frac{\Pi_3}{|\varepsilon| - L\Pi_3} <\alpha> \cdot \delta_{mm''} \, \delta_{jj''} \, \delta(k_3 - k_3'') \, \delta(|\varepsilon| - |\varepsilon''|) \tag{4.33}$$

We now make an important remark about the normalization of solutions $\psi(x)$ and ψ_F. As a result of the normalization integral (4.10), we see that is exactly the zeroth component $\Pi_0 = |\varepsilon|$ which appears as a factor in (4.12) and characterizes its behavior under the local Lorentz transformations. We shall therefore normalize all solutions with the remaining factor in (4.12),

$$R^2 = \frac{4<\alpha>}{\omega^4} \, \frac{1}{(|\varepsilon| - L\Pi_3)} \tag{4.34}$$

This corresponds to have (dropping δ-factors)

$$\psi_F^+ \, \psi_F = |\varepsilon| \quad , \quad <\psi|\psi> = |\varepsilon| \tag{4.35}$$

By using (4.35) or (4.34), the expression (4.33) of $j_F^{(3)}$ results

$$j_F^{(3)} = 4(2\pi)^3 \, \Pi_3 \, \delta_{mm''} \, \delta_{jj''} \, \delta(k_3 - k_3'') \, \delta(|\varepsilon| - |\varepsilon''|) \tag{4.36}$$

We use the expression (4.36) in the next section to discuss the microscopic assimetry of neutrino emission in the presence of a local vorticity field.

5. SYMMETRY TRANSFORMATIONS FOR NEUTRINO AMPLITUDES AND MICROSCOPY ASYMMETRY OF NEUTRINO EMISSION

In order to examine question connected to neutrino-antineutrino symmetry of some processes, we shall try to define amplitudes for particle and anti-particle states. To this end we obtain transformations which can be interpreted as leading from particle to anti-particle amplitudes, and which are actually symmetry transformations for the present neutrinos in the sense that they preserve the Hilbert space of neutrinos solutions generated by the basis (4.1) - (4.4). These transformations can be reasonably understood as corresponding locally to known symmetries of particle physics.

The use of tetrads is practically unavoidable to describe the interaction of fermions with gravitation [22,12] and, in this context, the theory has two groups involved: the local Lorentz rotation (3.2) of the tetrads and the isometry group of the manifold. Spinors are defined with respect to the local Lorentz structure, in the sense that they provide a basis space for a spinoral representation of the local Lorentz group. On the other hand, these spinors provide a basis space for a scalar representation of the isometry group of the manifold. For the present case of neutrinos, we are restricted to a subspace of spinor functions which are eigenstates of γ^5, namely the Hilbert space of solutions generated by (4.1) - (4.4).

In the definition of neutrino and anti-neutrino amplitudes, both groups are involved; for instance the energy eigenmodes are related to the Killing vector $\partial/\partial t$ of the isometry group, while the charge conjugation operation must take into account the local spinor structure. Our procedure here will be obtain consistent neutrino-antineutrino symmetry transformations of the Hilbert space of neutrino amplitudes generated by (4.1) - (4.4) and which then necessarily takes into account the two group structures present.

Starting from a negative-energy solution (4.3)/(4.4)

$$\psi_{(-)}(k_3,L) = \begin{pmatrix} \phi_{(-)}(k_3,L) \\ L\,\phi_{(-)}(k_3,L) \end{pmatrix} e^{-im\phi}\, e^{-ik_3 z}\, e^{i|\varepsilon|t}$$

we define the transformation

$$\psi_{(-)}(k_3,L) \rightarrow S^{-1}\, \bar{\psi}^{T}_{(-)}(k_3,L) \tag{5.1}$$

where S is a matrix of the algebra of Dirac matrices, which satisfies

$$S\gamma^A\, S^{-1} = -\gamma^{A^T} \tag{5.2}$$

In the present representation [14], (5.2) is satisfied by

$$S \sim \gamma^2\, \gamma^0 \tag{5.3}$$

where $\sim$ denotes equality up to a constant phase factor. An ex plicit calculation of (5.1) gives

$$\gamma^2 \gamma^0 \overline{\psi}^T_{(-)}(k_3,L) = \psi_{(+)}(-k_3,-L) \tag{5.4}$$

Transformation (5.1) has the following properties: (i) it is a symmetry transformation of the Hilbert space of neutrino amplitudes, since it takes a negative-energy solution (4.3) to a positive energy-solution (4.1), and vice-versa; (ii) the S matrix (5.2) and (5.3) has the character of a charge-conjugation operator on the amplitudes (4.1), (4.3) (in case of charged particles it relates solutions with distinct signs of the charge); (iii) neutrinos amplitudes related by (5.1) have opposite helicity L and momentum k_3 - the local momentum $\vec{\Pi}$ (cf. (4.28), (4.31)) change sign under (5.1). We note that (5.4) is precisely the symmetry (4.8) between positive- and negative-energy solutions. From the above properties we interpret (5.1) as a charge-conjugation-parity (CP) transformation for neutrino amplitudes, and hence we have the independent positive-energy wave functions interpreted as

$$\begin{aligned} \psi_{(+)}(k_3,L) &= \text{neutrino amplitude} \\ \psi_{(+)}(-k_3,-L) &= \text{corresponding antineutrino amplitude} \end{aligned} \tag{5.5}$$

The positive-energy amplitudes (5.5) are said CP related in the sense that the corresponding negative-energy amplitude $\psi_{(-)}(k_3,L)$ $[\psi_{(-)}(-k_3,-L)]$ of one is transformed into the other $\psi_{(+)}(-k_3,-L)$ $[\psi_{(+)}(k_3,L)]$ under (5.1). From the local CP invariance of neutrino physics (only negative helicity neutrinos exist)we take L=-1 for neutrinos which implies $L=+1$ for anti-neutrinos (cf.(5.5)). Neutrino and antineutrino amplitudes (5.5) have their respective momentum $\vec{\Pi}$ with opposite sign.

We can now discuss the microscopic asymmetry of neutrino emission along the direction determinated by the vorticity vector field.

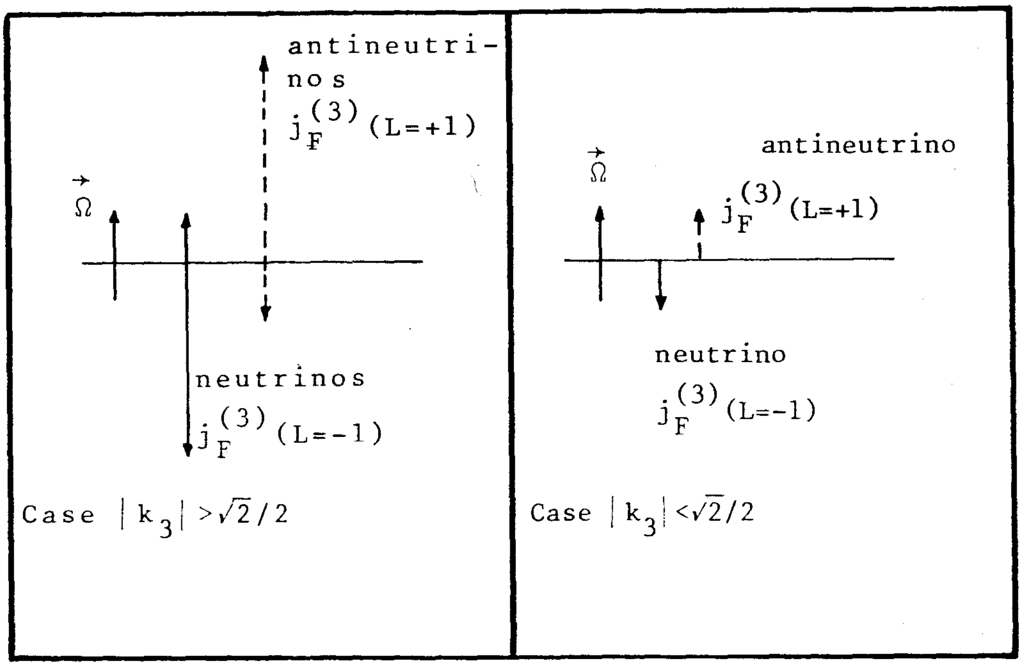
From the expression (4.36) for the component of the local Fourier current $\vec{J}_F$ along $\vec{\Omega}$, we take the relevant factor

$$j_F^{(3)} \simeq \Pi_3 = -k_3 + L\,\frac{\sqrt{2}}{2} \qquad (5.6)$$

and we distinguish the two cases

(1) $|k_3|>\sqrt{2}/2$: for neutrinos (L=-1) we have that $\vec{j}_F$ is larger along the direction antiparallel to $\vec{\Omega}$ than along the parallel direction; for antineutrinos (L=+1), $\vec{j}_F$ is larger along the direction parallel to $\vec{\Omega}$.

(2) $|k_3|<\sqrt{2}/2$: for neutrinos (L=-1), the component of $\vec{j}_F$ along $\vec{\Omega}$ is always negative ($\vec{j}_F$ has only antiparallel component along $\vec{\Omega}$); for antineutrinos (L=+1), the component $\vec{j}_F$ along $\vec{\Omega}$ is always positive. The following diagram is illustrative[24]:



As for the local $j^{(A)}(x) = \overline{\psi}(x)\, \gamma^A\, \psi(x)$, we calculate the component $j^{(3)}_{(x)}$ (that is, along $\vec{\Omega}$) at the origin $x = 1$. In the normalization (4.35), we have

$$j^{(3)}(x) = \frac{2L}{R^2} \left[(|\varepsilon| + L\Pi_3)^2 \beta^2 - \alpha^2 \right] \tag{5.7}$$

where R^2 is given by (4.34), and α and β have their expression in (4.17). We note that $j^{(A)}(x)$ depends on the coordinate $x = \cosh 2r$ only. At the origin $x = 1$, we can see that for a given $j \geq 1/2$ only the modes $m = \pm 1/2$ contribute to (5.7), namely for a given $j \geq 1/2$

$$\left(j^{(3)}(x=1) \right)_{m=-1/2} = \frac{L(|\varepsilon| + L\Pi_3)^2}{R^2}\; 2^{-\sqrt{2}|\varepsilon|} \tag{5.8}$$

(cf. Ref.[23]) and

$$\left(j^{(3)}(x=1) \right)_{m=+1/2} = \frac{16L}{R^2}\; 2^{-\sqrt{2}|\varepsilon|} \tag{5.9}$$

The total local current along $\vec{\Omega}$ (at the origin x=1) for a given mode $j \geq 1/2$,

$$j^{(3)}(x=1) = \sum_{m=-1/2,1/2} \left(j^{(3)}(x=1) \right)_m \quad ,$$

is then calculated to be

$$j^{(3)}(x=1) = \frac{(\sqrt{2}|\varepsilon|-2j-1)}{2} \quad \Pi_3 = \frac{(\sqrt{2}|\varepsilon|-2j-1)}{2}(-k_3+L\sqrt{2}/2) \tag{5.10}$$

The same analysis and diagram for the asymmetry of the Fourrier current (5.6) applies to (5.10).

A special case is the mode $j = -1/2$ for which

$$j^{(3)}(x=1)\Big|_{j=m=-1/2} = \frac{\sqrt{2}}{4}|\varepsilon|(\Pi_3 + L|\varepsilon|)$$

Finally we draw some interesting conclusions concerning the number density of neutrino and antineutrino states, CP violation and lepton asymetry, for the present problem. To this end we note that the number density of states - which we denote by $n(Lk_3)$ and is proportional to

$$N(Lk_3) \sim \frac{|\varepsilon|-\sqrt{2}(2j+1)}{k_3-\frac{\sqrt{2}}{2}L} R^{-2}$$

where R^2 is given by (4.34) - depends strongly on the sign of Lk_3 (through $|\varepsilon|$ and $L\Pi_3$), <u>for</u> $|k_3|$ <u>of the order of</u> $\sqrt{2}/2$. Consequently for a given value of (j,m,k_3), such that $|k_3|$ of the order of $\sqrt{2}/2$, we could have a number density of states different for L=-1 and L=+1. This fact can be significative in the presence of CP-violating interactions, as we shall discuss now for the case of creation of neutrino-antineutrino pairs in the presence of a CP-violating perturbation, when a neutrino-antineutrino number asymmetry may possibly occur.

Having in mind the CP-symmetry (5.5) [cf also remarks below (5.5)] and that $\Pi_3 = -k_3 + L\sqrt{2}/2$ we can draw the diagram shown in Fig.2 for the amplitudes (5.5) according to the sign of Lk_3.

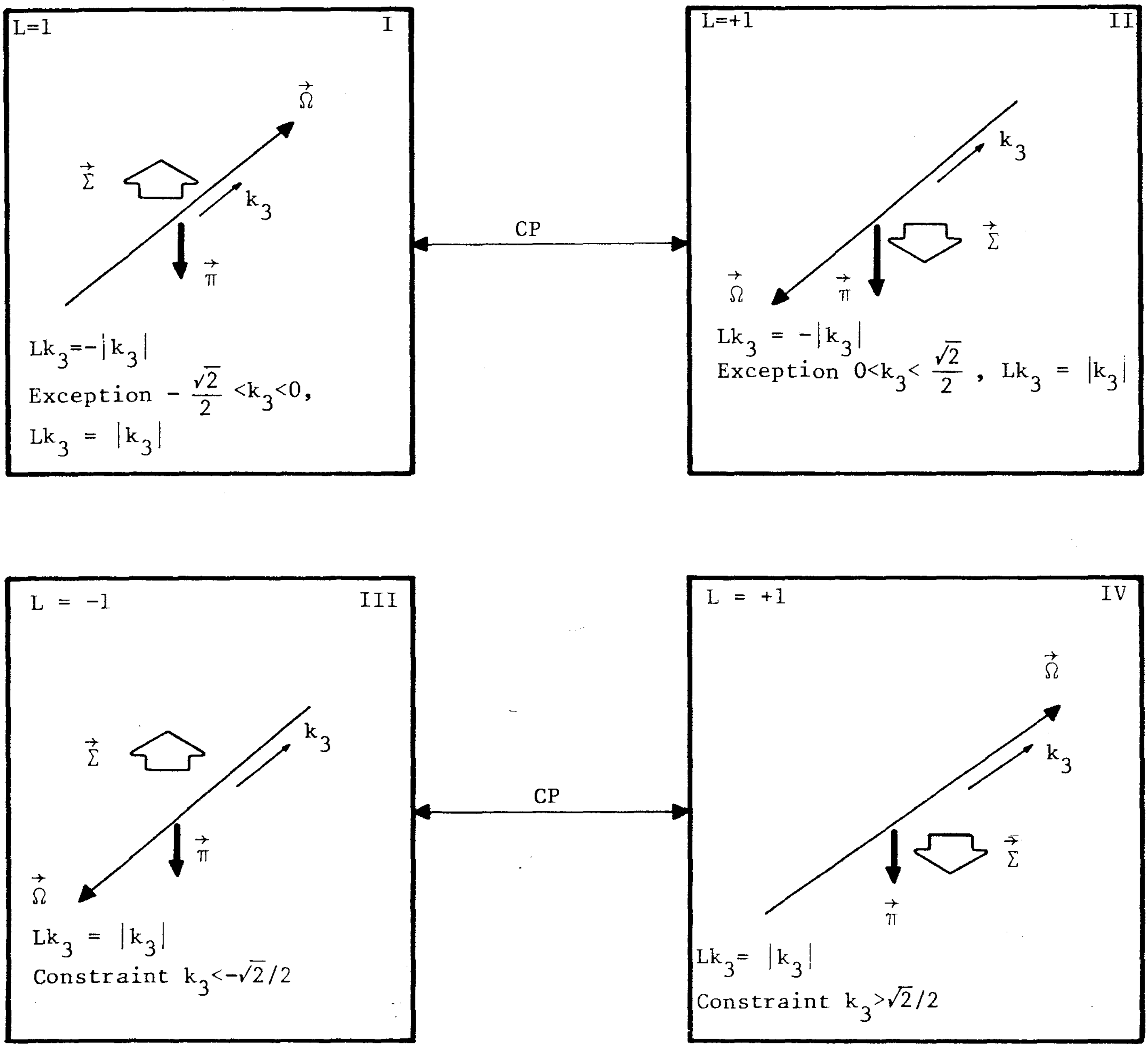


Fig. 2

In the diagram of currents in Fig.1, the large components of neutrino and antineutrino currents corresponds to amplitudes I and II and are CP related. The small components correspond to CP-related amplitudes III and IV, which clearly shows that the asymmetric emission of neutrinos is CP invariant.

In case of creation of neutrino-antineutrino pairs in the present universe, we can distinguish two possibilities.

(i) Neutrino-antineutrino pairs whose amplitudes are CP related, namely $(\nu_I \bar{\nu}_{II})$ or $(\nu_{III} \bar{\nu}_{IV})$ according to the above diagram; for each case the corresponding current diagram is CP invariant, and the number density of neutrino states is equal to the number density of antineutrino states.

(ii) Neutrino-antineutrino pairs whose amplitudes are not CP related, namely $(\nu_I \bar{\nu}_{IV})$ or $(\nu_{III} \bar{\nu}_{II})$. In both cases we note that Lk_3 has opposite signs for neutrino and antineutrino amplitudes, which corresponds to a number density of states different for neutrinos and antineutrinos. For $(\nu_I \bar{\nu}_{IV})$ or $(\nu_{III} \bar{\nu}_{II})$ we have, respectively, the number densities of states $(n(-|k_3|), n(|k_3|))$ and $(n(|k_3|), n(-|k_3|))$. Nevertheless, if the creation of pairs is due to a CP-invariant perturbation both cases will be equally probable since $(\nu_I \bar{\nu}_{IV}) \xleftrightarrow{CP} (\nu_{III} \bar{\nu}_{II})$ and no net asymmetry in neutrino-antineutrino number is possible. A net asymmetry (due to different density of states available for neutrinos and antineutrinos) will appear if the pair production perturbation violates CP. Indeed if pairs $(\nu_I \bar{\nu}_{IV})$ are produced, the pairs $(\nu_{III} \bar{\nu}_{II})$ are then forbidden and a net asymmetry between neutrino and antineutrino number will appear, proportional to the ratio

$$\delta_{k_3} = \frac{n(k_3)-n(-k_3)}{n(k_3)+n(-k_3)} \qquad (5.7)$$

for positive values of k_3. The ratio (5.7) is significantly nonzero only for $|k_3|$ of the order of $\sqrt{2}/2$. We also remark that the above discussion is independent of the space time point considered, since in our analysis we have dealt with scalar quantities only.

6. CONCLUSIONS

The main conclusion of our investigation is that the presence of a vorticity field of matter produces, via gravitation, microscopic asymmetries in neutrino physics. These results can also be extended to massive spin-1/2 fermions and this will be the subject of a future publication. We have proved these results in the context of the Einstein theory of gravitation, and for operational simplicity we have considered Gödel universe as the gravitational background, because it is the simplest known solution of Einstein field equations which is stationary and in which the matter content has a non-null vorticity. The basic results follow:

(1) The local dynamics of neutrinos is obtained from the Dirac equation in the given background. The spin of the neutrino precesses locally about the direction of the vorticity field. The direction of the angular velocity vector is parallel to the vorticity field, both for neutrino and antineutrino, and the absolute value of the angular velocity of precession depends on the energy of the neutrino/antineutrino. The Hamiltonian which determines the local dynamics of neutrinos is defined with respect to the global timelike Killing vector $\partial/\partial t$, and we have that the helicity L of neutrino (defined with respect to the local Lorentz frames of the tetrads) is conserved.

(2) We solve Dirac equation by separation into invariant modes defined by the global Killing vector fields of the spacetime, and we obtain a complete set of solutions of neutrino amplitudes in the hyperbolic harmonic modes (j,m,k_3,ε,L).

These modes provide two distinct representation bases for the algebra of the total angular momentum of the system (neutrino coupled to gravitation), one finite dimensional and the other infinite-dimensional. For both cases the space of angular-momentum appears to be polarized along the direction determined by the local vorticity field $\vec{\Omega}$. We construct the Fourier space associated to these complete bases and the complete unitarity relations for the kernel of the transformation are obtained.

(3) From the symmetry properties of the Hilbert space of neutrino solutions and its corresponding Fourier space we are able to define neutrino amplitudes, which are CP related as expected from the laws of neutrino physics.

(4) The Fourier current associated with the neutrino amplitude as well as the local neutrino current calculated at the origin $x = 1$ (for a given $j \geq 1/2$, summed over all all contributions $-1/2 \leq m \leq j$) are asymmetric along the direction determined by the vorticity field: the component of neutrino current along the direction antiparell to the vorticity field is larger than the component along the opposite direction. Also the Fourier current associated with the antineutrino amplitude as well as the local antineutrino current calculated at the origin $x = 1$ (summed over all contributions $-1/2 \leq m \leq j$) are asymmetric, since the component along the direction antiparallel to the vorticity vector is smaller than the component along the direction parallel to the vorticity vector. Therefore, at the microscopic level, neutrinos are preferentially emitted antiparallel to the local vorticity field;

as well, antineutrinos are preferentially emitted parallel to the local vorticity field. This result is CP invariant. In case of production of pairs under CP violation, a net number asymmetry appears between neutrinos and antineutrinos, which is significantly nonzero for k_3 of the order of the vorticity value $\sqrt{2}/2$.

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[13] Capital Latin indices are tetrad indices and run from to 3; they are raised and lowered with the Minkowski metric η^{AB}, η_{AB} = diag(+1, -1, -1, -1). Greek indices run from 0 to 3 and are raised and lowered with $g^{\alpha\beta}$, $g_{\alpha\beta}$; throughout the paper we use units such that $\hbar = c = 1$.

[14] γ^A are the constant Dirac matrices; we use a representation such that $\gamma^{A^\dagger} = \gamma^0\gamma^A\gamma^0$, with $(\gamma^0)^2 = -(\gamma^K)^2 = \mathbb{1}$, K= 1,2,3

and $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Explicitly $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^K = \begin{pmatrix} 0 & \sigma^K \\ -\sigma^K & 0 \end{pmatrix}$.

We use Pauli matrices in the representation $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

[15] Equations of the type

$$(x^2-1)\ \frac{d^2y}{dx^2} + (2x+M'-M)\ \frac{dy}{dx} + (K + \frac{K'}{x+1} - \frac{MM'}{x^2-1})y = 0$$ can be

solved by the standard substitutions $y = \left(\frac{x+1}{x-1}\right)^A (x^2-1)^B (x+1)^C$, where A, B and C are constants to be determined.

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[17] Angular momentum space has thus a preferred direction defined locally by the vorticity vector $\vec{\Omega}$. This is characterized by the fact that the projection of $\vec{J}$ along $\vec{\Omega}$ is Hermitian while any of its components along a direction orthogonal to $\vec{\Omega}$ is not Hermitian. The allowed rotations in this space maintain the direction $\vec{\Omega}$ invariant.

[18] The stronger condition (3.42) is assumed to garantee that $J_{(1)}$ and $J_{(2)}$ are antihermitian in the case $\sigma = 0$. Milder conditions as $\lim_{x\to\infty} \psi^+\psi = 0$ would produce nothing new.

[19] This analysis follows directly from the asymptotic formula $\lim_{z\to\infty} F(a,b,c,z) = (-1)^a \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{-a} + (-1)^b \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} z^{-b}$

[20] We remark that even for the case $j = m = -1/2$, N^2 is a non-zero finite number. We note that

$$|\varepsilon|^2 - (-K_3+L\frac{\sqrt{2}}{2})^2 = 4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)$$

[21] In the remaining of this section we take for simplicity $\omega = 1$.

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[23] We use the result $\lim_{c \to 0} \{cF(a,b,c;\lambda)\} = ab\, \lambda\, F(a+1, b+1, 2; \lambda)$

[24] The preferential emission of neutrino (antineutrino) along the direction antiparallel (parallel) to the local vorticity field $\vec{\Omega}$ has a macroscopic analog in the case of neutrino evaporation by a rotating black hole. A basic difference however lies in the local character of the vorticity field of matter flow as well in the local interpretation of L as the helicity of neutrino spinor fields, for the present case, in contrast to the asymptotic meaning of rotation and other quantities in the space-time of a rotating black hole. Cf. A. Vilenkin, Phys. Rev. Lett. 41, 1575(1978); Phys. Rev. D20, 1807(1979) and D.A. Leahy and W.G Unruh Phys. Rev. D19, 3509(1979).