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ABSTRACT

The critical frontier (as well as the thermal-type critical exponents) associated to the quenched bond-dilute spin-$\frac{1}{2}$ Ising ferromagnet in the simple cubic lattice is approximately calculated within a real space renormalization group framework in two different versions. Both lead to qualitatively satisfactory critical frontiers, although one of them provides an unphysical fixed point (which seem to be related to the three-dimensionality of the system) besides the expected pure ones; its effects tend to disappear for increasingly large clusters. Through an extrapolation procedure the (unknown) critical frontier is approximately located.

I - Introduction

Much effort is presently being dedicated to the theoretical study of random magnetic systems (site or bond, dilute or mixed, quenched or annealed models). Concerning the random Ising ferromagnets, techniques like Monte Carlo[1,2], high-temperature expansions[3,4], variational method[5], perturbative methods[6-9] (effective-medium, coherent potential, random phase approximations), duality and/or replica trick arguments[10-16], exact arguments[17] and finally real space renormalization group (RG) approaches[18-26] have been used.

The latter have been applied almost exclusively to planar lattices, where they have led to a certain success. The purpose of the present paper is to use the same type of RG approach for a three-dimensional lattice, namely the simple cubic (SC) one, in order to calculate the critical frontier and thermal-type exponents.

II - Renormalization group approach

The model we adopt is the quenched first-neighbour bond-dilute $\frac{1}{2}$- spin Ising ferromagnet; its Hamiltonian is given by

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij}\, \sigma_i \sigma_j \qquad (\sigma_i = \pm 1 \quad \forall\, i) \qquad (1)$$

where J_{ij} is a random variable whose probalility law is

$$P(J_{ij}) = (1-p)\,\delta(J_{ij}) + p\,\delta(J_{ij}-J) \qquad (J>0) \qquad (2)$$

By introducing the variables (hereafter referred as *thermal transmissivities*)

$$t \equiv \text{th}\,\frac{J_{ij}}{k_B T} \qquad (3)$$

and

$$t_0 \equiv \text{th}\,\frac{J}{k_B T} \qquad (4)$$

the law (2) can be substituted by

$$P(t) = (1-p)\,\delta(t) + p\,\delta(t-t_0) \qquad (2')$$

To construct the RG we shall adopt the cluster indicated in Fig.1(a), which can be considered as the first graph (associated to a renormalization lattice expansion factor $b = 2$) of a family whose last term ($b \to \infty$) is expected to be equivalent (thermodynamically speaking) to the infinite SC lattice. The use of this family of clusters has led to satisfactory results for the SC pure bond percolation [27-29] and pure Ising [30,31] problems. Once we associate the law (2') to each bond of the cluster, we obtain, for the transmissivity between the two terminal sites of the cluster, the following overall distribution law

$$P_c(t) = \sum_{\ell=0}^{N_b} p^{N_b-\ell}\,(1-p)^{\ell} \sum_m M_{\ell m}\,\delta(t - t_{\ell m}) \qquad (5)$$

where N_b is the total number of bonds of the cluster (in our case $N_b = 12$), $t_{\ell m}$ is given by

$$t_{\ell m} = \frac{\sum_{r=0}^{N_b-\ell} n_{\ell m}^{(r)} \; t_o^r}{1+\sum_{r=1}^{N_b-\ell} d_{\ell m}^{(r)} \; t_o^r} \tag{6}$$

and represents the transmissivity of a particular bond occupancy configuration of the cluster with ℓ absent bonds ($\{n_{\ell m}^{(r)}\}$ and $\{d_{\ell m}^{(r)}\}$ are integer coefficients), $M_{\ell m}$ is the number of such configurations which share the same transmissivity $t_{\ell m}$, and m runs over the set of such classes associated to a given value of ℓ. The multiplicity factors $\{M_{\ell m}\}$ clearly satisfy

$$\sum_{\ell=0}^{N_b} \sum_m M_{\ell m} = 2^{N_b} \tag{7}$$

Let us illustrate these statements through the cluster indicated in Fig.1(b); we obtain[22-24]

$$\begin{aligned} P_c(t) = {} & p^5 \delta \left(t - \frac{2t_o^2 + 2t_o^3}{1 + 2t_o^3 + t_o^4} \right) \\ & + p^4(1-p) \left\{ \delta \left(t - \frac{2t_o^2}{1+t_o^4} \right) + 4\,\delta \left(t - \frac{t_o^2 + t_o^3}{1+t_o^3} \right) \right\} \\ & + p^3(1-p)^2 \left\{ 2\,\delta(t-t_o^3) + 6\,\delta(t-t_o^2) + 2\,\delta(t) \right\} \\ & + p^2(1-p)^4 \left\{ 2\,\delta(t-t_o^2) + 8\,\delta(t) \right\} \\ & + p(1-p)^4\, 5\,\delta(t) + (1-p)^5\, \delta(t) \end{aligned} \tag{8}$$

For the case we are interested in (Fig. 1.a) we obtain the results presented in the Table 1. In order to establish the recursive relations which renormalize (p,t_o) into (p',t_o') we propose, as in Refs. [22-24],

$$\langle t\rangle_{P'} = \langle t\rangle_{P_C} \equiv F_1(p,t_o) \tag{9}$$

$$\langle t^2\rangle_{P'} = \langle t^2\rangle_{P_C} \equiv F_2(p,t_o) \tag{10}$$

where

$$P'(t) = (1-p')\delta(t) + p'\delta(t-t_o') \tag{11}$$

hence

$$p' = [F_1(p,t_o)]^2 / F_2(p,t_o) \tag{12}$$

$$t_o' = F_2(p,t_o) / F_1(p,t_o) \tag{13}$$

The associated flow diagram (see Fig. 2.a) presents four trivial fixed points, namely (0,0), (1,0) and (1,1) which are fully stable and (0,1) which is semi-stable, as well as three non trivial ones, namely $(p_c,1)$ (pure percolation point; fully unstable), $(1,t_c)$ (pure Ising point; semi-stable) and $(p_o,0)$ (unphysical point; semi-stable). The values p_c, t_c and p_o respectively satisfy

$$p_c = [F_1(p_c,1)]^2 / F_2(p_c,1) \tag{14}$$

$$t_c = F_2(1,t_c) / F_1(1,t_c) \tag{15}$$

and

$$p_o = \lim_{t\to 0} [F_1(p_o,t)]^2 / F_2(p_o,t) \tag{16}$$

This last equation becomes, in the present case (cluster of Fig. 1.a),

$$\frac{p_o \left[\sum_{\ell=0}^{10} p_o^{10-\ell} (1-p_o)^{\ell} \sum_m M_{\ell m} n_{\ell m}^{(2)} \right]^2}{\sum_{\ell=0}^{10} p_o^{10-\ell} (1-p_o)^{\ell} \sum_m M_{\ell m} (n_{\ell m}^{(2)})^2} = 1 \qquad (16')$$

The percolation and Ising correlation length critical exponents are (through diagonalization of the jacobian matrix $\partial(p',t_o')/\partial(p,t_o)$) respectively given by

$$\nu_p = \frac{\ell n\, b}{\ell n \left[dF_1(p,1)/dp \right]_{p=p_c}} \qquad (17)$$

and

$$\nu_t = \frac{\ell n\, b}{\ell n \left[dF_1(1,t)/dt \right]_{t=t_c}} \qquad (18)$$

with $b = 2$ in the present case. The set of the main results is presented in Tables 2 and 3, where FI(PI) refers to the frontier between the ferromagnetic (paramagnetic) and intermediate regions.

Let us now discuss what happens, as b increases, with the unphysical fixed point $(p_o, 0)$. In the limit $t_o \to 0$, only the shortest paths between the two terminals of the graph have to be considered, therefore we can reduce our analysis to a cluster made by b^2 branches in parallel, each of them consisting in b bonds in series. The probability law associated to each one of the b^2 branches is given by

$$(1-p^b)\,\delta(t) + p^b \delta(t-t_o^b)$$

hence

$$P_c(t) \sim \sum_{s=0}^{b^2} \binom{b^2}{s} p^{b(b^2-s)} (1-p^b)^s \, \delta\left[t - (b^2-s)t_o^b\right] \qquad (19)$$

in the limit $t_o \to 0$. Finally by using equation (16) we obtain

$$p_o = \frac{\left[\sum_{s=0}^{b^2} \binom{b^2}{s} p_o^{b(b^2-s)} (1-p_o^b)^s \, (b^2-s)\right]^2}{\sum_{s=0}^{b^2} \binom{b^2}{s} p_o^{b(b^2-s)} (1-p_o^b)^s \, (b^2-s)^2}$$

hence

$$\sum_{s=0}^{b^2-1} \binom{b^2-1}{s} (b^2-s) \, p_o^{b(b^2-s-1)} (1-p_o^b)^s = b^2 p_o^{b-1} \qquad (20)$$

This equation is equivalent, for $b=2$, to Eq.(16') and for $b \to \infty$, leads to $p_o \to 1$. And what happens with the PI- frontier? We have no closed argument to answer this question but if we take into account that (a) the fixed point $(p_o,0)$ coalesces with the (1,0) one, (b) the $b=2$ FI- and PI- frontiers are quite close among them in the neighbourhood $p \simeq p_c$, (c) the $b=2$ derivative of the PI- frontier at $p=p_o$ is quite high (10 in absolute value), (d) p_c and t_c present, as b grows, the correct tendency towards the (almost) exact values, (e) the flow sense of the PI- frontier coincides with that of the FI- frontier and that of the low branch ($t < t_c$) of the $p=1$ axis, one can speculate that, for $b \to \infty$, the PI- frontier coalesces with the FI- one (and both with the exact one) as indicated in Fig. 3 (such a shrinkage of an unphysical region has already been observed [38] within a bond percolation RG approach). If it is so, the flow diagram will become similar to the two-dimensional case (see in Fig. 2.b the flow diagram associated to the cluster of Fig. 1.b).

Before going further along this line let us wonder if the anomalous renormalization flow behaviour we found for the family of clusters we are concerned with is restricted to this family or rather is typical of any other similar three-dimensional cluster within the present formalism. To have an idea on these grounds we studied, within the same framework, the simplest three-dimensional cluster, namely the one indicated in Fig. 1.c (let us stress that within the present context a two-terminal graph can be considered planar if and only if it can be embeded in the plane in such a way that *both* terminals remain planarly accessible from the "outside" of the graph; clearly the cluster of Fig. 1.c does not satisfy this restriction). The results obtained with this cluster are qualitatively identical to those obtained with the cluster of Fig. 1.a (for the main numerical results see Table 2). As the anomalous flow behaviour they both present is absent from the two-dimensional case (see Figs. 1.b and 2.b) we are inclined to believe that it is related to the three-dimensionality (to be precise, to the fact that the dimensionality $d > 2$). This could be an (insatisfactory) indication that at $d = 3$ and according to Harris criterium [3], a new universality class (associated to a dilute fixed point) is expected to appear.

Before closing this Section let us formulate, for the cluster of Fig. 1.a, a different RG (a parametric one [23,29,37]) whose results will later on enable us to roughly estimate extrapolation errors. It essentially consists in maintaining Eq. (9) while Eq. (10) is substituted by an imposition on the renormalization space (or path), presently straight lines converging at $(p,t_o) = (1,1)$, i.e.

$$\frac{1 - t_o'}{1 - p'} = \frac{1 - t_o}{1 - p} \qquad (21)$$

in other words, the renormalization group acts now on a one-dimensional space (parametrized in our case by the angle of the straight line). Let us stress [29] that within this new framework the particular choice for the paths (Eq. (21) in our case) affects the approximate values for the critical exponents but does not affect the approximate critical frontier which, through Eq. (9) is given by

$$pt_o = F_1(p,t_o) \qquad (22)$$

The main results associated to this approach appear in Tables 2 and 3: we notice that the discrepancy with the previous treatment is quite small in what concerns the PF critical frontier. Remark also that the parametric RG provides no unphysical critical line contrarily to what happens with the canonical RG.

III - Extrapolated Critical Frontier

Let us introduce here an extrapolation procedure (quite similar to that introduced in Ref. [26]) which is expected to provide a satisfactory approximation for the critical frontier in the p-t_o space. This procedure satisfies by construction good available values for the critical points associated to the pure cases ($p_c = 0.247 \pm 0.003$ [32] and $t_c = 0.21811$ [34]). First we define new variables through

$$x \equiv \frac{1 - p}{1 - p_c(b)} \qquad (23)$$

and

$$y \equiv \frac{1 - t_o}{1 - t_c(b)} \qquad (24)$$

where $p_c(b)$ and $t_c(b)$ respectively denote the pure percolation and Ising approximate critical points provided by the RG associated to the cluster with size b. We then assume that the critical frontier in the x - y space depends *only slightly* on b (i.e. we assume an approximative law of "corresponding states" in what concerns the size of the cluster that has been used). Within this respect it is interesting to remark that for $b = 2$ the p- and t_o- contraction factors are numerically almost coincident and given respectively by

$$\frac{1 - p_c(\infty)}{1 - p_c(2)} = 0.9514 \qquad (25)$$

and

$$\frac{1 - t_c(\infty)}{1 - t_c(2)} = 0.9549 \qquad (26)$$

The above assumption clearly closes the extrapolation procedure as it leads to

$$\frac{1 - p}{1 - p_c(2)} = \frac{1 - \bar{p}}{1 - p_c(\infty)} \qquad (27)$$

and

$$\frac{1 - [t_o(p)]_{b=2}}{1 - t_c(2)} = \frac{1 - \bar{t}_o(\bar{p})}{1 - t_c(\infty)} \qquad (28)$$

where $\bar{t}_o(\bar{p})$ denotes the extrapolated critical frontier. In particular Eqs. (27) and (28) lead, for the pure critical points ($p=1$ or $t_o=1$), to

$$\frac{d\bar{t}_o(\bar{p})}{d\bar{p}} = \frac{1-p_c(b)}{1-p_c(\infty)} \quad \frac{1-t_c(\infty)}{1-t_c(b)} \quad \frac{dt_o(p)}{dp} \qquad (29)$$

which for $b=2$ becomes (see Eqs. (25) and (26))

$$\frac{d\bar{t}_o(\bar{p})}{d\bar{p}} = 1.0037 \quad \frac{dt_o(p)}{dp} \qquad (29')$$

The main extrapolation results are presented in Tables 2 and 3. In the latter we notice that the extrapolated canonical RG critical frontier (from Eqs. (9) and (10)) is very close to the extrapolated parametric RG one (from Eq. (22)). It is not easy to decide which one of the two present extrapolations is expected to be closer to the unknown exact one, and we are rather inclined to believe that their discrepancy is of the same order of magnitude as the discrepancy they both have with the exact one.

IV - Conclusion

Within a real space renormalization group framework (in two different versions, namely the canonical and the parametric ones) we have approximatively calculated the critical frontier associated to the quenched bond-dilute spin-$\frac{1}{2}$ Ising ferromagnet in the simple cubic lattice (the critical exponents ν_p and ν_t have been calculated as well). Both versions lead to qualitatively satisfactory critical frontiers, which are very close among them. However the canonical renormalization group provides also an unphysical region between the para- and ferro-magnetic ones. There are several reasons for believing that this region shrinks until complete disappearance as larger and larger clusters are considered. This unphysical region seems to persist for any other similar three-dimensional *finite* cluster, and could be an indication of the existence of a random fixed point (different from the pure percolation and Ising ones) in accordance with expectations from Harris criterium[3].

In order to obtain a critical frontier which might numerically be of some utility we have performed extrapolations for both renormalization group versions: the results are close among them and suggest that the unknown exact critical frontier lies, in the temperature-(bond) concentration space, *lower* than indicated by a previous treatment, namely the Effective Medium Approximation[9]. More specifically the present approach provides for the derivative $(-dt_o/dp)_{t_o=1}$ (with $t_o \equiv \text{th}\,\frac{J}{k_B T}$) a value which is at least 2% larger than the EMA one, and for the derivative $(-dt_o/dp)_{p=1}$ a value which is at least 18% smaller than the EMA one.

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CAPTION FOR FIGURES AND TABLES

FIG. 1 Renormalization group clusters ($b=2$): (a) for the simple cubic lattice; (b) for the square lattice ; (c) for the simplest three-dimensional cluster. The internal (●) and terminal (o) sites are indicated.

FIG. 2 Flow diagrams associated: (a) to cluster of Fig. 1.a; (b) to cluster of Fig. 1.b. (P), (F) and (I) respectively denote the para- , ferro- magnetic and intermediate (unphysical) regions.

FIG. 3 Out of scale possible evolution of the FI- and PI- frontiers as b grows up to infinity (by "exact" we mean the commonly expected critical frontier for the dilute Ising ferromagnet). (P), (F) and (I) respectively denote the para-, ferro- magnetic and intermediate (unphysical) regions.

Table 1 The set of coefficients $\{M_{\ell m}\}$, $\{n_{\ell m}^{(r)}\}$ (up row) and $\{d_{\ell m}^{(r)}\}$ (down row) associated to the cluster of Fig. 1.a. All missing coefficients vanish (excepting of course $d_{\ell m}^{(0)}$ which equals unity for all (ℓ,m)).

Table 2 Critical points, exponents and derivatives. The results (+) and (++) coincide with those obtained respectively in Refs. [29] and [30,31] in the treatment of *pure* bond percolation and Ising problems. The results quoted for $b=3$ have been obtained (excepting p_0) for the *pure* problems and have not been re-obtained within the present context (*dilute* problem) in order to avoid long computing times (same reason for not calculating the derivatives for b=3). Values § have been calculated with data of Ref. [9] (EMA means Effective Medium Approximation). The present extrapolations for the critical derivatives have been performed through use of Eq. (29') for the RG associated to Eqs. (9) and (10) (up value) as well as that associated to Eqs. (9) and (21) (down value).

Table 3 Values $t_o \equiv \mathrm{th}\,\frac{J}{k_B T}$ on the critical frontier associated to the simple cubic lattice (SC). The critical frontier corresponding to the RG defined through Eqs. (9) and (21) is in fact given by Eq. (22).

Table 1

ℓ	$M_{\ell m}$	t_o^2	t_o^3	t_o^4	t_o^5	t_o^6	t_o^7	t_o^8	t_o^9	t_o^{10}	t_o^{11}
0	1	4 0	8 8	8 15	24 24	40 32	24 24	8 15	8 8	4 0	0 1
1	4	4 0	6 6	4 10	14 14	20 16	10 10	4 5	2 2	0 0	0 0
1	8	3 0	6 6	6 10	14 14	20 16	10 10	2 5	2 2	1 0	0 0
2	2	4 0	4 4	8 6	12 8	4 8	0 4	0 1	0 0	0 0	0 0
2	4	3 0	4 4	4 8	8 8	8 4	4 4	0 3	0 0	1 0	0 0
2	4	2 0	4 4	6 6	8 8	6 8	4 4	2 1	0 0	0 0	0 0
2	4	2 0	4 4	4 8	8 8	10 4	4 4	0 3	0 0	0 0	0 0
2	4	4 0	4 4	2 8	8 8	8 4	4 4	2 3	0 0	0 0	0 0
2	8	2 0	5 4	4 6	7 8	10 8	3 4	0 1	1 0	0 0	0 0
2	8	2 0	4 5	4 6	8 7	10 8	4 3	0 1	0 1	0 0	0 0
2	16	3 0	4 4	3 6	8 8	9 8	4 4	1 1	0 0	0 0	0 0
2	16	3 0	5 5	3 6	7 7	9 8	3 3	1 1	1 1	0 0	0 0
3	4	4 0	2 2	4 6	4 4	0 2	2 1	0 0	0 0	0 0	0 0
3	8	1 0	4 2	2 5	4 4	5 2	0 2	0 0	0 0	0 0	0 0
3	8	2 0	2 2	4 6	4 4	2 2	2 1	0 0	0 0	0 0	0 0
3	8	1 0	2 4	2 5	4 4	5 2	2 0	0 0	0 0	0 0	0 0
3	8	2 0	2 3	2 3	6 4	4 4	0 1	0 0	0 0	0 0	0 0
3	8	2 0	4 4	2 3	2 3	4 4	2 1	0 0	0 0	0 0	0 0
3	8	2 0	3 2	2 3	4 6	4 4	1 0	0 0	0 0	0 0	0 0
3	8	3 0	2 2	2 5	4 4	3 2	2 2	0 0	0 0	0 0	0 0

ℓ	$M_{\ell m}$	t_o^2	t_o^3	t_o^4	t_o^5	t_o^6	t_o^7	t_o^8
3	8	2 0	4 4	2 3	3 2	4 4	1 2	0 0
3	8	3 0	2 2	2 5	4 4	3 2	2 2	0 0
3	16	1 0	3 3	4 3	4 4	3 4	1 1	0 0
3	16	2 0	3 4	2 3	4 2	4 4	1 2	0 0
3	16	2 0	4 3	2 3	2 4	4 4	2 1	0 0
3	16	2 0	3 3	3 3	4 4	2 4	1 1	1 0
3	16	3 0	3 3	1 4	4 4	3 2	1 1	1 1
3	16	3 0	3 3	4 3	5 4	1 4	0 1	0 0
3	32	2 0	3 3	2 4	4 4	4 2	1 1	0 1
3 4	16 4	3 0	4 4	2 5	4 4	3 2	0 0	0 0
4	1	4 0	0 0	0 6	0 0	4 0	0 0	0 1
4	2	0 0	0 0	0 0	0 0	0 0	0 0	0 0
4	2	0 0	4 0	0 5	4 0	0 2	0 0	0 0
4	2	2 0	0 0	4 6	0 0	2 0	0 0	0 1
4	4	0 0	2 2	4 1	2 2	0 2	0 0	0 0
4	4	2 0	2 2	2 2	2 2	0 0	0 0	0 1
4	4	3 0	2 0	0 5	0 0	3 2	0 0	0 0
4	8	1 0	2 2	0 3	2 2	3 0	0 0	0 0
4	8	2 0	2 2	2 3	2 2	0 0	0 0	0 0
4	8	2 0	2 2	1 1	2 2	0 2	0 0	0 0

ℓ	$M_{\ell m}$	t_o^2	t_o^3	t_o^4	t_o^5	t_o^6	t_o^7
4	8	1 0	2 2	2 1	2 2	1 2	0 0
4	16	1 0	2 1	2 3	2 2	1 0	0 1
4	16	1 0	1 2	2 3	2 2	1 0	1 0
4	16	2 0	2 1	1 2	1 3	1 1	1 0
4	16	3 0	1 1	2 3	1 2	1 0	0 1
4	16	1 0	2 3	1 2	1 1	2 1	1 0
4	16	1 0	1 3	1 2	3 1	2 1	0 0
4	16	1 0	2 3	2 1	2 0	1 2	0 1
4	16	1 0	2 2	2 1	2 2	1 2	0 0
4	16	2 0	2 2	1 2	1 1	1 1	1 1
4	16	1 0	3 2	2 1	0 2	1 2	1 0
4	16	1 0	3 2	1 2	1 1	2 1	0 1
4	16	2 0	1 1	2 3	2 2	1 1	0 0
4	16	1 0	3 1	1 2	1 3	2 1	0 0
4	16	1 0	2 2	2 1	2 2	1 2	0 0
4	16	2 0	1 1	1 2	3 3	1 1	0 0
4	32	2 0	2 2	2 1	2 2	0 2	0 0
4 5	8 8	1 0	2 0	0 1	0 2	1 0	0 0
4 5	32 8	2 0	2 2	2 3	2 2	0 0	0 0
4 5	48 8	3 0	2 2	0 3	2 2	1 0	0 0

ℓ	$M_{\ell m}$	t_o^2	t_o^3	t_o^4	t_o^5	t_o^6	t_o^7
4 5	72 16	2 0	3 3	1 2	1 1	1 1	0 0
5	4	2 0	0 0	0 1	2 2	0 0	0 0
5	8	0 0	2 0	0 3	2 0	0 0	0 0
5	8	2 0	1 0	0 1	0 2	0 0	1 0
5	8	0 0	3 0	0 2	1 0	0 1	0 0
5	8	0 0	2 1	2 0	0 1	0 1	0 0
5	8	0 0	1 2	2 0	1 0	0 1	0 0
5	8	1 0	0 0	2 3	0 0	1 0	0 0
5	16	0 0	0 0	0 0	0 0	0 0	0 0
5	16	1 0	1 1	1 0	1 1	0 1	0 0
5	16	2 0	0 0	1 2	0 0	1 1	0 0
5	16	1 0	1 1	0 1	1 1	1 0	0 0
5	16	0 0	1 1	2 1	1 1	0 0	0 0
5	16	1 0	1 0	1 1	1 2	0 0	0 0
5	16	1 0	1 1	1 0	1 1	0 1	0 0
5	16	1 0	1 1	1 1	1 0	0 0	0 1
5	16	1 0	1 1	1 1	0 1	0 0	1 0
5	16	0 0	2 1	1 1	0 1	1 0	0 0
5	16	1 0	0 1	1 1	2 1	0 0	0 0
5	16	1 0	1 1	1 1	1 1	0 0	0 0

ℓ	$M_{\ell m}$	t_o^2	t_o^3	t_o^4	t_o^5	t_o^6
5	16	0 0	1 2	1 1	1 0	1 0
5	32	1 0	2 1	0 0	0 1	1 1
5 6	16 4	3 0	0 0	0 3	0 0	1 0
5 6	16 4	2 0	0 0	2 3	0 0	0 0
5 6	32 8	1 0	2 2	1 0	0 0	0 1
5 6	32 8	1 0	2 2	0 1	1 0	0 0
5 6	64 16	1 0	2 1	1 1	0 1	0 0
5 6	64 16	1 0	1 2	1 1	1 0	0 0
5 6	128 32	2 0	1 1	0 1	1 1	0 0
6	4	0 0	0 0	2 1	0 0	0 0
6	8	0 0	1 0	0 1	1 0	0 0
6	8	0 0	2 0	0 0	0 0	0 1
6	8	1 0	0 0	1 0	0 0	0 1
6	16	0 0	1 0	1 0	0 1	0 0
6	16	1 0	0 0	0 0	1 1	0 0
6	16	0 0	1 0	0 1	1 0	0 0
6	16	0 0	0 1	1 0	1 0	0 0
6 7	32 8	0 0	2 0	0 1	0 0	0 0
6 7	64 16	1 0	1 0	0 0	0 1	0 0
6 7	64 16	1 0	0 0	1 1	0 0	0 0

ℓ	$M_{\ell m}$	t_o^2	t_o^3	t_o^4	t_o^5
6 7	64 16	0 0	1 1	1 0	0 0
7	8	0 0	0 0	0 0	1 0
4 5	8 60	2	2	0	0
6 7	28 4	0	2	1	0
5 6	8 92	2	0	0	0
7 8	44 6	0	0	1	0
5 6	32 240	1	1	0	0
7 8	112 16	0	1	0	0
7 8	32 8	0 0	0 0	1 0	0 0
6 7	16 120	0	1	0	0
8 9	56 8	0	0	0	0
5 6	8 76	1	0	0	0
7 8	240 152	0	0	0	0
9 10	40 4				
6 7	68 176				
8 9	257 172	0	0	0	0
10 11	62 12	0	0	0	0
12	1				

	b = 2 (Fig. 1.a) Eqs.(9) and (10)	b = 3 (same family as cluster of Fig. 1.a) Eqs.(9) and (10)	b = 2 (Fig. 1.a) Eqs.(9) and (21)	S C	b = 2 (Fig. 1.c) Eqs.(9) and (10)
p_c	0.2085[+]	0.225 [29]	0.2085[+]	0.247 ± 0.003 [32] (series)	0.2761
ν_p	1.031[+]	0.97 [29]	1.031[+]	0.825 ± 0.05 [33] (series)	1.104
t_c	0.1812[++]	0.1955 [30,31]	0.1812[++]	0.21811 [34] (series)	0.2309
ν_t	0.8705[++]	0.8189 [30,31]	0.8705[++]	$0.638 {+0.002 \atop -0.008}$ [35] 0.630 ± 0.0015 [36] (series)	0.9013
p_o	1/3	0.4215	∄	1 ($b \to \infty$)	1/2
$-dt_o/dp \Big\vert_{p=p_c}^{FI}$	5.76	?	4.79	4.702[§] (EMA) 5.78, 4.81 } present extrapolations	4.50
$-dt_o/dp \Big\vert_{p=1}^{FI}$	0.185	?	0.189	0.2238[§] (EMA) 0.186, 0.190 } present extrapolations	0.244
$-dt_o/dp \Big\vert_{p=p_c}^{PI}$	8.5	?	∄	same as for FI (?)	5.4
$-dt_o/dp \Big\vert_{p=p_o}^{PI}$	10	?	∄	∞ (?)	8.6

Table 2

b	b = 2 (Fig. 1.a) Eqs.(9) and (10)	Associated extrapolation for SC	b = 2 (Fig. 1.a) Eqs.(9) and (21)	Associated extrapolation for SC
0.2085	1	-	1	-
0.247 [32]	0.818	1	0.837	1
0.25	0.801	0.983	0.825	0.986
0.30	0.654	0.770	0.673	0.792
0.35	0.550	0.636	0.564	0.653
0.40	0.474	0.542	0.484	0.554
0.45	0.417	0.474	0.424	0.482
0.50	0.373	0.423	0.378	0.429
0.55	0.337	0.383	0.340	0.387
0.60	0.307	0.350	0.310	0.353
0.65	0.283	0.324	0.285	0.326
0.70	0.262	0.301	0.263	0.303
0.75	0.244	0.282	0.245	0.283
0.80	0.228	0.266	0.229	0.267
0.85	0.214	0.252	0.215	0.253
0.90	0.202	0.239	0.202	0.239
0.95	0.191	0.228	0.191	0.228
1	0.1812	0.21811 [34]	0.1812	0.21811 [34]

Table 3

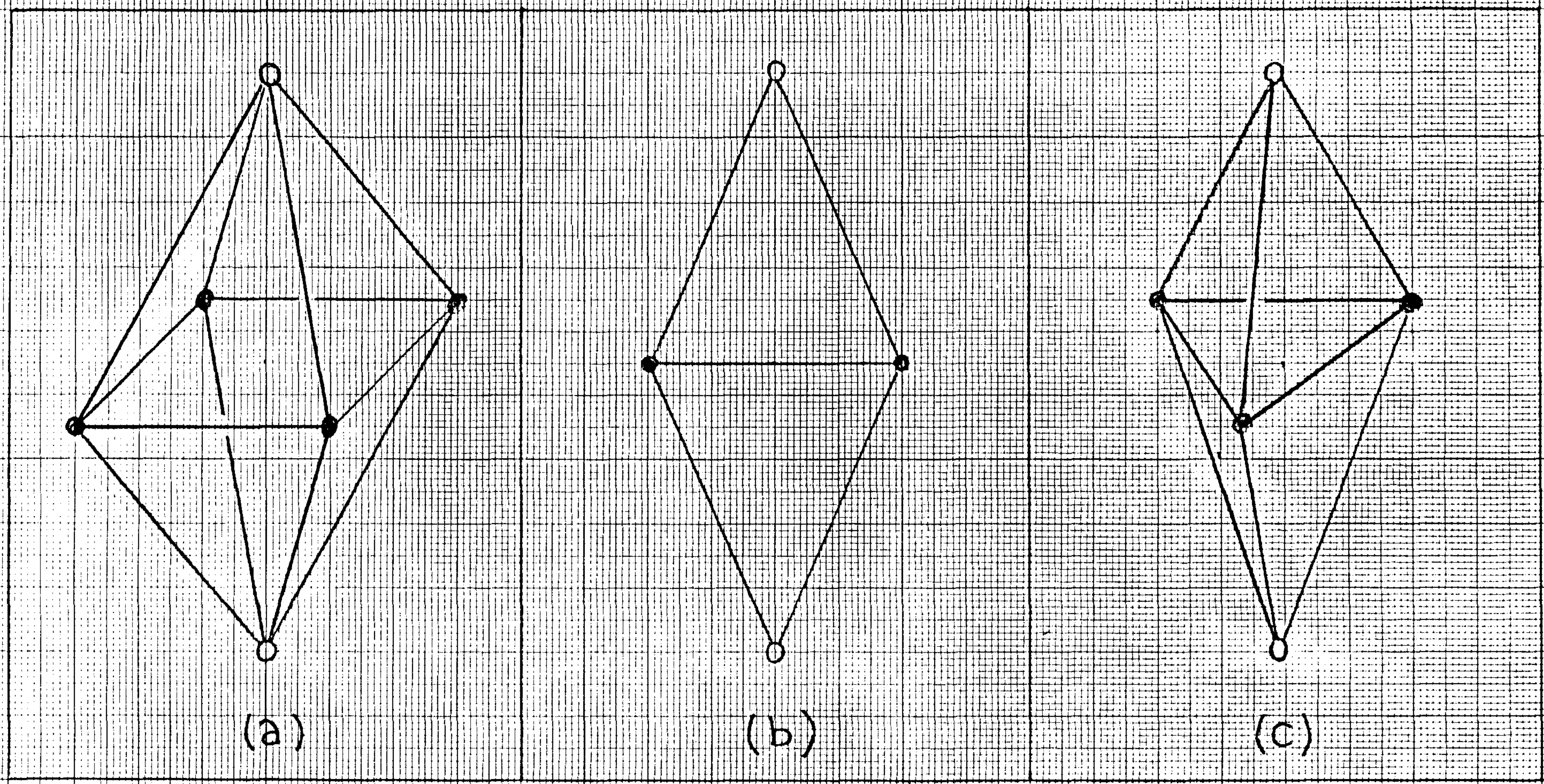
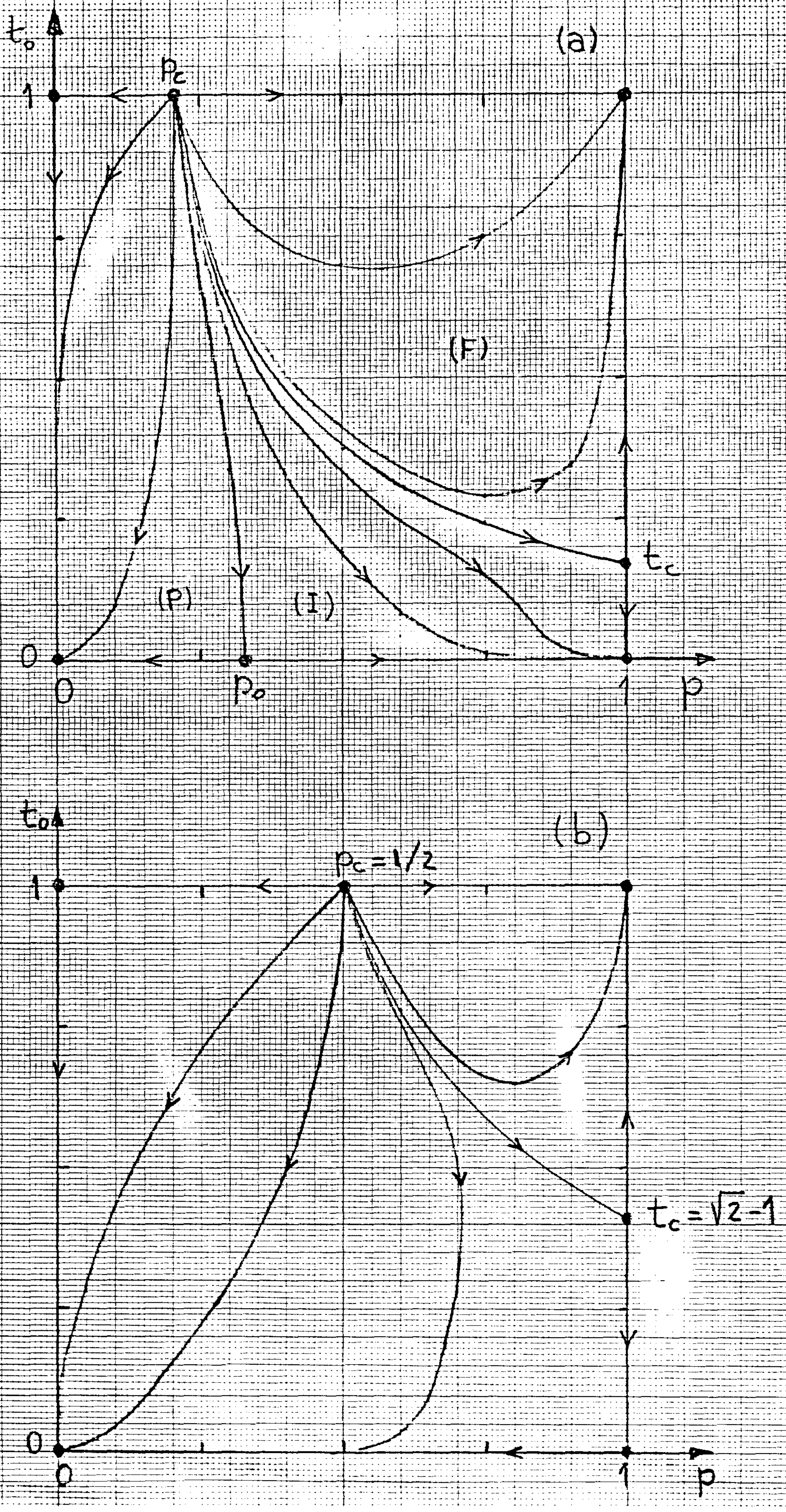


FIG. 1

FIG. 2



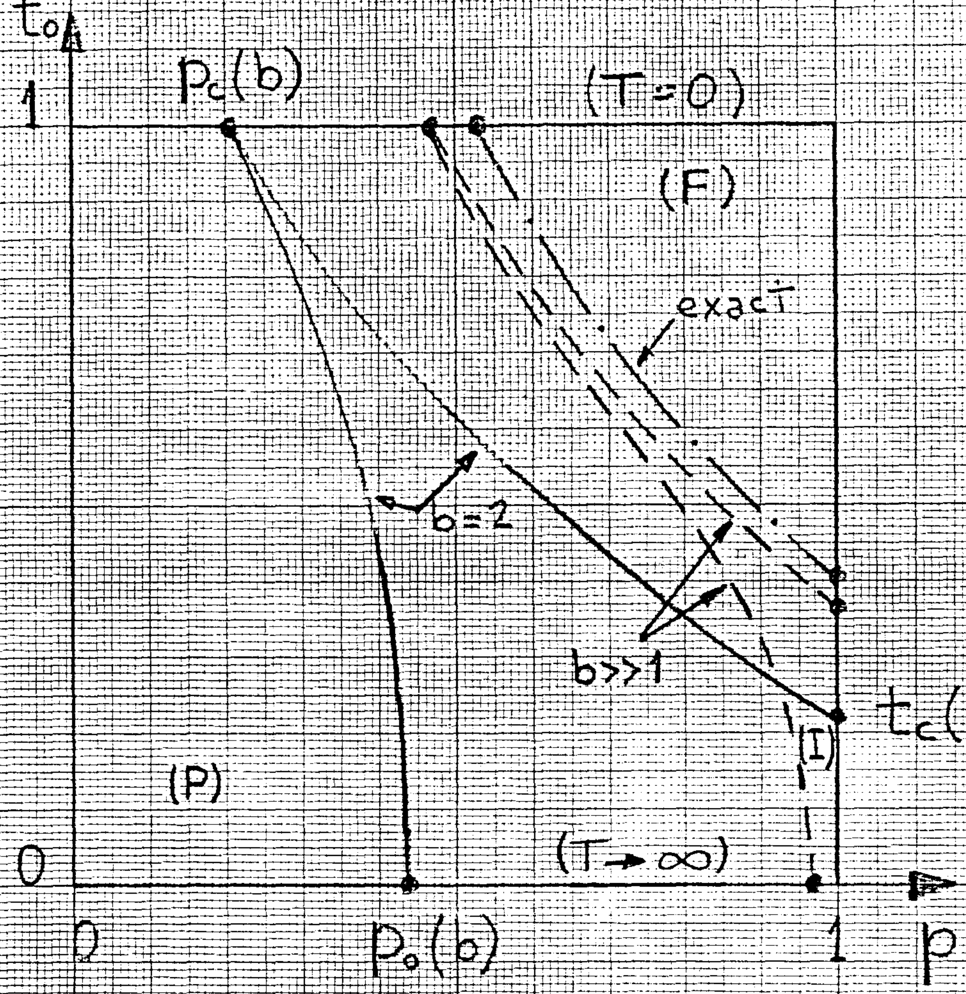


FIG. 3