

NOTAS DE FÍSICA

VOLUME XI

Nº 14

THE HIGH ENERGY BEHAVIOUR OF THE SCATTERING AMPLITUDE
FOR NEGATIVE MOMENTUM TRANSFER

by

S. W. McDowell

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Av. Wenceslau Braz, 71

RIO DE JANEIRO

1964

THE HIGH ENERGY BEHAVIOUR OF THE SCATTERING AMPLITUDE
FOR NEGATIVE MOMENTUM TRANSFER*

S. W. McDowell**

Institute for Advanced Study, Princeton, New Jersey

(Received September 15, 1964)

ABSTRACT

The high energy behaviour of the scattering amplitude is investigated in the real negative region of momentum transfer $-t$, below the threshold $t = -4m^2$ of the crossed channel. If one assumes the existence of bound states in the crossed t -channel with angular momenta larger than one, one can show that the high energy scattering amplitude behaves as if dominated by a Regge trajectory $\alpha(t)$ of even signature and the quantum numbers of the vacuum. It is shown that $\alpha(t)$ is continuous in the open interval $(0, 4m^2)$, and an upper bound for $\alpha(t)$ is given under the assumption of analyticity in the domain $\text{Re } \sqrt{t} \leq 2m$.

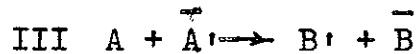
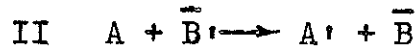
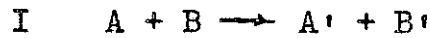
* This work was performed under the auspices of the National Science Foundation.

** On leave of absence from Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brazil.

It is known that the Froissart bound ¹ for the relativistic scattering amplitude $F(s,t)$ can be deduced from analyticity in the Lehmann ellipse plus the weak assumptions that the absorptive part of $F(s,t)$ is analytic in t , in the neighbourhood of some finite positive interval $(0, t_0)$ and is bounded there by a power of s ^{2, 3}. It has now also been proved,⁴ by using, in addition, analyticity in the s -plane, that if $F(s,t)$ has no poles in t corresponding to bound states with angular momentum larger than one in the interval $(0, 4m^2)$ the dispersion integrals are actually convergent with only two subtractions. We shall discuss here the asymptotic behaviour of $F(s,t)$ assuming the existence of poles with angular momentum larger than one. Although no elementary bosons exist with spin higher than one, this analysis has interest in itself as it discloses a connection between the high energy behaviour of the scattering amplitude and the angular momenta of the assumed bound states according to the pattern of a leading Regge trajectory $\alpha(t)$ of even signature and the quantum numbers of the vacuum. It is shown that $\alpha(t)$ is continuous in the open interval $(0, 4m^2)$.

We have also obtained an upper bound for $\alpha(t)$, assuming analyticity inside the parabola $\text{Re } \sqrt{t} = 2m$. This parabola is the limit as $k^2 \rightarrow \infty$, of the ellipse of convergence of the Legendre polynomial expansion.

Let $F(s, u, t)$ be the scattering amplitude describing three processes:



where A and B are two scalar particles of mass M_A and M_B respectively. The first two processes are elastic scattering and the last one is a collision in a state with the quantum numbers of the vacuum. The variables s, u and t are related by:

$$s + t + u = 2(M_A^2 + M_B^2) \quad (1)$$

We assume as in reference 4 that $F(s, u, t)$ is an analytic function of t in a certain domain \mathcal{D} as required to derive the Froissart bound, is bounded by a power s^N of s and, in addition, for fixed t inside \mathcal{D} , it is an analytic function of s with cuts along $s = (M_A + M_B)^2$ to $+\infty$ and $u = (M_A + M_B)^2$ to $+\infty$. The domain \mathcal{D} includes a neighbourhood of the positive real axis from $t = 0$ to $t = 4m^2$ with the exception of a finite number of points where $F(s, t, u)$ has simple poles. Here m is the mass of the least massive particle, say the pion mass. One can show that given a positive $\epsilon < 1$ one can find a real $t_\epsilon > 0$ and independent of s such that for $t < t_\epsilon$, $F(s, u, t)$ is bounded by $s^{1+\epsilon}$. Therefore for fixed $t < t_\epsilon$ one can write a dispersion relation for $F(s, t, u)$ with only two subtractions:

$$F(s, u, t) = C_0(t) + C_1(t)(s-u) + \frac{s^2}{\pi} \int \frac{A_1(s', t)}{(s'-s)s'^2} ds' + \frac{u^2}{\pi} \int \frac{A_2(u', t)}{(u'-u)u'^2} du' \quad (2)$$

The dispersion integrals may extend below the elastic threshold $s_0(u_0) = (M_A + M_B)^2$ but above this threshold each absorptive amplitude and all its derivatives with respect to t are positive definite, for t in the interval $(0, 4m^2)$. Now for t in this interval, $F(s, t, u)$ is bounded by s^N so that one can write a dispersion relation with $N+1$ subtractions:

$$F(s, u, t) = C_0(t) + C_1(t)(s-u) + \sum_{n=2}^N I_{1n}(t)s^n + I_{2n}(t)u^n + \frac{s^{N+1}}{\pi} \int \frac{A_1(s', t)}{(s'-s)s'^{N+1}} ds' + \frac{u^{N+1}}{\pi} \int \frac{A_2(u', t)}{(u'-u)u'^{N+1}} du' \quad (3)$$

For $t < t_c$ a comparison of (2) and (3) shows that $C_0(t)$ and $C_1(t)$ are the same in the two expressions and:

$$I_{1n}(t) = \frac{1}{\pi} \int \frac{A_1(s', t)}{s'^{n+1}} ds' \quad (4)$$

with a similar expression for $I_{2n}(t)$.

Now let us introduce the variable,

$$z = \frac{s-u}{4k_1 k_2} \quad (5)$$

where $k_1 = \frac{1}{2} \sqrt{t-4M_A^2}$, $k_2 = \frac{1}{2} \sqrt{t-4M_B^2}$ are the initial and final momenta in the center of mass system for process III and $z = \cos \theta$ where θ is the scattering angle. In the region we are considering both k_1 and k_2 are pure imaginary and the product is real and negative. One can express s and u in terms of z and t by:

$$2k_1 k_2 z = s + k_1^2 + k_2^2 = - (u + k_1^2 + k_2^2) \quad (6)$$

Therefore, since k_1^2 and k_2^2 are negative in the expansion of s^n or u^n in power series of z all the coefficients of even powers are positive. On the other hand one can expand z^p in Legendre polynomials of order $l \leq p$ and $(l-p)$ even. Again in this expansion all the coefficients are positive. Therefore one can finally write:

$$\sum_{n=2}^N \left(I_{1n}(t) s^n + I_{2n}(t) u^n \right) = \sum_{l=0}^N C_l(t) P_l(z) \quad (7)$$

where

$$C_l(t) = \sum_{n=l}^N \mu_{ln}(t) \left[I_{2n}(t) + (-1)^l I_{1n}(t) \right] \quad (8)$$

and for even l , all the μ_{ln} are positive. (Actually the μ_{ln} 's are all positive definite for both even and odd l .) In the real interval $0 < t < 4m^2$ the only singularities of $F(s, u, t)$ as a function of t are poles corresponding to bound states in the crossed channel III. Let t_1, t_2, \dots, t_k be the energies of these bound states, l_1, l_2, \dots, l_k the corresponding angular momenta. In the neighbourhood of $t = t_r$ all the coefficients $C_l(t)$ are regular except $C_{l_r}(t)$, which has a pole at $t = t_r$. It is then clear, by the result of Jin and Martin⁴ that the representation (2) is valid all through the interval $0 \leq t < t_1'$, where t_1' is the first bound state with angular momentum larger than one. Since the residue at this pole behaves like $|z|^{l_1}$ and, at least in the direction of the imaginary axis $|F(s, t_1' - \epsilon)| < c|s|^2$ it follows that $l_1' = 2$.

Let us next consider the sequence of bound states with increasing energies t_1', t_2', \dots, t_n' and even angular momenta l_1', l_2', \dots, l_n' such that l_i' is larger than the angular momenta of all bound states preceding t_i' . Let us suppose that in the interval $0 \leq t < t_1'$ the representation (3) is valid with $N = l_1' - 1$. Then by a slight generalization of the argument of reference 4 one can show that in the interval $0 \leq t < t_{i+1}'$ the representation (3) is valid with $N = l_i' + 1$. We shall give the main steps in the proof.

For $t < t_i'$, $I_{1,2n}(t)$ is given by (4) when $n \geq l_i'$. Since $A_{1,2}(s', t)$ and all its derivatives with respect to t are positive (for $s' > s_0$) one can expand $A(s', t)$ in power series of t with positive coefficients. It is then allowed to interchange the order of summation and integration in (4).⁵ One, thus, obtains a power series representation for $I_n(t)$ with positive coefficients. If t' is the radius of convergence of this series then it is also the first singularity of $I_n(t)$ and vice-versa, and for $t < t'$ the integral representation still holds.⁵ Now since the coefficients $\mu_{ln}(t)$ in (8) are all positive analytic functions of t then, for even $l > l_i'$, t' is also a singularity of $C_l(t)$. Since by hypotheses, all $C_l(t)$ with even $l > l_i'$ are singular in the interval $0 \leq t < t_{i+1}'$, it follows that the representation (4) holds for $n \geq l_i' + 2$ and therefore $F(s, u, t)$ may be represented by (3) with $N = l_i' + 1$. Thus our assertion is proved. Since this result is true in the interval $0 \leq t < t_1'$ its validity in general follows by complete induction. Now using

the same argument as before one deduces that:

$$l'_{i+1} = l'_i + 2 \quad (9)$$

It may happen that in the interval (t'_i, t'_{i+1}) there exists a bound state t_j with angular momentum $l_j = l'_i + 1$. Since for odd l the expression (8) involves the difference of the two functions $I_{1n}(t)$ and $I_{2n}(t)$ it is not in general true that for $t < t_j$ (3) holds with $N = l_j - 1$. It is however obvious that, for $t \gg t_j$, at least $l_j + 1$ subtractions are required.

From the above considerations it is clear that if the angular momentum l_j (even or odd) of a bound state t_j is larger than all the preceding ones the angular momentum of the next bound state with the same property is either $l_j + 1$, or $l_j + 2$ if l_j is even.

Another result which emerges from this analysis is that for all the bound states t'_i as previously defined, the residues are negative. In fact as one approaches the pole t'_i from below, $C_{l'_i}(t)$ will be given by (8) and is positive. Therefore the residue is negative.

Let us now define a function $\alpha(t)$ as the limiting value of the set of real numbers α_i for which both integrals

$$I_{1,2\alpha_i} = \int_{s_0}^{\infty} \frac{A_{1,2}(s', t)}{s'^{\alpha_i + 1}} ds' \quad (10)$$

are convergent. ⁶ We shall first show that $\alpha(t)$ is continuous

in the open interval $(0, 4m^2)$. Let us take in the t plane three circles with origin at $t = 0$ and increasing radii t , $t + \delta$ and $t_0 = 4m^2$ respectively. These circles are inside the domain \mathcal{D} of analyticity in t of $A(s, t)$ and on each circle $|A(s, t)|$ is maximum on the positive real axis. Then applying to $A(s, t)$, Hadamard's three circles theorem ⁷ one obtains:

$$A(s, t + \delta) < A(s, t)^{\xi_1} A(s, t_0)^{\xi_2} \quad (11)$$

where

$$\xi_1 = \ln \left(\frac{t_0}{t + \delta} \right) / \ln \left(\frac{t_0}{t} \right); \quad \xi_2 = \ln \left(\frac{t + \delta}{t} \right) / \ln \left(\frac{t_0}{t} \right) \quad (12)$$

and $\xi_1 + \xi_2 = 1$.

Since we are excluding the points $t = 0$ and $t = t_0$ one can take $\delta_0 < t < t_0 - \delta_0$ where δ_0 is arbitrarily small. Then for $\delta < \delta_0$ one has:

$$\xi_2 < \delta \left[t \ln \frac{t_0}{t} \right]^{-1} < 2 \frac{\delta}{\delta_0} \quad (13)$$

But $A(s, t_0)$ is bounded by $\left(\frac{s}{s_0} \right)^N$, therefore (11) gives:

$$A(s, t + \delta) < A(s, t) \left(\frac{s}{s_0} \right)^{k\delta} \quad (14)$$

where $k = 2N/\delta_0$. Therefore given an ϵ one can choose a $\delta_1 = \epsilon/k$ such that for $\delta < \min. \{ \delta_0, \delta_1 \}$ one has:

$$\int_{s_0}^{\infty} \frac{A(s, t + \delta)}{s^{\alpha(t) + \epsilon + 1}} ds < s_0^{-k\delta} \int_{s_0}^{\infty} \frac{A(s, t)}{s^{\alpha(t) + k(\delta_1 - \delta) + 1}} ds < \infty \quad (15)$$

Hence $|\alpha(t + \delta) - \alpha(t)| < \epsilon$ so that $\alpha(t)$ is continuous. If $F(s, u, t)$ has a Regge behaviour, $\alpha(t)$ coincides with the Pomeranchuk

trajectory. However, even in the general sense as defined above $\alpha(t)$ has the properties of the Pomeranchuk trajectory in the interval $(0, 4m^2)$, namely that, in the (λ, t) -plane the leading poles with even angular momentum and quantum numbers of the vacuum lie on $\alpha(t)$ and all the others lie on or below this curve.

Finally let us assume that $A(s, t)$ is actually bounded by $s^{\alpha(t)+\epsilon}$ for whatever small ϵ , and that $\alpha(t)$ is analytic inside the parabola:

$$\operatorname{Re} \sqrt{t} = \sqrt{t_0} = 2m \quad (16)$$

This parabola is the limit as $k^2 \rightarrow \infty$, of the ellipse of convergence of the Legendre polynomial expansion. Since in the Legendre polynomial expansion of $A(s, t)$ all the coefficients are positive, for all t on or inside the parabola (10) $\operatorname{Re} \alpha(t)$ has an absolute maximum at $t = t_0$. Then for λ real and positive

$$\varphi(t) = \exp \lambda [\alpha(t) - \alpha(t_0)]$$

is bounded by one in the same region. Now the interior of the parabola is analytically mapped into the interior of the unit circle by the transformation ⁸:

$$z = \operatorname{tg}^2 \left(\frac{\pi}{4} \sqrt{\frac{t}{t_0}} \right) \quad (17)$$

Therefore one can apply Pick's inequality ⁹ to the function $\varphi[t(z)]$. One obtains (for real positive t):

$$e^{\lambda \alpha(t)} < \frac{e^{\lambda \alpha(0)} + z e^{\lambda \alpha(t_0)}}{1 + z e^{\lambda [\alpha(0) - \alpha(t_0)]}} \quad (18)$$

In the limit $\lambda \rightarrow 0$, (12) becomes:

$$\alpha(t) < \alpha(0) + \frac{2z}{1+z} [\alpha(t_0) - \alpha(0)]$$

or

$$\alpha(t) < \alpha(t_0) - \cos\left(\frac{\pi}{2} \sqrt{\frac{t}{t_0}}\right) [\alpha(t_0) - \alpha(0)] \quad (19)$$

which is an upper bound for $\alpha(t)$ joining the values at $t = 0$ and $t = t_0$. Considering that the absence of $\pi - \pi$ bound states imply $\alpha(t_0) < 2$ and since $\alpha(0) \leq 1$, an absolute upper bound for the Pomeranchuk trajectory in the interval $(0, 4m^2)$ is:

$$\alpha(t) = 2 - \cos\left(\frac{\pi}{2} \sqrt{\frac{t}{t_0}}\right) \quad (20)$$

It is a pleasure to acknowledge Dr. A. Martin for most helpful discussions and suggestions. The author is grateful to Professor Oppenheimer for the kind hospitality extended to him at the Institute for Advanced Study.

* * *

REFERENCES

1. M. Froissart, Phys. Rev. 123, 1054 (1961).
2. A. Martin, Phys. Rev. 129, 1432 (1963).
3. M. Greenberg and F. E. Low, Phys. Rev. 124, 2047 (1961).
4. Y. Jin and A. Martin, to be published.
5. E. C. Titchmarsh. The Theory of Functions, 2nd edition, p. 44 (Oxford 1939).
6. This definition was suggested by A. Martin.
7. E. C. Titchmarsh. Ibid, p. 172.
8. A. R. Forsyth. Theory of Functions of a Complex Variable, 3rd edition, p. 619 (Cambridge 1918).
9. G. Caratheodory. Theory of Functions of a Complex Variable, 2nd edition, Vol. II, p. 14 (New York 1960).