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THE HIGH ENERGY BEHAVIOUR OF THE SCATTERING AMPLITUDE FOR NEGATIVE MOMENTUM TRANSFER*

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ABSTRACT

The high energy behaviour of the scattering amplitude is investigated in the real negative region of momentum transfer $-t$, below the threshold $t = 4m^2$ of the crossed channel. If one assumes the existence of bound states in the crossed t-channel with angular momenta larger than one, one can show that the high energy scattering amplitude behaves as if dominated by a Regge trajectory $\alpha(t)$ of even signature and the quantum numbers of the vacuum. It is shown that $\alpha(t)$ is continuous in the open interval $(0, 4m^2)$, and an upper bound for $\alpha(t)$ is given under the assumption of analyticity in the domain $\mathrm{Re}\sqrt{t} \leqslant 2m$.

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It is known that the Froissart bound [1] for the relativistic scattering amplitude $F(s,t)$ can be deduced from analyticity in the Lehmann ellipse plus the weak assumptions that the absorptive part of $F(s,t)$ is analytic in t, in the neighbourhood of some finite positive interval $(0, t_o)$ and is bounded there by a power of s [2, 3]. It has now also been proved,[4] by using, in addition, analyticity in the s-plane, that if $F(s,t)$ has no poles in t corresponding to bound states with angular momentum larger than one in the interval $(0,4m^2)$ the dispersion integrals are actually convergent with only two subtractions. We shall discuss here the asymptotic behaviour of $F(s,t)$ assuming the existence of poles with angular momentum larger than one. Although no elementary bosons exist with spin higher than one, this analysis has interest in itself as it discloses a connection between the high energy behaviour of the scattering amplitude and the angular momenta of the assumed bound states according to the pattern of a leading Regge trajectory $\alpha(t)$ of even signature and the quantum numbers of the vacuum. It is shown that $\alpha(t)$ is continuous in the open interval $(0,4m^2)$.

We have also obtained an upper bound for $\alpha(t)$, assuming analyticity inside the parabola $\mathrm{Re}\sqrt{t} = 2m$. This parabola is the limit as $k^2 \longrightarrow \infty$, of the ellipse of convergence of the Legendre polynomial expansion.

Let $F(s, u, t)$ be the scattering amplitude describing three processes:

$$\begin{array}{ll} \text{I} & A + B \longrightarrow A' + B' \\ \text{II} & A + \bar{B}' \longrightarrow A' + \bar{B} \\ \text{III} & A + \bar{A}' \longrightarrow B' + \bar{B} \end{array}$$

where A and B are two scalar particles of mass M_A and M_B respectively. The first two processes are elastic scattering and the last one is a collision in a state with the quantum numbers of the vacuum. The variables s, u and t are related by:

$$s + t + u = 2(M_A{}^2 + M_B{}^2) \tag{1}$$

We assume as in reference 4 that F(s, u, t) is an analytic function of t in a certain domain $\mathscr{D}$ as required to derive the Froissart bound, is bounded by a power s^N of s and, in addition, for fixed t inside $\mathscr{D}$, it is an analytic function of s with cuts along $s = (M_A + M_B)^2$ to $+\infty$ and $u = (M_A + M_B)^2$ to $+\infty$. The domain $\mathscr{D}$ includes a neighbourhood of the positive real axis from $t = 0$ to $t = 4m^2$ with the exception of a finite number of points where F(s, t, u) has simple poles. Here m is the mass of the least massive particle, say the pion mass. One can show that given a positive $\epsilon < 1$ one can find a real $t_\epsilon > 0$ and independent of s such that for $t < t_\epsilon$, F(s, u, t) is bounded by $s^{1+\epsilon}$. Therefore for fixed $t < t_\epsilon$ one can write a dispersion relation for F(s, t, u) with only two subtractions:

$$F(s,u,t) = C_o(t) + C_1(t)(s-u) + \frac{s^2}{\pi}\int^{\infty} \frac{A_1(s',t)}{(s'-s)s'^2}\, ds' + \frac{u^2}{\pi}\int^{\infty} \frac{A_2(u',t)}{(u'-u)u'^2}\, du' \tag{2}$$

The dispersion integrals may extend below the elastic threshold $s_o(u_o) = (M_A + M_B)^2$ but above this threshold each absorptive amplitude and all its derivatives with respect to t are positive definite, for t in the interval $(0, 4m^2)$. Now for t in this interval, F(s, t, u) is bounded by s^N so that one can write a dispersion relation with N+ 1 subtractions:

$$F(s,u,t) = C_o(t) + C_1(t)(s-u) + \sum_{n=2}^{N} I_{1n}(t)s^n + I_{2n}(t)u^n) +$$

$$+ \frac{s^{N+1}}{\pi} \int^{\infty} \frac{A_1(s',t)}{(s'-s)s'^{N+1}} ds' + \frac{u^{N+1}}{\pi} \int^{\infty} \frac{A_2(u',t)}{(u'-u)u'^{N+1}} du' \qquad (3)$$

For $t < t_\epsilon$ a comparison of (2) and (3) shows that $C_o(t)$ and $C_1(t)$ are the same in the two expressions and:

$$I_{1n}(t) = \frac{1}{\pi} \int \frac{A_1(s',t)}{s'^{n+1}} ds' \qquad (4)$$

with a similar expression for $I_{2n}(t)$.

Now let us introduce the variable,

$$z = \frac{s-u}{4k_1k_2} \qquad (5)$$

where $k_1 = \frac{1}{2}\sqrt{t-4M_A{}^2}$, $k_2 = \frac{1}{2}\sqrt{t-4M_B{}^2}$ are the initial and final momenta in the center of mass system for process III and z = = cos θ where θ is the scattering angle. In the region we are considering both k_1 and k_2 are pure imaginary and the product is real and negative. One can express s and u in terms of z and t by:

$$2k_1k_2z = s + k_1^2 + k_2^2 = -(u+k_1^2+k_2^2) \tag{6}$$

Therefore, since k_1^2 and k_2^2 are negative in the expansion of s^n or u^n in power series of z all the coefficients of even powers are positive. On the other hand one can expand z^p in Legendre polynomials of order $\ell \leqslant p$ and $(\ell - p)$ even. Again in this expansion all the coefficients are positive. Therefore one can finally write:

$$\sum_{n=2}^{N} \left(I_{1n}(t)s^n + I_{2n}(t)u^n \right) = \sum_{\ell=0}^{N} C_\ell(t)\; P_\ell(z) \tag{7}$$

where

$$C_\ell(t) = \sum_{n=\ell}^{N} \mu_{\ell n}(t) \left[I_{2n}(t) + (-1)^\ell\; I_{1n}(t) \right] \tag{8}$$

and for even ℓ, all the $\mu_{\ell n}$ are positive. (Actually the $\mu_{\ell n}$'s are all positive definite for both even and odd ℓ.) In the real interval $0<t<4m^2$ the only singularities of $F(s, u, t)$ as a function of t are poles corresponding to bound states in the crosssed channel III. Let $t_1, t_2, \ldots t_k$ be the energies of these bound states, $\ell_1, \ell_2, \ldots \ell_k$ the corresponding angular momenta. In the neighbourhood of $t = t_r$ all the coefficients $C_\ell(t)$ are regular except $C_{\ell_r}(t)$, which has a pole at $t = t_r$. It is then clear, by the result of Jin and Martin [4] that the representation (2) is valid all through the interval $0 \leqslant t < t_1'$, where t_1' is the first bound state with angular momentum larger than one. Since the residue at this pole behaves like $|z|^{\ell_1'}$ and, at least in the direction of the imaginary axis $|F(s,t_1'-\epsilon)|<c|s|^2$ it follows that $\ell_1' = 2$.

Let us next consider the sequence of bound states with increasing energies $t_1', t_2', \ldots t_n'$ and <u>even</u> angular momenta $\ell_1', \ell_2', \ldots \ell_n'$ such that ℓ_i' is larger than the angular momenta of all bound states preceding t_i'. Let us suppose that in the interval $0 \leqslant t < t_1'$ the representation (3) is valid with $N = \ell_i' - 1$. Then by a slight generalization of the argument of reference 4 one can show that in the interval $0 \leqslant t < t_{i+1}'$ the representation (3) is valid with $N = \ell_i' + 1$. We shall give the main steps in the proof.

For $t < t_i'$, $I_{1,2n}(t)$ is given by (4) when $n \geqslant \ell_i'$. Since $A_{1,2}(s'',t)$ and all its derivatives with respect to t are positive (for $s' > s_o$) one can expand $A(s',t)$ in power series of t with positive coefficients. It is then allowed to interchange the order of summation and integration in (4).[5] One, thus, obtains a power series representation for $I_n(t)$ with positive coefficients. If t' is the radius of convergence of this series then it is also the first singularity of $I_n(t)$ and vice-versa, and for $t < t'$ the integral representation still holds.[5] Now since the coefficients $\mu_{\ell n}(t)$ in (8) are all positive analytic functions of t then, for even $\ell > \ell_i'$, t' is also a singularity of $C_\ell(t)$. Since by hypotheses, all $C_\ell(t)$ with even $\ell > \ell_i'$ are singular in the interval $0 \leqslant t < t_{i+1}'$, it follows that the representation (4) holds for $n \geqslant \ell_i' + 2$ and therefore $F(s,u,t)$ may be represented by (3) with $N = \ell_i' + 1$. Thus our assertion is proved. Since this result is true in the interval $0 \leqslant t < t_1'$ its validity in general follows by complete induction. Now using the same argument as before one deduces that:

$$\ell'_{i+1} = \ell'_i + 2 \qquad (9)$$

It may happen that in the interval $\left(t'_i, t'_{i+1}\right)$ there exists a bound state t_j with angular momentum $\ell_j = \ell'_i + 1$. Since for odd ℓ the expression (8) involves the difference of the two functions $I_{1n}(t)$ and $I_{2n}(t)$ it is not in general true that for $t < t_j$ (3) holds with $N = \ell_j - 1$. It is however obvious that, for $t \geqslant t_j$, at least $\ell_j + 1$ subtractions are required.

From the above considerations it is clear that if the angular momentum ℓ_j (even or odd) of a bound state t_j is larger than all the preceding ones the angular momentum of the next bound state with the same property is either $\ell_j + 1$, or $\ell_j + 2$ if ℓ_j is even.

Another result which emerges from this analysis is that for all the bound states t'_i as previously defined, the residues are negative. In fact as one approaches the pole t'_i from below, $C_{\ell'_i}(t)$ will be given by (8) and is positive. Therefore the residue is negative.

Let us now define a function $\alpha(t)$ as the limiting value of the set of real numbers α_i for which both integrals

$$I_{1,2\alpha_i} = \int_{s_0}^{\infty} \frac{A_{1,2}(s',t)}{s'^{\alpha_i + 1}}\, ds' \qquad (10)$$

are convergent.[6] We shall first show that $\alpha(t)$ is continuous in the open interval $(0,4m^2)$. Let us take in the t plane three circles with origin at $t = 0$ and increasing radii t, $t+\delta$ and $t_o = 4m^2$ respectively. These circles are inside the domain $\mathcal{D}$ of analyticity in t of $A(s,t)$ and on each circle $|A(s,t)|$ is maximum on the positive real axis. Then applying to $A(s,t)$, Hadamard's three circles theorem [7] one obtains:

$$A(s,t+\delta) < A(s,t)^{\xi_1} A(s,t_o)^{\xi_2} \tag{11}$$

where

$$\xi_1 = \ln\left(\frac{t_o}{t+\delta}\right)/\ln\left(\frac{t_o}{t}\right); \quad \xi_2 = \ln\left(\frac{t+\delta}{t}\right)/\ln\left(\frac{t_o}{t}\right) \tag{12}$$

and $\xi_1 + \xi_2 = 1$.

Since we are excluding the points $t = 0$ and $t = t_o$ one can take $\delta_o < t < t_o - \delta_o$ where δ_o is arbitrarily small. Then for $\delta < \delta_o$ one has:

$$\xi_2 < \delta \left[t \ln \frac{t_o}{t}\right]^{-1} < 2\,\frac{\delta}{\delta_o} \tag{13}$$

But $A(s,t_o)$ is bounded by $\left(\frac{s}{s_o}\right)^N$, therefore (11) gives:

$$A(s,t+\delta) < A(s,t)\left(\frac{s}{s_o}\right)^{k\delta} \tag{14}$$

where $k = 2N/\delta_o$. Therefore given an ϵ one can choose a $\delta_1 = \epsilon/k$ such that for $\delta < \min.\{\delta_o, \delta_1\}$ one has:

$$\int_{s_o}^{\infty} \frac{A(s,t+\delta)}{s^{\alpha(t)+\epsilon+1}}\,ds < s_o^{-k\delta}\int_{s_o}^{\infty} \frac{A(s,t)}{s^{\alpha(t)+k(\delta_1-\delta)+1}} < \infty \tag{15}$$

Hence $|\alpha(t+\delta) - \alpha(t)| < \epsilon$ so that $\alpha(t)$ is continuous. If $F(s,u,t)$ has a Regge behaviour, $\alpha(t)$ coincides with the Pomeranchuk trajectory. However, even in the general sense as defined above $\alpha(t)$ has the properties of the Pomeranchuk trajectory in the interval $(0, 4m^2)$, namely that, in the (ℓ, t)- plane the leading poles with even angular momentum and quantum numbers of the vacuum lie on $\alpha(t)$ and all the others lie on or below this curve.

Finally let us assume that $A(s,t)$ is actually bounded by $s^{\alpha(t)+\epsilon}$ for whatever small ϵ, and that $\alpha(t)$ is analytic inside the parabola:

$$\mathrm{Re}\,\sqrt{t} = \sqrt{t_o} = 2m \qquad (16)$$

This parabola is the limit as $k^2 \longrightarrow \infty$, of the ellipse of convergence of the Legendre polynomial expansion. Since in the Legendre polynomial expansion of $A(s,t)$ all the coefficients are positive, for all t on or inside the parabola (10) $\mathrm{Re}\,\alpha(t)$ has an absolute maximum at $t = t_o$. Then for λ real and positive

$$\varphi(t) = \exp \lambda \left[\alpha(t) - \alpha(t_o)\right]$$

is bounded by one in the same region. Now the interior of the parabola is analytically mapped into the interior of the unit circle by the transformation [8]:

$$z = \mathrm{tg}^2 \left(\frac{\pi}{4}\sqrt{\frac{t}{t_o}}\right) \qquad (17)$$

Therefore one can apply Pick's inequality [9] to the function $\varphi[t(z)]$. One obtains (for real positive t):

$$e^{\lambda\alpha(t)} < \frac{e^{\lambda\alpha(0)} + z e^{\lambda\alpha(t_o)}}{1 + z e^{\lambda\left[\alpha(0) - \alpha(t_o)\right]}} \qquad (18)$$

In the limit $\lambda \rightarrow 0$, (12) becomes:

$$\alpha(t) < \alpha(0) + \frac{2z}{1+z}\left[\alpha(t_o) - \alpha(0)\right]$$

or

$$\alpha(t) < \alpha(t_o) - \cos\left(\frac{\pi}{2}\sqrt{\frac{t}{t_o}}\right)\left[\alpha(t_o) - \alpha(0)\right] \qquad (19)$$

which is an upper bound for $\alpha(t)$ joining the values at $t = 0$ and $t = t_o$. Considering that the absence of $\pi - \pi$ bound states imply $\alpha(t_o) < 2$ and since $\alpha(0) \leqslant 1$, an absolute upper bound for the Pomeranchuk trajectory in the interval $(0, 4m^2)$ is:

$$\alpha(t) = 2 - \cos\left(\frac{\pi}{2}\sqrt{\frac{t}{t_o}}\right) \qquad (20)$$

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