

Functional Methods in the Generalized Dicke Model

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Abstract

The Dicke model describes an ensemble of N identical two-level atoms (qubits) coupled to a single quantized mode of a bosonic field. The fermion Dicke model should be obtained by changing the atomic pseudo-spin operators by a linear combination of Fermi operators. The generalized fermion Dicke model is defined introducing different coupling constants between the single mode of the bosonic field and the reservoir, g_1 and g_2 for rotating and counter-rotating terms respectively. In the limit $N \rightarrow \infty$, the thermodynamic of the fermion Dicke model can be analyzed using the path integral approach with functional method. The system exhibits a second order phase transition from normal to superradiance at some critical temperature with the presence of a condensate. We evaluate the critical transition temperature and present the spectrum of the collective bosonic excitations for the general case ($g_1 \neq 0$ and $g_2 \neq 0$). There is quantum critical behavior when the coupling constants g_1 and g_2 satisfy $g_1 + g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$, where ω_0 is the frequency of the mode of the field and Ω is the energy gap between energy eigenstates of the qubits. Two particular situations are analyzed. First, we present the spectrum of the collective bosonic excitations, in the case $g_1 \neq 0$ and $g_2 = 0$, recovering the well known results. Second, the case $g_1 = 0$ and $g_2 \neq 0$ is studied. In this last case, it is possible to have a superradiant phase when only virtual processes are introduced in the interaction Hamiltonian. Here also appears a quantum phase transition at the critical coupling $g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$, and for larger values for the critical coupling, the system enter in this superradiant phase with a Goldstone mode.

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1 Introduction

The aim of the present paper is to investigate the thermodynamic in a generalization of the Dicke model [1], using the path integral approach with the functional integration method. We are considering the question of how does the counter-rotating terms of the interaction Hamiltonian contribute to the system exhibits a phase transition from normal to superradiance at some critical temperature β_c^{-1} .

We are particularly interested in consider N identical two-level atoms interacting with a quantized bosonic field, i.e., the spin-boson problem. The most discussed situation in the literature is a system of atoms interacting with a reservoir which consists of a surrounding quantized electromagnetic field. In this situation the frequencies associated with the electromagnetic field are continuously distributed and the atomic systems are characterized by a denumerable set of frequencies. Nevertheless, different situations has been extensively studied. One is the case of electromagnetic field in cavities where the field modes are described by a set of denumerable frequencies [2] [3]. One particular example is the Jaynes-Cummings model [4], where was investigated the dynamics of a single two-level atom interacting with one mode of a quantized bosonic field in an ideal lossless cavity. The eigenstates of the model, i.e., the dressed states of the two-level atom-bosonic mode were first obtained by these authors. After these important papers, the spin-boson problem has been studied by many authors, receiving considerable discussions. The development of new techniques has lead to increase the interest in spontaneous radiations effects in such systems. More recently, this kind of model has been used also to analyze dissipation in quantum computers. A model describing a two-level atom coupled to a reservoir of harmonic oscillators was introduced by Di Vincenzo [5], to study decoherence in quantum computers. This model presents destruction of quantum coherence without decay of populations, owing to a quite particular coupling between the qubit and the reservoir.

In the following we are discussing the Dicke model [1], where a single quantized mode of a bosonic field interacts with a reservoir of N identical two-level atoms (qubits) at temperature β^{-1} . One method to obtain information from the small system of interest, eliminating the reservoir degrees of freedom is to use the master equations, introducing the reduced density operator [6] [7]. An alternative and non-perturbative powerful tool to deal with the elimination of the reservoir degrees of freedom, is the path integral approach with functional integration method [8]. In a general situation, this method allow us to obtain the semiclassical equation of motion and also gives a systematic way for the evaluation of quantum corrections.

Although functional integrals have been used in different area of physics, as for example quantum field theory and also statistical mechanics, in the past this technique have no found much use in quantum optics. An exception is the Zardecki paper [9]. Other paper that deserves to call attention is the Hillery and Zubaire [10] work. In this paper the path integral approach in quantum optics was developed, where systems with a few bosonic modes were studied. Using the Schrodinger-Glauber-Sudarshan's coherent states [11] [12] [13] [14] [15] [16] representation of these modes, these authors studied the propagator in a single mode system and also the case of N modes of a bosonic field. Since, in the Dicke model, the coupling between the N qubits and the

single quantized mode of the bosonic field is linear, in order to analyze the nonanalytic behavior of thermodynamic quantities, the integration over the degrees of freedom of the environment can be performed exactly.

To apply the path integral approach with functional method, as a technique to investigate the thermodynamic of the Dicke model, two steps are mandatory. First, it is necessary to change the atomic pseudo-spin operators of the model by a linear combination of Fermi operators to define the fermion Dicke model [17] [18]. A generalization of this model can be achieved introducing the counter-rotating terms in the interaction Hamiltonian. Second, the thermodynamic limit ($N \rightarrow \infty$), where N is the number of qubits must be taken [19] [20]. Therefore, studying the fermion Dicke model with counter-rotating terms, describing virtual processes, we are interested to evaluate the critical transition temperature of the model. In this generalized model, one may ask the importance of the contribution of each of these processes, real ones and virtual ones in the formation of the condensate.

A point that it is worth commenting upon is the relationship between our questions concerning the contributions of the real and virtual processes in the superradiant phase transition and how does the vacuum fluctuations and the self-reaction contribute to the radiative processes in atoms. Many authors [21] [22] [23] used the Heisenberg picture to identify the vacuum and the self-reaction contributions in radiative processes in atoms. Although there was some attempts to separate the contribution of the vacuum fluctuations and radiative reaction in the spontaneous decay [24] [25], Dalibard et al [26] [27] discussed how does the magnitude of these separate effects can varied by means of the particular ordering chosen for atomic and creation and annihilation field operators. When normal ordering is used, the source field gives the total contribution to the decay, and for anti-normal ordering, both the source field and vacuum field gives equal contributions to the spontaneous decay. Finally, if the symmetric ordering is chosen, the vacuum and the source field gives different contributions for the decay. There is no ordering where the spontaneous emission can be entirely associated to the vacuum field [28].

As we already emphasized, we are interested to study the nonanalytic behavior of thermodynamics quantities near a phase transition in the fermion Dicke model where the counter-rotating terms are also presented. Introducing different coupling constants, g_1 and g_2 for rotating and counter-rotating terms respectively, we are able to identify the contribution of each of the processes, real ones and virtual ones in the formation of the condensate. The order parameter of the transition is the expectation value of the number of excitations associated to the mode of the bosonic field per atom. We evaluate the critical transition temperature of the model and present the spectrum of the collective bosonic excitations in the general case, for the case $g_1 \neq 0$ and $g_2 = 0$ and also $g_1 = 0$ and $g_2 \neq 0$. In both cases it appear a critical behavior with Goldstone (gapless) modes. In the last case, it appears a quantum phase transition [29] at critical coupling $g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$. Therefore, in the fermion Dicke model with the counter-rotating terms it is possible to have a quantum phase transition and a superradiant phase, where the radiation rate on the number N of atoms becomes a quadratic dependence [30], in opposite to the situation of N atoms radiating incoherently, where the radiation rate is proportional on the number of the atoms N .

At this point we would like to make a summary of results concerning the thermodynamic of the Dicke model. An important result were obtained by Hepp and Lieb [31]. These author presented the free energy of the model in the thermodynamic limit. For a sufficiently large value for the coupling constant between the qubits and the single quantized mode of the bosonic field, the model present a second order phase transition from the normal to the superradiant phase. Latter, using a coherent state representation they generalize some results for atoms of more than two-levels without assume the rotating-wave approximation. They also investigated the stability of the model with an infinite number of bosonic modes [32]. The study of the phase transitions in the Dicke model was presented also by Wang and Hioe [33], where some of the results obtained by Hepp and Lieb were rederived. The generalized Dicke model, where the counter-rotating terms are also present in the interaction Hamiltonian, was investigated by Hioe [34] and also Duncan [35]. Hioe studied the thermodynamic of the generalized Dicke model with two different coupling constants using the Schrodinger-Glauber-Sudarshan's coherent states. Duncan proved that the critical temperature of the generalized model is higher than the temperature of the original model. Also a bosonization procedure was employed to study the phase transitions in the generalized Dicke model. Employing a Holstein-Primakoff mapping [36] [37] [38], which express the angular momentum in terms of a single bosonic mode, Emary and Brandes [39] [40] were able to express the generalized Dicke model in terms of two mode bosonic field. These authors discussed the relation between the quantum phase transition and the chaotic behavior that appear in the model for finite N . An indication of quantum chaos is the change in energy level-spacing statistics from Poissonian to being described by the Gaussian ensemble of random matrix theory. This chaotic behavior was discussed also in the Jaynes-Cummings model by Graham and Hoherbach [41] [42] and Lewenkopf et al. [43], in the situation where the counter-rotating terms are present in the interaction Hamiltonian [44] [45], since the seminal paper of Milonni et al. [46]

We would like to stress that we are not doing any distinction between the elementary unit of quantum information, the qubit and the two-level atom. Our intention in this paper is to review some important spin-boson models and also the basic ideas of the path integral approach with functional methods that can be used in these systems. This paper is organized as follows. In section II we discuss two-level atoms-boson field interaction Hamiltonians. In section III the Hilbert space for the fermion Dicke model is discussed. In the section IV the path integral with functional integral method is applied to study the thermodynamic of the generalized fermion Dicke model. We evaluate the critical transition temperature of the model for the general case ($g_1 \neq 0$ and $g_2 \neq 0$) and also present the spectrum of the collective bosonic excitations, for the case $g_1 \neq 0$ and $g_2 = 0$ and also for the case $g_1 = 0$ and $g_2 \neq 0$. Conclusions are given in section V. In the paper we use $k_B = c = \hbar = 1$.

2 The two-level atoms-Bose field Hamiltonians

In order to describe the dynamics of the reservoir and the small system, we have to introduce the Hamiltonian governing the interaction of the quantized bosonic field with free non-identical two-level atoms. Free means that there is no interaction between the two-level atoms (qubits). We are assuming that the qubits are enclosed in a very large lossless cavity, and the distance between the qubits is large enough, so that the interaction between the qubits can be neglected.

The main purpose of this section is to review a number of relevant qubit-Bose field Hamiltonians. This discussion is standard in the literature. For a recent treatment see for example the Ref. [47]. It is worth mentioning that there are a variety of theoretical models of reservoirs. The first situation is when the system S is coupled to an infinite number of harmonic oscillators. In this situation there are two kinds of reservoir of common interest. The first one is a thermal reservoir, where we assume that the harmonic oscillators are in thermal equilibrium at temperature β^{-1} . The second one is a squeezed reservoir. The specific system-reservoir model which is appropriate for the study of several interesting situations is when the harmonic oscillator bath is constituted by a bosonic field in free space also in the presence of macroscopic structures.

Therefore let us consider a bosonic quantum system S , with Hilbert space $\mathcal{H}^{(S)}$ which is coupled with the reservoir of qubits, with Hilbert space $\mathcal{H}^{(B)}$. Let us assume that the reservoir is in thermal equilibrium at temperature β^{-1} . The bosonic quantum system is a sub-system of the total system living in the tensor product space $\mathcal{H}^{(S)} \otimes \mathcal{H}^{(B)}$.

Let us denote by H_S the Hamiltonian of the quantized bosonic field (where the "S" means small system), by H_B the free Hamiltonian of the N -qubits (where the "B" means bath) and H_I the Hamiltonian describing the interaction between the quantized bosonic field and the qubits reservoir. The Hamiltonian for the total system can be written as

$$H = H_S \otimes I_B + I_S \otimes H_B + H_I, \quad (1)$$

where I_S and I_B denotes the identities in the Hilbert spaces of the quantized bosonic field and the qubit reservoir.

The free j -th qubit Hamiltonian will be denoted by $H_D^{(j)}$. Therefore we have

$$H_D^{(j)} |i\rangle_j = \omega_i^{(j)} |i\rangle_j, \quad (2)$$

where $|i\rangle_j$ are orthogonal energy eigenstates accessible to the j -th qubit and $\omega_i^{(j)}$ are the respective eigenfrequencies. Using Eq. (2) and the orthonormality of the energy eigenstates we can write the j -th qubit Hamiltonian $H_D^{(j)}$ as

$$H_D^{(j)} = \sum_{i=1}^2 \omega_i^{(j)} (|i\rangle \langle i|)_j. \quad (3)$$

Throughout the article, for the sake of simplicity in the notation, we are assuming that the operator $\left(|i\rangle\langle i|\right)_j \equiv |i\rangle_j\langle i|$, and the same convention is used for the other operators.

Let us define the pseudo-spin operators for each qubit $\sigma_{(j)}^z$, and the raising and lowering pseudo-spin operators $\sigma_{(j)}^+$ and $\sigma_{(j)}^-$, where each of these operators are given respectively by

$$\sigma_{(j)}^z = \left(|2\rangle\langle 2| - |1\rangle\langle 1|\right)_j, \quad (4)$$

$$\sigma_{(j)}^+ = \left(|2\rangle\langle 1|\right)_j \quad (5)$$

and finally

$$\sigma_{(j)}^- = \left(|1\rangle\langle 2|\right)_j. \quad (6)$$

This representation is a second quantization of the qubits. Combining Eq. (3) and Eq. (4), the j -th qubit Hamiltonian can be written as

$$H_D^{(j)} = \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \frac{1}{2} \left(\omega_1^{(j)} + \omega_2^{(j)} \right), \quad (7)$$

where the energy gap between the energy eigenstates of the j -th qubit is given by

$$\Omega^{(j)} = \omega_2^{(j)} - \omega_1^{(j)}. \quad (8)$$

Shifting the zero of energy to $\frac{1}{2}(\omega_1^{(j)} + \omega_2^{(j)})$ for each qubit, the j -th qubit Hamiltonian given by Eq. (7) can be rewritten as

$$H_D^{(j)} = \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z. \quad (9)$$

Note that the pseudo-spin operators $\sigma_{(j)}^+$, $\sigma_{(j)}^-$ and $\sigma_{(j)}^z$ satisfy the standard angular momentum commutation relations corresponding to spin $\frac{1}{2}$ operators, i.e.,

$$\left[\sigma_{(j)}^+, \sigma_{(j)}^- \right] = \sigma_{(j)}^z, \quad (10)$$

$$\left[\sigma_{(j)}^z, \sigma_{(j)}^+ \right] = 2 \sigma_{(j)}^+, \quad (11)$$

and finally

$$\left[\sigma_{(j)}^z, \sigma_{(j)}^- \right] = -2 \sigma_{(j)}^-. \quad (12)$$

To introduce the coupling between the bosonic field and the qubits, let us assume a single quantized mode of a bosonic field with a linear coupling with the qubits. Therefore, the Hamiltonian of the j -th qubit $H_D^{(j)}$, with the contribution of the single quantized mode of the bosonic field H_S , and the interaction Hamiltonian $H_I^{(j)}$ is given by

$$\begin{aligned} I_S \otimes H_D^{(j)} + H_S \otimes I_B + H_I^{(j)} = \\ I_S \otimes \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + g (b + b^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-), \end{aligned} \quad (13)$$

where the second term in the Eq. (13) has the contribution from the single quantized mode of the bosonic field Hamiltonian and the last term is the interaction Hamiltonian of the j -th qubit with the one-mode quantized field. In the above equation g is a small coupling constant between the qubit and the single mode of the bosonic field. The b and b^\dagger are the boson annihilation and creation operators of mode excitations that satisfy the usual commutation relation rules. The generalization to N qubits is described by

$$\begin{aligned} I_S \otimes \sum_{j=1}^N H_D^{(j)} + H_S \otimes I_B + \sum_{j=1}^N H_I^{(j)} = \\ I_S \otimes \sum_{j=1}^N \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + \frac{g}{\sqrt{N}} \sum_{j=1}^N (b + b^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \end{aligned} \quad (14)$$

This Hamiltonian is a possible generalization of the Dicke model. The interaction Hamiltonian given by Eq.(14) is simplified if we ignore the counter-rotating terms. Considering this approximation we have

$$\begin{aligned} I_S \otimes \sum_{j=1}^N H_D^{(j)} + H_S \otimes I_B + \sum_{j=1}^N H_I^{(j)} = \\ I_S \otimes \sum_{j=1}^N \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + \frac{g}{\sqrt{N}} \sum_{j=1}^N (b \otimes \sigma_{(j)}^+ + b^\dagger \otimes \sigma_{(j)}^-). \end{aligned} \quad (15)$$

This Hamiltonian describes the original Dicke model in the situation $\Omega^{(j)} = \Omega, \forall j$. This approximation is known as the rotating-wave approximation. In the rotating-wave approximation we ignore energy non-conserving terms in which the emission (absorption) of a quantum of a quantized field is accompanied by the transition of one qubit from its lower (upper) to its upper (lower) state. The rotating-wave approximation ignores terms in which the j -th qubit raising (lowering) operators multiplies the mode of the field creation (annihilation) operator. Although the rotating-wave approximation has been used to describes the interaction of radiation with matter, as we

stressed, there are many situations where we can not assume the above mentioned approximation, as for example if we go beyond the dipole approximation, to describe Casimir-Polder forces [48], the Lamb-shift and also when atomic systems generate high intensity fields.

Situations where we can not assume the rotating-wave approximation has been also extensively discussed by one of the authors. First Svaiter and Svaiter [49] [50] evaluated the transition rates of a two-level atom in different kinematic situations assuming a weak coupling between this small system and a real massless scalar field. These authors studied the Unruh-Davies effect [51] [52], where an accelerated two-level atom measures a thermal spectrum even if the field is prepared in the vacuum state. The origin of this effect is the presence of an event horizon that change virtual processes in real ones. In the Ref. [53] still studying the same two-level system coupled to a real scalar field, Ford et al assumed the presence of two infinite perfectly reflecting plates which change the vacuum fluctuations associated to the quantized bosonic field. The image method and the imaginary time formalism [54] [55] [56], was used to study radiative processes at finite temperature. In the Refs. [57] [58] one of the authors continue to investigate radiative processes associated to the Unruh-Dewitt detector [59], in interaction with a massless scalar field. Being more precise, in the Ref. [58], it was presented the detector's excitation rate when it is uniformly rotating around some fixed point, when the scalar field is prepared in the Minkowski vacuum.

Another model, where the behavior is quite interesting from the mathematical and physical point of view is the one where the coupling between the N qubits and the bosonic field in a lossless cavity is intensity dependent. We have

$$\begin{aligned}
 I_S \otimes \sum_{j=1}^N H_D^{(j)} + H_S \otimes I_B + \sum_{j=1}^N H_I^{(j)} = \\
 I_S \otimes \sum_{j=1}^N \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + \frac{g}{\sqrt{N}} \sum_{j=1}^N \left(b (b^\dagger b)^{\frac{1}{2}} \otimes \sigma_{(j)}^+ + b^\dagger (b^\dagger b)^{\frac{1}{2}} \otimes \sigma_{(j)}^- \right). \quad (16)
 \end{aligned}$$

Going back to the case of only one qubit, the two-level atom coupled to a single mode quantized electromagnetic field is known as the Jaynes-Cummings model. The Jaynes-Cummings model for one qubit, can be written as

$$\begin{aligned}
 I_S \otimes H_D^{(j)} + H_S \otimes I_B + H_I^{(j)} = \\
 I_S \otimes \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + g \left(b \otimes \sigma_{(j)}^+ + b^\dagger \otimes \sigma_{(j)}^- \right). \quad (17)
 \end{aligned}$$

Practical realization of this model in the laboratory is not a problem. First we consider only one two-level atom in the presence of a quantized bosonic field in a cavity. Assuming that the frequency ω_0 of one of the cavity modes is near-resonant with the energy gap Ω of the two-level atom, such situation generates the following physical model. The two-level atom effectively interacts only with that mode, and all the other bosonic modes do not couple with the two-level atom.

This model provides a useful means of studying nonclassical effects in the interaction between fields and matter, as for example the phenomenon of collapses and revivals of the Rabi oscillations in a quantized field that is not in a pure number state [60] [61] [62] [63]. It is important to point out that the behavior of the two-level atom depends if the single field mode is quantized or not. In the case of a classical field the model presents atomic population inversion with monotonic periodic oscillations. Assuming that the single field mode is quantized the system presents two different dynamical behavior. If we prepare the atom in the ground state and the field mode in a pure number state, it can be shown that the expectation value of the pseudo-spin operator σ^z is the same as in the case of a classical field. If we prepare the atom in the excited state and the field mode in a pure number state, the atom periodically returns to the upper state with a definite Rabi frequency. For the particular case of the atom in an empty cavity ($n = 0$), the two-level system undergoes periodic, reversible spontaneous decay. Finally, preparing the field mode in a coherent state, the dynamics of the two-level system present quantum collapse and also revival effects.

It is possible to generalize the Jaynes-Cummings model, still using the rotating-wave approximation in the following way

$$\begin{aligned} I_S \otimes H_D^{(j)} + H_S \otimes I_B + H_I^{(j)} = \\ I_S \otimes \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + g \left(b^2 \otimes \sigma_{(j)}^+ + (b^\dagger)^2 \otimes \sigma_{(j)}^- \right). \end{aligned} \quad (18)$$

In the literature the above model is known as the two-photon Jaynes-Cummings model [64] [65].

The same idea that has been used in the Eq. (16) can also be implemented in the Jaynes-Cummings model. The Hamiltonian for the Jaynes-Cummings model with a intensity-dependent coupling constant where we are still using the rotating-wave approximation is given by [66] [67] [68]

$$\begin{aligned} I_S \otimes H_D^{(j)} + H_S \otimes I_B + H_I^{(j)} = \\ I_S \otimes \frac{\Omega^{(j)}}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + g \left(b (b^\dagger b)^{\frac{1}{2}} \otimes \sigma_{(j)}^+ + b^\dagger (b^\dagger b)^{\frac{1}{2}} \otimes \sigma_{(j)}^- \right). \end{aligned} \quad (19)$$

So far we have discussed N qubits interacting with a single quantized mode of the bosonic field. Our aim is now to discuss the interaction of a system of N identical qubits with energy gap ($\Omega = \omega_2 - \omega_1$), with an infinite number of harmonic oscillators which defines the reservoir. Let a_k^\dagger and a_k be the creation and annihilation operators of the k -th harmonic oscillator of frequency ω_k . These creation and annihilation operators satisfy the standard commutation relations $[a_k^\dagger, a_q] = \delta_{kq}$, $[a_k, a_q] = 0$ and also $[a_k^\dagger, a_q^\dagger] = 0$. The total Hamiltonian, i.e., the Hamiltonian of the combined system of the reservoir and the N identical qubits interacting with the reservoir reads

$$H = I_R \otimes \frac{\Omega}{2} \sum_{j=1}^N \sigma_{(j)}^z + \sum_k \omega_k a_k^\dagger a_k \otimes I_S + \frac{g}{\sqrt{N}} \sum_{j=1}^N \sum_k \left(a_k \otimes \sigma_{(j)}^+ + a_k^\dagger \otimes \sigma_{(j)}^- \right). \quad (20)$$

In the Eq. (20) the first term in the right side is the free Hamiltonian of N identical qubits, the second term is the free harmonic oscillators reservoir Hamiltonian and finally the third term is the interaction Hamiltonian between the reservoir and the N identical qubits. Note that we shift the zero of energy for each qubit, as we did before, and we are assuming the rotating-wave approximation, where $g(\sqrt{N})^{-1}$ is the j -th qubit, k -th harmonic oscillator coupling constant. We also can use a different interaction Hamiltonian to study the influence of decoherence in quantum computers as was introduced by Di Vincenzo [5]. This author proposed the following model describing a system of one qubit coupled to a reservoir of harmonic oscillators:

$$I_R \otimes H_S + H_R \otimes I_S + H_I = I_R \otimes \frac{\Omega}{2} \sigma^z + \sum_k \omega_k a_k^\dagger a_k \otimes I_S + g \sum_k (a_k^\dagger + a_k) \otimes \sigma^z, \quad (21)$$

where Ω is the usual energy level spacing of the qubit, a_k^\dagger and a_k are respectively the boson creation and annihilation operators associated to the harmonic oscillators. Note the particular coupling between the reservoir and the qubit. It is well known that this model allow for an exact analytic solution and also exhibits the destruction of quantum coherence without decay of population. There are two straightforward generalizations for this model given by Eq. (21). The first one is to introduce N identical qubits and the total Hamiltonian for the composed system reads

$$I_R \otimes H_S + H_R \otimes I_S + H_I = I_R \otimes \frac{\Omega}{2} \sum_{j=1}^N \sigma_{(j)}^z + \sum_k \omega_k a_k^\dagger a_k \otimes I_S + \frac{g}{\sqrt{N}} \sum_{j=1}^N \sum_k (a_k^\dagger + a_k) \otimes \sigma_{(j)}^z. \quad (22)$$

Other generalization for the Di Vincenzo model is to introduce a arbitrary mode-dependent coupling constant [69]. Therefore we have the following model describing a system of one qubit coupled to a reservoir of harmonic oscillators:

$$I_R \otimes H_S + H_R \otimes I_S + H_I = I_R \otimes \frac{\Omega}{2} \sigma^z + \sum_k \omega_k a_k^\dagger a_k \otimes I_S + \sum_k (\lambda_k a_k^\dagger + \lambda_k^* a_k) \otimes \sigma^z, \quad (23)$$

Working in this line, the Hamiltonian describing a system of N qubit coupled to a reservoir of harmonic oscillators reads [70]

$$I_R \otimes H_S + H_R \otimes I_S + H_I = I_R \otimes \frac{\Omega}{2} \sum_{j=1}^N \sigma_{(j)}^z + \sum_k \omega_k a_k^\dagger a_k \otimes I_S + \frac{1}{\sqrt{N}} \sum_{j=1}^N \sum_k (g_{jk} a_k^\dagger + g_{jk}^* a_k) \otimes \sigma_{(j)}^z, \quad (24)$$

where g_{jk} describes the coupling between the j -th qubit with the reservoir.

After this briefly discussion of many models describing the two-level atoms-Bose field coupled system we would like to go back to Eq.(20). Going back to Eq. (20), another possibility is not assume the rotating-wave approximation in the interaction Hamiltonian. Without the rotating-wave approximation, the interaction Hamiltonian between the N qubits and the reservoir of harmonic oscillators reads

$$H_I = \frac{g}{\sqrt{N}} \sum_{j=1}^N \sum_k (a_k + a_k^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \quad (25)$$

Our aim is to show that the generalized fermion Dicke model where we are not assuming the rotating-wave-approximation also present a phase transition and present the spectrum of the collective bosonic excitations of the model. In the next section we will study the Hilbert space for the fermion Dicke model.

3 The Hilbert space for the fermion Dicke model

It should be emphasized that there are different routs to proceed in the study of cooperative spontaneous emission of a radiation field in the Dicke model. The first one is to introduce collective atomic operators

$$J^z = \sum_{i=1}^N \sigma_{(i)}^z, \quad (26)$$

$$J^+ = \sum_{i=1}^N \sigma_{(i)}^+, \quad (27)$$

and

$$J^- = \sum_{i=1}^N \sigma_{(i)}^-. \quad (28)$$

These operators J^z , J^+ and J^- obey the standard angular momentum commutation relations. The Hilbert space of this algebra is spanned by the Dicke states $|j, m\rangle$ which are eigenstates of J^2 and J^z . This method allow us to describe the N two-level atoms by a single $(N + 1)$ level system. A second one method as we stressed before, is to change the atomic pseudo-spin operators of the Dicke model by a linear combination of Fermi operators to define the fermion Dicke model. The dimensionality of the space where the combinations of Fermi operators act are greater than the dimensionality of the space where the pseudo-spin operators act, and therefore we have a problem of the elimination of the superfluous states. We start with the Hamiltonian of the Dicke model, H_D given by

$$H_D = I_S \otimes \frac{\Omega}{2} \sum_{i=1}^N \sigma_{(i)}^z + \omega_0 b^\dagger b \otimes I_B + \frac{g}{\sqrt{N}} \sum_{i=1}^N (b \otimes \sigma_{(i)}^+ + b^\dagger \otimes \sigma_{(i)}^-). \quad (29)$$

Note that the H_D Hamiltonian contains the pseudo-spin operators used to obtain a second quantized version for the free two-level atoms and also the creation and annihilation operators of the single quantized mode of the bosonic field. As we discussed before, the pseudo-spin operators $\sigma_{(i)}^z$, $\sigma_{(i)}^+$ and $\sigma_{(i)}^-$, for the same or different qubits obey the standard commutation relations

$$\left[\sigma_{(i)}^+, \sigma_{(j)}^- \right] = \sigma_{(j)}^z \delta_{(i)}^{(j)}, \quad (30)$$

$$\left[\sigma_{(i)}^z, \sigma_{(j)}^+ \right] = 2 \sigma_{(j)}^+ \delta_{(i)}^{(j)}, \quad (31)$$

and finally

$$\left[\sigma_{(i)}^z, \sigma_{(j)}^- \right] = -2 \sigma_{(j)}^- \delta_{(i)}^{(j)}, \quad (32)$$

where $\delta_{(i)}^{(j)}$ is the Kronecker delta.

Let us define the Fermi raising and lowering operators α_i^\dagger , α_i , β_i^\dagger and β_i , that satisfy the anti-commutator relations $\alpha_i \alpha_j^\dagger + \alpha_j^\dagger \alpha_i = \delta_{ij}$ and $\beta_i \beta_j^\dagger + \beta_j^\dagger \beta_i = \delta_{ij}$. We can also define the following bilinear combination of Fermi operators, $\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i$, $\alpha_i^\dagger \beta_i$ and finally $\beta_i^\dagger \alpha_i$. Note that $\sigma_{(i)}^z$, $\sigma_{(i)}^+$ and $\sigma_{(i)}^-$ obey the same commutation relations as the above presented bilinear combination of Fermi operators. This suggests that we can change the pseudo-spin operators of the Dicke model by the bilinear combination of Fermi operators

$$\sigma_{(i)}^z \longrightarrow (\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i), \quad (33)$$

$$\sigma_{(i)}^+ \longrightarrow \alpha_i^\dagger \beta_i, \quad (34)$$

and finally

$$\sigma_{(i)}^- \longrightarrow \beta_i^\dagger \alpha_i. \quad (35)$$

From now on we use the usual notation instead of the notation stressing the tensor product space of the total Hilbert space of the system. With the substitutions that we defined in the Eq. (33), Eq.(34) and Eq.(35) we shall call the resulting Hamiltonian as the fermion Dicke model, i.e., H_F . The Hamiltonian of the fermion Dicke model can be written as

$$H_F = \omega_0 b^\dagger b + \frac{\Omega}{2} \sum_{i=1}^N (\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i) + \frac{g}{\sqrt{N}} \sum_{i=1}^N (b \alpha_i^\dagger \beta_i + b^\dagger \beta_i^\dagger \alpha_i). \quad (36)$$

Note that in the Eq. (36) we are still adopting the rotating-wave approximation. We can also define the fermion Dicke model, without assume the rotating-wave approximation. The interaction Hamiltonian H_I of the single quantized mode of the bosonic field in the presence of the N qubits without the rotating-wave approximation is given by

$$H_I = \frac{g}{\sqrt{N}} \sum_{i=1}^N (b^\dagger + b) (\alpha_i^\dagger \beta_i + \beta_i^\dagger \alpha_i). \quad (37)$$

In effect, this interaction Hamiltonian includes four kind of processes that correspond to the absorption or emission of a mode-excitation of the field with the transition of the qubit from its lower (upper) to its upper (lower) state. Therefore the interaction Hamiltonian includes virtual processes or the vacuum fluctuations contributions. A simple generalization of Eq. (37) that we are adopting in the paper is to introduce two different coupling constants, the first one for the rotating-wave terms and another one for the counter-rotating terms. The advantage of this method is that it is possible to identify the contributions of the virtual processes and real ones in the formation of the bosonic condensate.

After this discussion, let us analyze the Hilbert space of the fermion Dicke model. The fermion Hilbert space for each atom is four dimensional. The Hilbert space for the i -th atom is generated by the following term vectors. The first one is the vacuum state

$$\phi_0 = |0, 0\rangle_i. \quad (38)$$

Next, we can define more two vectors applying the operators α_i^\dagger and β_i^\dagger on the vacuum state, i.e., $\alpha_i^\dagger\phi_0$ and $\beta_i^\dagger\phi_0$. Therefore we have

$$\alpha_i^\dagger|0, 0\rangle_i = |1, 0\rangle_i \quad (39)$$

and

$$\beta_i^\dagger|0, 0\rangle_i = |0, 1\rangle_i. \quad (40)$$

Finally, the bilinear combination of the Fermi operators ($\alpha_i^\dagger\beta_i^\dagger$) acting on the vacuum state yields

$$\alpha_i^\dagger\beta_i^\dagger|0, 0\rangle_i = |1, 1\rangle_i. \quad (41)$$

The vectors $|1, 0\rangle_i$ and $|0, 1\rangle_i$ generate a two-dimensional subspace, which is characterized by the following condition with the bilinear combination of Fermi operators

$$(\alpha_i^\dagger\alpha_i + \beta_i^\dagger\beta_i)|0, 1\rangle_i = |0, 1\rangle_i \quad (42)$$

and

$$(\alpha_i^\dagger\alpha_i + \beta_i^\dagger\beta_i)|1, 0\rangle_i = |1, 0\rangle_i. \quad (43)$$

We define $N_i = (\alpha_i^\dagger\alpha_i + \beta_i^\dagger\beta_i)$ as the fermion number operator acting in the Hilbert space corresponding to the i -th atom. With the four dimensional Hilbert space we can construct a physical Hilbert space and a nonphysical one generated by the vectors $|\Psi\rangle_i$ and $|\Phi\rangle_i$ respectively. Therefore we have

$$c_1^i|0, 1\rangle_i + c_2^i|1, 0\rangle_i = |\Psi\rangle_i \quad (44)$$

and

$$d_1^i|0, 0\rangle_i + d_2^i|1, 1\rangle_i = |\Phi\rangle_i. \quad (45)$$

Next we will obtain a formula which connect the partition function of spin Dicke model to the partition function of the fermion Dicke model. Using the definition of the fermion number operator of all qubits, and that $N_i = (\alpha_i^\dagger \alpha_i + \beta_i^\dagger \beta_i)$ we have

$$N = N_i + \sum_{j \neq i} N_j. \quad (46)$$

Since the number operator and the Hamiltonian operator commute and using the same notation we have

$$H_F = H_i + \sum_{j \neq i} H_j. \quad (47)$$

Using the fact that

$$H_i |0, 0\rangle_i = H_i |1, 1\rangle_i = 0, \quad (48)$$

we have that

$$H_i |\Phi\rangle_i = 0, \quad (49)$$

where $|\Phi\rangle_i$ is the general vector state in the nonphysical subspace of the i -th qubit. The trace over the nonphysical states of the i -th qubit vanishes. Therefore we have

$${}_i \langle \Phi | \exp \left[-\beta \left(H_i + \sum_{j \neq i} H_j + \frac{i\pi}{2\beta} N \right) \right] | \Phi \rangle_i = \quad (50)$$

$${}_i \langle \Phi | \exp \left[-\beta \left(H_i + \sum_{j \neq i} H_j \right) + \frac{i\pi}{2} \left(N_i + \sum_{j \neq i} N_j \right) \right] | \Phi \rangle_i = 0. \quad (51)$$

Therefore we have

$$Tr \exp \left[-\beta \left(H_F + \frac{i\pi}{2\beta} N \right) \right] = (-i)^N Tr_{phys} \exp[-\beta H_F] = (-i)^N Tr \exp[-\beta H_\sigma], \quad (52)$$

and we can present the relation between the partition function of the spin Dicke model to the partition function of the fermion Dicke model. A simple formula is given by

$$Tr \exp[-\beta H_\sigma] = i^N Tr \exp \left[\left(-\beta H_F + \frac{i\pi}{2} N \right) \right]. \quad (53)$$

According the Eq. (53), to study the spin Dicke model we can use the fermion Hamiltonian H_F adding $\frac{i\pi N}{2\beta}$, i.e. the fermion number operator with pure imaginary coefficient (chemical potential).

All the standard diagrammatic technique for the Fermi system can be generated using the Fourier representation for the Green function given by

$$G^0(\omega_F) = \frac{1}{i\omega_F - \epsilon + \mu} = \frac{1}{i\omega_F - \epsilon - \frac{i\pi}{2\beta}}, \quad (54)$$

where $\omega_F = \frac{2\pi}{\beta}(n + 1/2)$ is the fermion Matsubara frequency. After the above discussion we can consider the problem of define the partition function of the fermion Dicke model defined by Z_F .

4 The functional integral for the generalized fermion Dicke model

After the above discussion we can consider the problem of define the partition function Z_F of the generalized fermion Dicke model. First let us define the Euclidean action S of this model, which describes a single quantized mode of the field and the ensemble of N identical qubits. The Euclidean action S is given by

$$S = \int_0^\beta d\tau \left(b^*(\tau) \frac{\partial}{\partial \tau} b(\tau) + \sum_{i=1}^N \left(\alpha_i^*(\tau) \frac{\partial}{\partial \tau} \alpha_i(\tau) + \beta_i^*(\tau) \frac{\partial}{\partial \tau} \beta_i(\tau) \right) \right) - \int_0^\beta d\tau H_F(\tau) \quad (55)$$

where H_F is the full Hamiltonian for the generalized fermion Dicke model given by

$$\begin{aligned} H_F = & \omega_0 b^*(\tau) b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) - \beta_i^*(\tau) \beta_i(\tau) \right) + \\ & + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i^*(\tau) \beta_i(\tau) b(\tau) + \alpha_i(\tau) \beta_i^*(\tau) b^*(\tau) \right) + \\ & + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i(\tau) \beta_i^*(\tau) b(\tau) + \alpha_i^*(\tau) \beta_i(\tau) b^*(\tau) \right). \end{aligned} \quad (56)$$

Note that we are introducing two coupling constants, g_1 and g_2 , for the rotating and anti-rotating wave terms respectively. As we discussed before, the main reason for this is that we are interested to identify the contribution of the real and virtual processes in the phase transition with the formation of the condensate. Let us define the formal quotient of two functional integrals, i.e., the partition function of the generalized fermion Dicke model and the partition function of the free fermion Dicke model. Therefore we are interested in calculate the following quantity

$$\frac{Z_F}{Z_{F_0}} = \frac{\int [d\eta] e^S}{\int [d\eta] e^{S_0}}, \quad (57)$$

where $S = S(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the Euclidean action of the generalized fermion Dicke model given by Eq. (55), $S_0 = S_0(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the free Euclidean action for the free single bosonic mode and the free qubits, i.e., the expression of the complete action S taking $g_1 = g_2 = 0$ and finally $[d\eta]$ is the functional measure. The functional integrals involved in Eq. (57), are functional integrals with respect to the complex functions $b^*(\tau)$ and $b(\tau)$ and Grassmann Fermi fields $\alpha_i^*(\tau)$, $\alpha_i(\tau)$, $\beta_i^*(\tau)$ and $\beta_i(\tau)$. Since we are using thermal equilibrium boundary conditions, in the imaginary time formalism, the integration variables in Eq. (57) obey periodic boundary conditions for the Bose field, i.e., $b(\beta) = b(0)$ and anti-periodic boundary conditions for Fermi fields i.e., $\alpha_i(\beta) = -\alpha_i(0)$ and $\beta_i(\beta) = -\beta_i(0)$.

The free action for the single mode bosonic field $S_0(b)$ is given by

$$S_0(b) = \int_0^\beta d\tau \left(b^*(\tau) \frac{\partial b(\tau)}{\partial \tau} - \omega_0 b^*(\tau) b(\tau) \right). \quad (58)$$

Then we can write the action S of the generalized fermion Dicke model, given by Eq. (55), using the free action for the single mode bosonic field $S_0(b)$ given by Eq. (58), plus an additional term that can be expressed in a matrix form. Therefore the total action S can be written as

$$S = S_0(b) + \int_0^\beta d\tau \sum_{i=1}^N \rho_i^\dagger(\tau) M(b^*, b) \rho_i(\tau), \quad (59)$$

where $\rho_i(\tau)$ is a column matrix given in terms of fermion field operators

$$\begin{aligned} \rho_i(\tau) &= \begin{pmatrix} \beta_i(\tau) \\ \alpha_i(\tau) \end{pmatrix}, \\ \rho_i^\dagger(\tau) &= \begin{pmatrix} \beta_i^*(\tau) & \alpha_i^*(\tau) \end{pmatrix} \end{aligned} \quad (60)$$

and the matrix $M(b^*, b)$ is given by

$$M(b^*, b) = \begin{pmatrix} \partial_\tau + \Omega/2 & (N)^{-1/2} \left(g_1 b^*(\tau) + g_2 b(\tau) \right) \\ (N)^{-1/2} \left(g_1 b(\tau) + g_2 b^*(\tau) \right) & \partial_\tau - \Omega/2 \end{pmatrix}. \quad (61)$$

These fields $b(\tau)$, $\alpha_i(\tau)$ and $\beta_i(\tau)$ can be written as a Fourier expansion. Therefore we have

$$b(\tau) = \beta^{-1/2} \sum_{\omega} b(\omega) e^{i\omega\tau}, \quad (62)$$

and

$$\rho_i(\tau) = \beta^{-1/2} \sum_p \rho_i(p) e^{ip\tau}. \quad (63)$$

Since the field $b(\tau)$ obeys periodic boundary conditions, and the fields $\alpha_i(\tau)$ and $\beta_i(\tau)$ obey anti-periodic boundary conditions, we have that $\omega = \frac{2\pi n}{\beta}$ and $p = \frac{(2n+1)\pi}{\beta}$, where they are the boson and fermion Matsubara frequencies respectively. Substituting the Fourier expansions in the free action given by Eq. (58) we get

$$S_0(b) = \sum_{\omega} (i\omega - \omega_0) b^*(\omega) b(\omega). \quad (64)$$

Using the same procedure in the Eq. (59) allow us to write

$$S = S_0(b) + \sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(b^*, b) \rho_i(q), \quad (65)$$

where the matrix $M_{pq}(b^*, b)$ is given by

$$M_{pq}(b^*, b) = \begin{pmatrix} (ip + \Omega/2)\delta_{pq} & (N\beta)^{-1/2} \left(g_1 b^*(q-p) + g_2 b(p-q) \right) \\ (N\beta)^{-1/2} \left(g_1 b(p-q) + g_2 b^*(q-p) \right) & (ip - \Omega/2)\delta_{pq} \end{pmatrix}. \quad (66)$$

Using the above results, the ratio between the two functional integrals Z and Z_0 , i.e., $\frac{Z}{Z_0}$ is given by

$$\frac{\int [d\eta(b)] \exp\left(\sum_{\omega} (i\omega - \omega_0) b^*(\omega) b(\omega)\right) \int [d\eta(\rho)] \exp\left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(b^*, b) \rho_i(q)\right)}{\int [d\eta(b)] \exp\left(\sum_{\omega} (i\omega - \omega_0) b^*(\omega) b(\omega)\right) \int [d\eta(\rho)] \exp\left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(0, 0) \rho_i(q)\right)} \quad (67)$$

where the functional measures $[d\eta(b)]$ and $[d\eta(\rho)]$ in the above equation are defined respectively by

$$[d\eta(b)] = \prod_{\omega} db(\omega) db^*(\omega) \quad (68)$$

and

$$[d\eta(\rho)] = \prod_{i,p} d\rho_i(p) d\rho_i^\dagger(p). \quad (69)$$

We need impose cutoffs over the boson and fermion Matsubara frequencies on these measures. This procedure is necessary to be sure that the ratio between the two functional integrals given by $\frac{Z}{Z_0}$ does not diverge. After all, at the end, we must take these cutoffs to infinity. In order to define the effective action associated to the bosonic mode, we integrate out the fermionic degrees

of freedom. The integrals with respect to the Fermi fields are Gaussian and we may integrate over these Grassmann variables. This procedure yields

$$\int [d\eta(\rho)] \exp\left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(b^*, b) \rho_i(q)\right) = \det^N M(b^*, b), \quad (70)$$

where the matrix M is a block matrix of the following form

$$M(b^*, b) = \begin{pmatrix} (ip + \Omega/2) I & (N\beta)^{-1/2} Q^\dagger \\ (N\beta)^{-1/2} Q & (ip - \Omega/2) I \end{pmatrix}, \quad (71)$$

where I is the identity matrix and the components of matrix Q are

$$Q_{pq} = g_1 b(p - q) + g_2 b^*(q - p). \quad (72)$$

The following change of coordinates can simplify our calculations. Let us change variables in the following way:

$$b(\omega) \rightarrow \left(\frac{\pi}{(\omega_0 - i\omega)}\right)^{1/2} b(\omega) \quad (73)$$

and

$$b^*(\omega) \rightarrow \left(\frac{\pi}{(\omega_0 - i\omega)}\right)^{1/2} b^*(\omega). \quad (74)$$

We must note that Eq. (74) is not conjugate of Eq. (73). Nevertheless, is not difficult to justify this transformation, if we introduce polar coordinates instead of $b(\omega)$, $b^*(\omega)$: $b(\omega) = (\rho(\omega))^{1/2} e^{i\phi(\omega)}$, $b^*(\omega) = (\rho(\omega))^{1/2} e^{-i\phi(\omega)}$ and then perform a complex rotation of the integration counter when integrating with respect to $\rho(\omega)$: $\rho(\omega) \rightarrow \rho(\omega) [\pi/(\omega_0 - i\omega)]^{1/2}$. Its easy to see that after these changes of variables the denominator of the Eq. (67), turns out to be equal to unity

$$\int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) b(\omega)\right) = 1, \quad (75)$$

so we can express the ratio $\frac{Z}{Z_0}$ by the integral

$$\frac{Z}{Z_0} = \int [d\eta(b)] \exp\left(S_{eff}(b)\right), \quad (76)$$

where $S_{eff}(b)$ is the effective action of the bosonic mode which is given by

$$S_{eff} = -\pi \sum_{\omega} b^*(\omega) b(\omega) + N \ln \det(I + A). \quad (77)$$

The determinant in the above equation is given by

$$\det(I + A) = \det\left(M^{-1/2}(0, 0)M(b^*, b)M^{-1/2}(0, 0)\right) \quad (78)$$

and the matrix A is defined as follows

$$A = \begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}. \quad (79)$$

In the equation above the quantities B and C are matrices with components given by

$$B_{pq} = \left(\frac{\pi}{\beta N}\right)^{\frac{1}{2}} \left(ip + \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(\frac{g_1 b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} + \frac{g_2 b(p-q)}{\sqrt{\omega_0 - i(p-q)}}\right) \left(iq - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \quad (80)$$

and

$$C_{pq} = -\left(\frac{\pi}{\beta N}\right)^{\frac{1}{2}} \left(ip - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(\frac{g_1 b(p-q)}{\sqrt{\omega_0 - i(p-q)}} + \frac{g_2 b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}}\right) \left(iq + \frac{\Omega}{2}\right)^{-\frac{1}{2}}. \quad (81)$$

In Eq. (76) we may go to the limit $\omega_B, \omega_F \rightarrow \infty$ and the instead of a formal quotient of two infinite functional integrals we shall have only one finite functional integral. This representation turns out to be very useful for obtaining the asymptotic formula for Z/Z_0 at large N . There exists only one stationary phase point at $\beta^{-1} > \beta_c^{-1}$. If $\beta^{-1} < \beta_c^{-1}$, we have a circle of a stationary phase $|b(0)|^2 = \rho_0$, $b(\omega) = b^*(\omega) = 0$, if $\omega \neq 0$. There also exists an interpolation formula between these asymptotes. The presence of degenerate vacua is a feature of states with spontaneous symmetry breaking. As we will see, gapless excitation will appear.

We shall investigate the integral given by Eq. (76) for temperatures that satisfy $\beta^{-1} > \beta_c^{-1}$. First of all let us show that this integral converges. We use the estimate

$$|\det(I + A)| \leq \exp\left(\operatorname{Re}(trA) + \frac{1}{2} tr(AA^\dagger)\right). \quad (82)$$

where $\operatorname{Re}(trA)$ means the real part of trA . The matrix A have the form given by Eq. (79). Therefore we find that $trA = 0$ and $tr(AA^\dagger) = tr(BB^\dagger) + tr(CC^\dagger)$. Therefore, we obtain the estimate

$$\begin{aligned} \frac{Z}{Z_0} &\leq \int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) b(\omega) + N tr(BB^\dagger) + N tr(CC^\dagger)\right), \\ &\leq \int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) \left(1 - a_0(\omega)\right) b(\omega) + \right. \\ &\quad \left. + \pi \sum_{\omega} \left(b(\omega) c_0(\omega) b(-\omega) + b^*(\omega) c_0(\omega) b^*(-\omega)\right)\right), \end{aligned} \quad (83)$$

where the $a_0(\omega)$ and $c_0(\omega)$ are given respectively by

$$a_0(\omega) = \frac{g_1^2 + g_2^2}{\beta (\omega_0^2 + \omega^2)^{1/2}} \sum_{p-q=\omega} \frac{1}{(\frac{\Omega^2}{4} + q^2)^{1/2}} \frac{1}{(\frac{\Omega^2}{4} + p^2)^{1/2}}, \quad (84)$$

and

$$c_0(\omega) = \frac{\omega_0 g_1 g_2}{\beta (\omega_0^2 + \omega^2)} \sum_{p-q=\omega} \frac{1}{(\frac{\Omega^2}{4} + q^2)^{1/2}} \frac{1}{(\frac{\Omega^2}{4} + p^2)^{1/2}}. \quad (85)$$

Using the measure given in Eq. (68), we have that $\frac{Z}{Z_0} \leq F$, where $F = F_1 F_2$ and where F_1 and F_2 are given by

$$F_1 = \int db(0)db^*(0) \exp \left[-\pi b^*(0) (1 - a_0(0)) b(0) + \pi (b(0) c_0(0) b(0) + b^*(0) c_0(0) b^*(0)) \right] \quad (86)$$

and

$$F_2 = \int \prod_{\omega>0} db(\omega)db^*(\omega)db(-\omega)db^*(-\omega) \exp \left[-\pi \sum_{\omega>0} b^*(\omega) (1 - a_0(\omega)) b(\omega) - \pi \sum_{\omega>0} b^*(-\omega) (1 - a_0(\omega)) b(-\omega) + 2\pi \sum_{\omega>0} (b(\omega) c_0(\omega) b(-\omega) + b^*(\omega) c_0(\omega) b^*(-\omega)) \right]. \quad (87)$$

Note that in the case of the generalized fermion Dicke model we obtained a Gaussian integral that mixtures positive with negative frequencies. A straightforward calculation gives that the ratio $\frac{Z}{Z_0}$ obeys the following inequality

$$\frac{Z}{Z_0} \leq \left[\left(1 - a_0(0) + 2c_0(0) \right) \left(1 - a_0(0) - 2c_0(0) \right) \right]^{-1/2} \prod_{\omega>0} \left[\left(1 - a_0(\omega) + 2c_0(\omega) \right) \left(1 - a_0(\omega) - 2c_0(\omega) \right) \right]^{-1}. \quad (88)$$

In a similar way like Popov and Fedotov [20] proved, for the case of rotating wave approximation, we have that, $0 < a_0(\omega) + 2c_0(\omega) < a_0(0) + 2c_0(0)$ and $a_0(0) + 2c_0(0) = O(\omega^{-2} \ln \omega)$. Therefore

if $a_0(0) + 2c_0(0) < 1$, then Eq. (88) guarantees convergence of the expression $\frac{Z}{Z_0}$. The condition $a_0(0) + 2c_0(0) = 1$ is the equation for the transition temperature, then we have

$$a_0(0) + 2c_0(0) = \frac{(g_1 + g_2)^2}{\Omega \omega_0} \tanh\left(\frac{\beta_c \Omega}{4}\right) = 1. \quad (89)$$

The inverse of the critical temperature β_c is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1}\left(\frac{\Omega \omega_0}{(g_1 + g_2)^2}\right). \quad (90)$$

Note that there is a quantum phase transition where the coupling constants g_1 and g_2 satisfy $g_1 + g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$. For larger values for $(g_1 + g_2)$ the system enters in a superradiant phase.

To calculate the asymptotic behavior of the functional integrals at temperatures that satisfy $\beta^{-1} > \beta_c^{-1}$, we can do the following approximation

$$\det^N(I + A) = \det^N(I + BC) \rightarrow \exp\left(N \operatorname{tr}(BC)\right). \quad (91)$$

This substitute can be done and we can estimate the error if we divide all the functional space into two domains C_1 and C_2

$$\operatorname{tr}\left[(BC)(BC)^\dagger\right] \leq (4N)^{-1} \mapsto C_1, \quad (92)$$

$$\operatorname{tr}\left[(BC)(BC)^\dagger\right] \geq (4N)^{-1} \mapsto C_2. \quad (93)$$

Denoting

$$K_N = \det^N(I + A) - \exp\left(N \operatorname{tr}(BC)\right), \quad (94)$$

for the ratio $\frac{Z}{Z_0}$, we have the following identity

$$\begin{aligned} \frac{Z}{Z_0} &= \int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega)b(\omega) + N \operatorname{tr}(BC)\right) + \\ &+ \int_{C_1} [d\eta(b)] K_N \exp\left(-\pi \sum_{\omega} b^*(\omega)b(\omega)\right) + \\ &+ \int_{C_2} [d\eta(b)] K_N \exp\left(-\pi \sum_{\omega} b^*(\omega)b(\omega)\right). \end{aligned} \quad (95)$$

The first integral of the above equation is Gaussian, let us define it by I_0 . We use the Eq. (80) and the Eq. (81) in order to calculate the trace of BC, i.e., $\operatorname{tr}(BC)$ which is present in the expression

I_0 . A simple calculation gives

$$I_0 = \int [d\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) \left(1 - a(\omega)\right) b(\omega) + \pi \sum_{\omega} \left(b(\omega) c(\omega) b(-\omega) + b^*(\omega) c(\omega) b^*(-\omega)\right)\right), \quad (96)$$

where $a(\omega)$ and $c(\omega)$ of above equation are given respectively by

$$a(\omega) = \left(\frac{g_1^2 (\Omega - i\omega)^{-1} + g_2^2 (\Omega + i\omega)^{-1}}{(\omega_0 - i\omega)}\right) \tanh\left(\frac{\beta\Omega}{4}\right) \quad (97)$$

and

$$c(\omega) = \left(\frac{g_1 g_2 \Omega}{(\omega_0^2 + \omega^2)^{1/2} (\Omega^2 + \omega^2)}\right) \tanh\left(\frac{\beta\Omega}{4}\right). \quad (98)$$

Note that to recover the result obtained by Popov and Fedotov [19] [20] we have only to assume $g_2 = 0$. In this case, we have that $c(\omega) = 0$, which simplify the integration over the mode of the bosonic field in Eq. (96). Making the integration we obtain that I_0 is given by

$$I_0 = \prod_{\omega} \left(1 - a(\omega)\right)^{-1}. \quad (99)$$

After this observation lets go back to the general case, where g_1 and g_2 take arbitrary values. The expression I_0 given in Eq. (96) is a Gaussian integral, this expression is similar to the integral given in Eq. (83), so following the same steps we get that

$$I_0 = I_0(\omega = 0) \prod_{\omega > 0} \left[c(\omega)^2 - \left(1 - a(\omega)\right) \left(1 - a(-\omega)\right) \right]^{-1}, \quad (100)$$

where $I_0(\omega = 0)$ is the contribution of the condensate given by

$$I_0(\omega = 0) = \left[\left(1 - a(0) + 2c(0)\right) \left(1 - a(0) - 2c(0)\right) \right]^{-1/2}. \quad (101)$$

It is possible to estimate the error of I_0 which is given by the two last terms of Eq. (95). For details see the reference [8]. The errors depend on order N^{-1} . Therefore $\frac{Z}{Z_0}$ can be written as

$$\begin{aligned} \frac{Z}{Z_0} &= \left[\left(1 - a(0) + 2c(0)\right) \left(1 - a(0) - 2c(0)\right) \right]^{-1/2} \\ &\quad \prod_{\omega > 0} \left[\left(1 - a(\omega)\right) \left(1 - a(-\omega)\right) - c^2(\omega) \right]^{-1} + \\ &\quad + O(N^{-1}), \end{aligned} \quad (102)$$

therefore in the limit ($N \rightarrow \infty$) the equality $\frac{Z}{Z_0} = I_0$ is a good approximation. To find the collective excitation spectrum we have to use the equation

$$c^2(\omega) - \left(1 - a(\omega)\right) \left(1 - a(-\omega)\right) = 0, \quad (103)$$

and making the analytic continuation ($i\omega \rightarrow E$), we obtain the following equation

$$\begin{aligned} 1 = & - \left[\frac{g_1^4 + g_2^4}{(\omega_0^2 - E^2)(\Omega^2 - E^2)} \right] \tanh^2\left(\frac{\beta\Omega}{4}\right) + \\ & - \left[\frac{g_1^2 g_2^2}{(\omega_0^2 - E^2)} \left(\frac{1}{(\Omega - E)^2} + \frac{1}{(\Omega + E)^2} - \frac{4\Omega^2}{(\Omega^2 - E^2)^2} \right) \right] \tanh^2\left(\frac{\beta\Omega}{4}\right) + \\ & + \left[\frac{g_1^2(\Omega - E)^{-1} + g_2^2(\Omega + E)^{-1}}{(\omega_0 - E)} + \frac{g_1^2(\Omega + E)^{-1} + g_2^2(\Omega - E)^{-1}}{(\omega_0 + E)} \right] \tanh\left(\frac{\beta\Omega}{4}\right). \end{aligned} \quad (104)$$

Solving the above equation for the case $\beta^{-1} = \beta_c^{-1}$, we find the following roots

$$E_1 = 0 \quad (105)$$

and

$$E_2 = \left(\frac{g_1(\Omega + \omega_0)^2 + g_2(\Omega - \omega_0)^2}{(g_1 + g_2)} \right)^{1/2}. \quad (106)$$

Its low energy state of excitation is a Goldstone mode. Now, let us present the critical temperature and the spectrum of the collective bosonic excitations of the model with the rotating-wave approximation, where $g_1 \neq 0$ and $g_2 = 0$. The result obtained by Popov and Fedotov is recovered, where the equation

$$a(0) = 1 \quad (107)$$

and

$$\frac{g_1^2}{\omega_0\Omega} \tanh\left(\frac{\beta_c\Omega}{4}\right) = 1, \quad (108)$$

give the inverse of the critical temperature, β_c . It is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1}\left(\frac{\omega_0\Omega}{g_1^2}\right). \quad (109)$$

In this case, also there is a quantum phase transition, i.e., a zero temperature phase transition when $g_1 = (\omega_0\Omega)^{\frac{1}{2}}$. It is interesting to point out that there are two different ways to analyze

the phase transition. The first one is to follow the non-analytic behavior of the thermodynamic quantities as a function of temperature. A different way is to follow the non-analytic behavior of the thermodynamic quantities as a function of the coupling constant strength. Working in this second approach, we may expect that for large coupling constant g_1 there is a superradiant phase. The spectrum of the collective Bose excitations in this case is

$$E_1 = 0, \quad (110)$$

and

$$E_2 = \Omega + \omega_0. \quad (111)$$

Now we will show that it is possible to have a condensate with superradiance in a system of N qubits coupled with one mode of a Bose field where only virtual processes contribute. In the pure counter-rotating wave case, i.e., $g_1 = 0$ and $g_2 \neq 0$, the inverse of the critical temperature, β_c is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1} \left(\frac{\omega_0 \Omega}{g_2^2} \right). \quad (112)$$

and the spectrum of the collective Bose excitations given by

$$E_1 = 0, \quad (113)$$

and

$$E_2 = |\Omega - \omega_0|. \quad (114)$$

A comment is in order concerning the Bose excitations spectrum. In the both cases: using or not the rotating-wave approximation, there is a phase transition. In the case of the rotating-wave approximation $g_1 \neq 0$ and $g_2 = 0$, there is a Goldstone mode ($E = 0$). In the pure counter-rotating wave case $g_1 = 0$ and $g_2 \neq 0$, also there is a Goldstone (gapless) mode. The existence of Goldstone modes and the energy of the other mode was presented for both above mentioned cases. The spectrum in the general case is given by the Goldstone mode and also by a non-zero energy mode given by Eq. (106). It is interesting to stress that we obtained a critical behavior in both situations ($g_1 \neq 0, g_2 = 0$ and $g_1 = 0, g_2 \neq 0$), where the condensate has Goldstone (gapless) modes, with a superradiant state. Therefore we show that it is possible to have a condensate with superradiance in a system of N qubits coupled with one mode of a Bose field where only virtual processes contribute.

5 Conclusions

In the present paper we are first discussing Hamiltonians describing a quantized bosonic field interacting with two-level atoms. Without assuming the rotating-wave-approximation, with the

coupling constants g_1 and g_2 for rotating and counter-rotating terms respectively, we define the generalized fermion Dicke model. Changing the atomic pseudo-spin operators of the model by a linear combination of Fermi operators we define the fermion generalized Dicke model. Studying the case where identical two-level atoms act as a thermal reservoir ($N \rightarrow \infty$), we investigate the thermodynamic of the generalized Dicke model using the path integral approach with functional integration method. We are considering the question of how does the counter-rotating terms of the interaction Hamiltonian contribute in the formation of the condensate with a superradiant phase transition in the model.

We study the nonanalytic behavior of thermodynamic quantities of the generalized model for both situations, evaluating the critical transition temperature and presenting the spectrum of the collective bosonic excitations, for the case $g_1 \neq 0$ and $g_2 = 0$, $g_1 = 0$ and $g_2 \neq 0$ and also in the general case. Our result show that it is possible to have a condensate with superradiance in a system of N qubits coupled with one mode of a bosonic field where only virtual processes contribute. It is important to realize that the energy of the non-Goldstone mode in Eq.(111) is always larger than the energy of the non-Goldstone mode of Eq. (114), i.e., in the system where the condensate appears due to the virtual processes. Our conclusion from the above results is that both processes, real and virtual ones give different contributions to generate the condensate.

An important question is the way of practical realization of the generalized Dicke model in the laboratory. As was stressed by Dimer et al [71] it remains as a challenge to provide a physical system where the counter-rotating terms are dominant. Those authors proposed that in cavities with the N qubits, only one mode of quantized field and classical fields (lasers), it is possible to obtain a physical system that corresponds to the generalized Dicke model. Also, has been discussed in the literature the possibility of control the relative importance of the counter-rotating terms in the Jaynes-Cummings model in the laboratory using a ion trap [72]. Another mechanism to explore the importance of virtual processes was proposed by Ford [73] and Ford and Svaiter [74] [75], where the possibility of amplification of the vacuum fluctuations was discussed. These authors studied the renormalized vacuum fluctuations associated with a scalar and electromagnetic field near the focus of a parabolic mirror. Using the geometric optics approximation these authors found that the mirror geometry can produce large vacuum fluctuations near the focus.

It is not our proposal to discuss the practical realization of the generalized Dicke model in the laboratory, but only analyze the thermodynamic of the model using the path integral approach with functional method. An evidence in favor of our results is an experiment where it is possible to control the importance of the counter-rotating terms in the generalized model in such a way that an ideal $g_1 \approx 0$ situation is achieved. Experimental observation of the superradiant phase in this situation will improve our understanding of this phenomenon, when the atoms radiates spontaneously, at a radiation rate much higher than would be expected from an ensemble of independently radiating atoms.

There are many different continuations for this paper. The first one is to investigate the model introduced by DiVicenzo, defined by Eq. (22), at finite temperature, using also functional integral methods. Also the model defined by Eq. (16), with a coupling between the N qubits and the single quantized mode of a bosonic field, which characterize a intensity-dependent coupling can be

discussed using the conventional technique of the path integral and functional integration method. A similar calculation presented in this paper could be carried out for this model, although the calculations would be somewhat more laborious than the presented in the paper. The generalization of this model with the introduction of the counter-rotating terms deserves further investigations.

An outstanding question is the presence of the quantum phase transition and also the chaotic behavior for a finite N , where a crossover between Poisson and Wigner-Dyson behavior in the energy level-spacing statistics appears. The question that has been discussed in the literature is the distribution function of spacing of adjacent eigenenergies in a large number of systems. For systems where the classical dynamics is integrable, a Poissonian nearest neighbor spacing distribution is expected, i.e., the spacing rule for random levels. Some systems, the eigenenergies do not follow the Poisson law, but behave as the eigenvalues of a random matrix taken from a suitable ensemble. The set of all real random matrices with matrix elements obeying some distribution function defines the Gaussian orthogonal ensemble. Other ensembles are the Gaussian unitary and the Gaussian symplectic ensemble respectively. The chaotic behavior for a finite N in the above discussed model is under investigation by the authors.

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References

- [1] R. H. Dicke, *Phys. Rev.* **93**, 99 (1954).
- [2] D. Meshede, *Phys. Rep.* **211**, 201 (1991).
- [3] P. Meystre, *Phys. Rep.* **219**, 243 (1992).
- [4] E. T. Jaynes and F. W. Cummings, *Proc. Inst. Elect. Eng.* **51**, 89 (1963).
- [5] D. P. DiVincenzo, *Phys. Rev.* **A51**, 1015 (1995).
- [6] H. Dekker, *Phys. Rep.* **80**, 1 (1981).
- [7] K. Blum, *Density Matrix Theory and Applications*, Plenum Press, N. Y. (1981).
- [8] V. N. Popov, *Functional Integrals and Collective Excitations*, Cambridge University Press (1987).

- [9] A. Zardecki, Phys. Rev. **A22**, 1664 (1980).
- [10] M. Hillery and M. S. Zubaire, Phys. Rev. **A26**, 451 (1982).
- [11] E. Schrodinger, Naturwiss **14**, 664 (1926).
- [12] R. Glauber, Phys. Rev. Lett **10**, 84 (1963).
- [13] R. Glauber, Phys. Rev. **130**, 2529 (1963).
- [14] R. Glauber, Phys. Rev. **131**, 2766 (1963).
- [15] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 227 (1963).
- [16] P. L. Knight and L. Allen, "*Concepts in Quantum Optics*", Pergamon Press Inc. N.Y. (1983), pp. 54-69.
- [17] J. Kondo, Physica **125B**, 279 (1984).
- [18] L. D. Chiang and S. Chakravarty, Phys. Rev. **B**, 154 (1985).
- [19] V. N. Popov and S. A. Fedotov, Theor. Math. Phys. **51**, 363 (1982).
- [20] V. N. Popov and S. A. Fedotov, Sov. Phys. JETP. **67**, 535 (1988).
- [21] J. R. Ackerhart, P. L. Knight and J. H. Eberly, Phys. Rev. Lett **30**, 456 (1976).
- [22] P. W. Milonni, J. R. Ackerhart and W. A. Smith, Phys. Rev. Lett **31**, 958 (1976).
- [23] P. W. Milonni and W. A. Smith, Phys. Rev. **A11**, 814 (1976).
- [24] B. Fain, N. Cimento **B68**, 73 (1982).
- [25] J. R. Ackerhart and J. H. Eberly, Phys. Rev. **D10**, 3350 (1974).
- [26] J. Dalibard, J. Dupont Roc and C. Cohen Tannoudji, J. Phys. **43**, 1617 (1982).
- [27] J. Dalibard, J. Dupont Roc and C. Cohen Tannoudji, J. Phys. **45**, 637 (1984).
- [28] P. W. Milonni, "*The Quantum Vacuum*", Academic Press Inc., San Diego (1994), pp. 137-142.
- [29] S. Sachdev, "*Quantum Phase Transitions*", Cambridge University Press, Cambridge (1999).
- [30] V. B. Kir'yanov and V. S. Yarunin, Theor. Math. Phys. **43**, 340 (1980).
- [31] K. Hepp and E. H. Lieb, Ann. Phys. **76**, 360 (1973).

- [32] K. Hepp and E. H. Lieb, Phys. Rev. **A8**, 2517 (1973).
- [33] Y. K. Wang and F. T. Hioe, Phys. Rev. **A7**, 831 (1973).
- [34] F. T. Hioe, Phys. Rev. **A8**, 1440 (1973).
- [35] G. Comer Duncan, Rev. **A8**, 418 (1974).
- [36] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1949).
- [37] L. R. Meedand and N. Papanicolau, Phys. Rev. **B28**, 1633 (1983).
- [38] M. Hillery and L. D. Mlodinow, Phys. Rev. **A31**, 797 (1985).
- [39] C. Emary and T. Brandes, Phys. Rev. Lett **90**, 044101 (2003).
- [40] C. Emary and T. Brandes, Phys. Rev. **E67**, 066203 (2003).
- [41] R. Graham and M. Hoherbach, Phys. Lett. **101A**, 61 (1984).
- [42] R. Graham and M. Hoherbach, Phys. Rev. Lett. **57**, 1378 (1986).
- [43] C. H. Levenkopf, M. C. Nemes, V. Marvulle, M. P. Pato and W. F. Wreszinski, Phys. Lett. **A155**, 113 (1991).
- [44] L. Muller, J. Stolze , H Leschke and P. Nagel, Phys. Rev. **A44**, 1022 (1991).
- [45] G. A. Finney and J. Gra-Banacloche, Phys. Rev. **A50**, 2040 (1994).
- [46] P. W. Milonni, J. R. Ackerhalt and H. W. Galbraith, Phys. Rev. Lett. **50**, 966 (1983).
- [47] R. Kulloock and N. F. Svaiter, "*A Continuously Observed Two-level System Interacting with a Vacuum Field*", quant-ph/0703167 (2007).
- [48] H. B. G. Casimir and P. Polder, Phys. Rev. **73**, 360 (1948).
- [49] B. F. Svaiter and N. F. Svaiter, Phys. Rev. **D46**, 5267 (1992).
- [50] B. F. Svaiter and N. F. Svaiter, Phys. Rev. **D47**, 4802 (1993) (erratum).
- [51] P. C. W. Davies, J. Phys. **A8**, 365 (1975).
- [52] W. G. Unruh, Phys. Rev. **D14**, 870 (1976).
- [53] L. H. Ford, N. F. Svaiter and M. L. Lyra, Phys. Rev. **A49**, 1378 (1994).

- [54] R. Kubo, J. Phys. Soc. Jap. **12**, 570 (1959).
- [55] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
- [56] S. A. Fulling and S. N. M. Ruijsenaars, Phys. Rep. **152**, 135 (1987).
- [57] V. A. De Lorenci, R. De Paola and N. F. Svaiter, Class. Quant. Grav. **20**, 4241 (2000).
- [58] R. De Paola and N. F. Svaiter, Class. Quant. Grav. **18**, 1799 (2001).
- [59] B. S. Dewitt, Phys. Rep. **196**, 295 (1976).
- [60] J. H. Eberly, N. B. Narozhny and J. J. Sanches-Mongradon, Phys. Rev. Lett. **44**, 1323 (1980).
- [61] N. B. Narozhny and J. J. Sanches-Mongradon and J. H. Eberly, Phys. Rev. **A23**, 236 (1981).
- [62] H-I Yoo, J. J. Sanches-Mongradon and J. H. Eberly, J. Phys. **A14**, 1383 (1981).
- [63] Y. I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).
- [64] A. Joshi and R. R. Puri, Phys. Rev. **A42**, 4336 (1990).
- [65] A. H. Toor and M. S. Zubairy, Phys. Rev. **A45**, 4951 (1992).
- [66] B. Buck and C. V. Sukumar, Phys. Lett. **81A**, 132 (1981).
- [67] B. Buck and C. V. Sukumar, J. Phys. **A17**, 885 (1984).
- [68] V. Buzek, Phys. Rev. **A39**, 3196 (1989).
- [69] F. Benatti and R. Floreanini, quant-ph/507271.
- [70] G. M. Parma, K.-A. Seiominen and A. K. Ekert, Proc. R. Soc. Lond. **A452**, 567 (1996).
- [71] F. Dimer, B. Estienne, A. S. Parkins and H. J. Carmichael, Phys. Rev. **A75**, 013804 (2007).
- [72] J. C. Cirac, A. S. Parkins, R. Blat and P. Zoller, Phys. Rev. Lett. **570**, 556 (1993).
- [73] L. H. Ford, Phys. Rev. **A58**, 4279 (1990).
- [74] L. H. Ford and N. F. Svaiter, Phys. Rev. **A62**, 062105-1 (2000).
- [75] L. H. Ford and N. F. Svaiter, Phys. Rev. **A66**, 062106-1 (2002).