The Strong-Coupling Expansion and the Ultra-local Approximation in Scalar Theories

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Abstract

We discuss the strong-coupling expansion in Euclidean field theory. In a formal representation for the Schwinger functional, we treat the off-diagonal terms of the Gaussian factor as a perturbation about the remaining terms of the functional integral. Consequently, we are developing the perturbative expansion around the ultra-local model, where fields defined in different points of the Euclidean space are decoupled. We first study the strong-coupling expansion in the $(\lambda\varphi^4)_d$ theory. Assuming the ultra-local approximation, we study the singularities of this perturbative expansion, analysing the analytic structure of the zero-dimensional generating function in the complex coupling constant plane. Second, we discuss the ultra-local generating functional in two idealized field theory models defined by the following interaction Lagrangians: $\mathcal{L}_{\mathcal{II}}(g_1, g_2; \varphi) = g_1\varphi^p(x) + g_2\varphi^{-p}(x)$, and the sinh-Gordon model, i.e., $\mathcal{L}_{\mathcal{III}}(g_3, g_4; \varphi) = g_3 (\cosh(g_4 \varphi(x)) - 1)$. To control the divergences of the strong-coupling perturbative expansion two different steps are used through the paper. First we introduce a lattice structure only to give a meaning to the ultra-local generating functional. On the other hand, the resulting perturbative expansion is an expansion for the continuum and not for the lattice approximation. Thus, using an analytic regularization procedure we discuss briefly how it is possible to obtain a renormalized Schwinger functional associated with these models, going beyond the ultra-local approximation withouth using a lattice regularization procedure.

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1   Introduction

The purpose of this paper is twofold. First, in the context of the strong-coupling perturbative expansion, is to investigate the analytic structure of the ultra-local generating functional for the $(\lambda\varphi^4)_d$ model in the complex coupling constant plane. We also studying the ultra-local generating functional of two non-renormalizable scalar field models. The second purpose is to discuss how it is possible to obtain a renormalizable Schwinger functionals associated with the models going beyond the ultra-local approximation. This is performing using an analytic regularization procedure. This work is a natural extension of the program developed by Klauder [1], Rivers [2] and others that have been studying the strong-coupling expansion and the ultra-local generating functional in different scalar infrared free models in field theory.

The perturbative renormalization approach in quantum field theory, is an algoritm, where starting from the Feynman diagramatic representation of the perturbative series, two different steps are usually performed. In the first step, we control all the ultraviolet divergences of the theory, using a procedure to obtain well defined expressions for each Feynman diagram. In the second step, we have to implement a renormalization prescription where the divergent part of each Feynman diagram is canceled by a suitable counterterm. For complete reviews of this program see for example Ref. [3] or Ref. [4]. Concerning the first step, there are different ways to transform the Feynman diagrams in well defined finite quantities. The most simple way is to modify the field theory at short distances by introducing a sharp cut-off in momentum space or a more elaborated version as the Pauli-Villars regularization [5]. In this second case we simply modify the propagator for large momentum. We can also replace the continuum Euclidean space by a hypercubic lattice, with lattice space a, called the lattice regularization method. It is clear that the introduction of a lattice provides a cut-off in momentum space of the order of the inverse of the lattice spacing a. Finally, a more convenient regularization procedure is the dimensional regularization [6] [7] [8] [9], which is particulary well suited to deal with abelian and non-abelian gauge theories. In the second step, to implement the renormalization procedure, using dimensional regularization for example, we use the fact that the ultraviolet divergences of Feynman diagrams appear as poles of some function defined in a complex plane. Then, the perturbative renormalization is performed by the cancelation of the principal part of the Laurent series of the analytic regularized expressions. This cancelation is done introducing counterterms in the theory. There are different ways to disregard

the divergent part of each Feynman diagram. For example we can use the minimal subtraction scheme (MS) where the counterterms just cancel the principal part of the Laurent series of the analytic regularized expressions, or any different renormalization scheme. The arbitrariness of the method employed must be cured by the renormalization group equations [10] [11] [12].

In this framework, field theory models are classified as perturbatively super-renormalizables, renormalizables and non-renormalizables. In the stardard weak-coupling perturbative expansion, the fundamental difference between a perturbatively renormalizable and a non-renormalizable model in field theory is given by the following property. In a non-renormalizable model, the usual renormalization procedure, used to remove the infinities that arise in the usual perturbative expansion introduce infinite new empirical parameters in the theory. This comes from the fact that we need to specify the finite part of infinitely many counterterms. Consequently, the predictibility or the physical consistence of non-renormalizable models is missing. Thus, renormalizability of field theory models provides a valuable constraint on new theories aimed to describe elementary particles. See for instance the discussion developed in Ref. [13]. Nevertheless, there are some non-renormalizable models where it is possible to construct a physically sensible version of the theory. A well known example is the Gross-Neveu model, which is not renormalizable in the usual sense. However, this model is renormalizable in the $\frac{1}{N}$ expansion, for $d < 4$ [14] [15]. We would like to mention that there is a modern alternative attitude towards the renormalizability of physical theories, using the idea of effective field theories [16] [17], but in this paper we will not discuss these issues.

In the case of perturbatively renormalizable models, although the renormalization procedure can be implemented in a mathematical consistent way, it is still not clear how the renormalized perturbative series can be summed up in different models [18] [19] [20]. In the literature, there are many results showing that the series that we obtain in different perturbative renormalizable theories in $d = 4$ does not converge for any value of the coupling constants of the interacting theories. Well known theories with such kind of problems are scalar models with a $(\lambda \varphi^4)$ self-interaction and also quantum electrodynamics. If someone try to perform a partial resummation of the perturbative series in both theories, the Landau poles appear [21] [22] [23]. In a four dimensional spacetime, to circumvent the problem of non-hermitian Hamiltonians in the infinite cut-off limit, the $(\lambda \varphi^4)_4$ model, the $O(N)_4$ model, and also quantum electrodynamics must be trivial.

A new step in the development of quantum field theories was given by the construction of non-abelian gauge field theories and the discovery of the asymptotic freedom [24] [25] [26] after the construction of the renormalization group equations. From the renormalization group equations a important classification of different field theory models arises. The models are either asymptotically free or IR (infrared) stables. In the renormalization group approach, the triviality of $(\lambda\varphi^4)_4$ model, the $O(N)_4$ model and also quantum electrodynamics in a four dimensional spacetime is a reflection of the absence of a non-trivial ultraviolet stable fixed point in the Callan-Symanzik β-function. In the infrared-free theories, in a four dimensional spacetime, the problem of the singularities in the three and four-point connected functions for quantum electrodynamics and $(\lambda\varphi^4)_4$ respectively, for positive values of the coupling constants is related to the fact that, in the framework of the weak-coupling expansion, in these theories the high frequency fluctuations are more strongly coupled than the lower frequencies ones. As we discussed, it is well known that we found a completely distint behavior in a theory where the high frequency fluctuations are more weakly coupled than the lower frequencies ones. In this situation, at least the zero charge problem does not appears.

The unification of the statistical mechanics and some models in quantum field theory was achieved in progressive steps. First Schwinger introduced the idea of Euclidean fields, where the classical action must be continued to Euclidean time [27]. Then, Symanzik constructed the Euclidean functional integral where the vacuum persistence functional defined in Minkowski space-time becomes a statistical mechanics average of classical fields weighted by a Boltzmann probability [28] [29]. At the same time, a deeper insight into our understanding of the renormalization procedure in different models in field theory was given by the study of the critical phenomena and the Wilson version of the renormalization group equations [30]. Further, Osterwalder and Schrader proved that for scalar theories, the Euclidean Green's functions or the Schwinger functions, which are the moments of the Boltzman measure, are equivalent to the Minkowski Green's functions [31]. A new step in our understanding of the limitation of the perturbative approach in quantum field theories was achieved by Aizenman [32] and Frohlich [33]. These authors proved that the $(\lambda\varphi^4)_d$ model, with the use of a lattice regularization with nearest neighborhood realization of the Laplacian, leads to a trivial theory in the continuum limit for $d \geq 5$. For $d = 4$, with additional assumptions, it is also possible to obtain the triviality of the model. For an interesting review studying the triviality problem in quantum field theory, see for example Ref. [34].

It is important to point out that some authors claim that the triviality of $(\lambda\varphi^4)_d$, for $d \geq 4$ is a odd result, since for $d = 4$ the renormalized perturbative series is non-trivial and for $d \geq 5$ the theory is perturbatively non-renormalizable. For example, Klauder has been giving arguments pointing out that the triviality of $(\lambda\varphi^4)_d$ for $d \geq 4$ is still an open problem [35]. Making use of the correspondence principle, this author has been emphasizing that the quantization of a non-trivial classical theory can not be a non-interacting quantum theory. Furthermore, he claims that an alternative regularization procedure can gives a non-trivial theory in the infinite cut-off limit.

Some results going in this direction have been obtained by Gallavotti and Rivasseau [36]. These authors discussed the scalar $(\lambda\varphi^4)_d$ theory with more general regularized theories where the realization of the Laplacian is not restricted to the nearest neighbours and also the presence of antiferromagnetic couplings. They suggested that the ultraviolet limit of such lattice regularized field theories is not a Gaussian field theory model, which would open the possibility to construct scalar models with a a non-trivial ultraviolet limit. We shall emphasize that the ultraviolet behavior of the $(\lambda\varphi^4)_d$ model for $d \geq 4$ is a strong-coupling problem, since, in the weak coupling perturbative expansion, the high orders terms of the perturbative series are dominant in the large cut-off limit, as has been discussed by many authors. See for instance the discussion in Ref. [37]. Another situation in field theory where the weak-coupling perturbative expansion does not seem appropriate is in the large distance behavior of quantum chromodynamics.

In the case of a strong-coupling regime of a theory, we can try to obtain rigorous results by the use of constructive field theory, perform a partial resummation of the Feynman diagrams improving the Feynman-Dyson perturbative series, or perform a different perturbative expansion by using the following approaches. The first one is to introduce auxiliary fields in order to disconect the interaction part from the free part of the Lagrangian density. Further, perform a perturbative expansion of the Schwinger functional in inverse powers of the coupling constant [38]. For example, the $\frac{1}{N}$ expansion, where N is the number of the components of the field in some isotopic space (the dimension of the order parameter), is a realization of this approach [39]. Of course we are still using the standard perturbative scheme, performing a perturbative expansion with respect to the anharmonic terms of the theory. The basic idea of the second approach is to construct a formal representation for the generating functional of complete Schwinger functions of the theory, treating the off-diagonal terms of the Gaussian factor as a perturbation about the remaining terms in the functional integral. This approach it has been called in the literature the strong-coupling

expansion [40] [41] [42] [43] [44] [45].

The purpose of this paper is to discuss some of the problems of the strong-coupling expansion in different scalar models. We first investigate the singularities of the generating functional of complete Schwinger functions, for the $(\lambda\varphi^4)_d$ model, in the complex coupling constant plane. These informations can be easily obtained studying the analytic structure of the zero-dimensional generating function. Second we present two idealized interacting field theory models where the weak-coupling expansion can not be used. In this case the strong-coupling expansion is more adequate to investigate the properties of the models. Finally, using an analytic regularization procedure, we discuss briefly how it is possible to obtain a renormalized Schwinger functional, going beyond the ultra-local approximation. It is important to stress that the analytic structure of the Schwinger functions (in the weak-coupling perturbative expansion framework) and also of the Schwinger functional can be easily obtained using the ultra-local approximation derived in the strong-coupling perturbative expansion. As we will see, these non-perturbative results do not change if we go beyond the ultra-local approximation. We should say at this point that Bender et all [46] also consider the zero-dimensional field theory to obtain non-perturbative results in field theory.

There are three points that we would like to briefly discuss. First, it is interesting to ask whether the ultra-local approximation has been used in the literature in another context. In the Laudau theory of continuous phase transition, which reproduces the mean-field exponents, we have a simplified version of the ultra-local approximation, since in the partition function we drop the gradient term and the sum will be dominated by its largest term. Further, Landau and Ginzburg modified the the original Landau theory by introducing the gradient term into the energy density that discourages rapid fluctuations in the order parameter [47]. Before continue we would like to point out that the strong-coupling perturbative expansion is quite similar to the high-temperature series expansion in statistical mechanics. A similar idea is the Mayer expansion, a method for carrying out the cluster expansion, introduced into quantum field theory by Symanzik [48]. Finally, in the study of critical phenomena using lattice simulation there is a analog of the strong-coupling perturbative expansion; the hopping parameter expansion, where the perturbative expansion starts from the disordered lattice system [49]. For a complete review of the Linked cluster expansion, see for example Ref. [50].

Second, it is interesting to point out that the study of the ultra-local field theory models can

also bring us some insights to understand non-renormalizable field theory models. As a simple example, let us suppose a massive abelian vector field $W_\mu(x)$ coupled with fermions. Using a Fourier representation for the Euclidean two-point Schwinger function associated with the massive vector field $W_\mu(x)$, the Fourier coefficients $G_{\mu\nu}(k)$ are given by

$$G_{\mu\nu}(k) = \left(\delta_{\mu\nu} + \frac{k_\mu\, k_\nu}{m^2}\right)\frac{1}{k^2+m^2}.$$

The Fourier coefficients $G_{\mu\nu}(k)$ of the two-point Schwinger function does not vanish in the ultraviolet limit $k \to \infty$ and its large Euclidean moment behavior is roughly m^{-2}. Consequently the behavior in this limit is similar to the behavior for the two-point Schwinger function in the ultra-local $(\lambda\varphi^4)_d$ model.

Finally, we would like to call the attention of the reader that the study of the analytic structure of theories in the complex coupling constant plane has been used by many authors in quantum field theory. It is well known that the behavior of the standard perturbative series in powers of the coupling constant at large order is related to the analytic structure of the partition function in a neighborhhood of the origin in the complex coupling constant plane. For example, Bender and Wu [51] studied the anharmonic oscillator and pointed out that there is a relation between the nth Rayleigh-Schrodinger coefficients and the lifetime of the unstable states of a negative coupling constant anharmonic oscillator.

The organization of the paper is as follows: In section II we discuss the standard weak-coupling expansion for the $(\lambda\varphi^4)_d$ model. In section III we discuss the strong-coupling expansion for the $(\lambda\varphi^4)_d$ model. Section IV we perform the study of the analytic structure of the zero-dimensional $(\lambda\varphi^4)$ model in the complex coupling constant plane. In section V we presented idealized models where the strong-coupling perturbative expansion must be used. In section VI we sketch how it is possible to go beyond the ultra-local approximation in general scalar models. Finally, section VII contains our conclusions. To simplify the calculations we assume the units to be such that $\hbar = c = 1$, and also all the physical quantities are dimensionless. Consequently it is convenient to introduce an arbitrary parameter μ with mass dimension to define all the dimensionless physical quantities.

2   Weak coupling perturbative expansion for the scalar $(\lambda\varphi^4)_d$ model

Let us consider a neutral scalar field with a $(\lambda\varphi^4)$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time supported by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. Actually, the $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z(h)$ is formally defined by the following functional integral:

$$Z(h) = \int [d\varphi] \, \exp\left(-S_0 - S_I + \int d^d x \, h(x)\varphi(x)\right), \tag{1}$$

where the action that usualy describes a free scalar field is

$$S_0(\varphi) = \int d^d x \, \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\,\varphi^2\right), \tag{2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I(\varphi) = \int d^d x \, \frac{g_0}{4!}\,\varphi^4(x). \tag{3}$$

In Eq.(1), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_x d\varphi(x)$. The terms g_0 and m_0^2 are respectivelly the bare coupling constant and mass squared of the model. Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by functional derivatives.

In the weak-coupling perturbative expansion, which is the conventional procedure, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the n-point unrenormalized Schwinger functions are expressed in a powers series of the bare coupling constant g_0. Let us summarize how to perform the weak-coupling perurbative expansion in the $(\lambda\varphi^4)_d$ theory, and briefly discuss also the divergent

behavior of the perturbative series. The Gaussian functional integral $Z_0(h)$ associated with $Z(h)$ is

$$Z_0(h) = \mathcal{N} \int [d\varphi] \, \exp\left(-\frac{1}{2}\varphi \, K \, \varphi + h\varphi\right). \tag{4}$$

We are using the compact notation of Zinn-Justin [52] and the terms in Eq.(4) are given by

$$\varphi \, K \, \varphi = \int d^d x \int d^d y \, \varphi(x) K(m_0; \, x, y)\varphi(y), \tag{5}$$

and

$$h\varphi = \int d^d x \, \varphi(x) h(x). \tag{6}$$

The symmetric kernel $K(m_0; x, y)$ is defined by

$$K(m_0; x, y) = (-\Delta + m_0^2) \, \delta^d(x - y), \tag{7}$$

and Δ denotes the Laplacian in R^d. As usual, the normalization factor is defined using the condition $Z_0(h)|_{h=0} = 1$. Therefore $\mathcal{N} = [\det(-\Delta + m_0^2)]^{\frac{1}{2}}$ but, in the following, we are absorbing this normalization factor in the functional measure. It is convenient to introduce the inverse kernel $G_0(m_0; x - y)$ which satisfies the identity

$$\int d^d z \, G_0(m_0; x - z) K(m_0; z - y) = \delta^d(x - y). \tag{8}$$

Since Eq.(4) is a Gaussian functional integral, simple manipulations (performing only Gaussian integrals) gives

$$\int [d\varphi] \, \exp\left(-S_0 + \int d^d x \, h(x)\varphi(x)\right) = \exp\left(\frac{1}{2}\int d^d x \int d^d y \, h(x) \, G_0(m_0; x - y)h(y)\right). \tag{9}$$

Therefore, we have an expression for $Z_0(h)$ in terms of the inverse kernel $G_0(m_0; x - y)$, i.e., in terms of the free two-point Schwinger function.

This construction is fundamental to perform the weak-coupling Feynman-Dyson perturbative expansion with the Feynman diagramatic representation of the perturbative series. Using Eq.(1), Eq.(2), and Eq.(9), we are able to write the generator functional of all unrenormalized Schwinger functions $Z(h)$ as

$$Z(h) = \exp\left(-\int d^d x \, \mathcal{L}_\mathcal{I}(\frac{\delta}{\delta h})\right) \exp\left(\frac{1}{2}\int d^d x \int d^d y \, h(x) \, G_0(m_0; x - y)h(y)\right), \tag{10}$$

where $\mathcal{L}_{\mathcal{I}}$ is defined by the non-Gaussian contribution to the action. Consequently, in the conventional perturbative expansion the generating functional of complete Schwinger functions is formally given by the infinite series

$$Z(h) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int d^d x \ \mathcal{L}_{\mathcal{I}}(\frac{\delta}{\delta h}) \right)^n \exp \left(\frac{1}{2} \int d^d x \int d^d y \ h(x) \, G_0(m_0; x - y) h(y) \right). \tag{11}$$

To generate all the n-point Schwinger functions we have only to perform a suitable number of functional differentiations in $Z(h)$ with respect to the source $h(x)$ and set the source to zero in the end. The unrenormalized n-point Schwinger functions are defined by

$$G_n(x_1, x_2, .., x_n) = Z^{-1}(h = 0) \left[\frac{\delta}{\delta h(x_1)} ... \frac{\delta}{\delta h(x_n)} Z(h) \right] |_{h=0}. \tag{12}$$

This general method can be used to derive the weak-coupling perturbative expansion in different theories. Observe that it is possible to generalize this formalism including the product of composite sources [53] [54] [55], but in this paper we limit ourselves to models withouth composite operators. To generate only the connected diagrams $G_n^{(c)}(x_1, .., x_n)$, let us consider the the generating functional of the connected Schwinger functions (also called the free energy functional), defined as $F(h) = \ln Z(h)$. The functional Taylor expansion of the generating functional of connected Schwinger functions is

$$F(h) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{k=1}^{n} d^d x_k \prod_{k=1}^{n} h(x_k) \, G_n^{(c)}(x_1, .., x_n). \tag{13}$$

The functional Taylor expansion of the generating functional of all Schwinger functions $Z(h)$ in powers of the coupling constant g_0 is

$$Z(h) = \mathcal{N} \left(1 + \sum_{k=1}^{\infty} g_0^k \, A_k(h) \right), \tag{14}$$

where $A_k(h)$ are unrenormalizable perturbative coefficients. After a regularization and renormalization procedure it is possible to show that any physically measurable quantity $f(g)$ can be expanded in power series defined by

$$f(g) = \sum_{k=0}^{\infty} f_k \, g^k, \tag{15}$$

where g is the renormalized coupling constant and f_k are perturbative coefficients.

Since in general the series that we obtain from perturbatively renormalizable theories are divergent, the perturbative renormalization method is not enough to obtain well defined physical quantities, For example, for $P(\varphi)_2$ the renormalized perturbative series for any connected Schwinger function that can be obtained by a Wick ordering is divergent [57]. For the $(\lambda\varphi^4)_3$ model a similar divergent behavior was proved by de Calan and Rivasseau [58]. Although in general the series that we obtain from perturbatively renormalizable theories are divergent, there is an impressive agreement of the theoretical results with the experiments, when someone use the first terms of the weak-coupling perturbative expansion to extract predictable results. Therefore, these divergent series shall be an asymptotic expansion of the solutions of the theories. In other words, in a specific theory even though the renormalized perturbative series diverges, a finite number of terms of the series is still a good approximation of the functions in question. The Borel resummation is this tool that allow us to obtain the solutions of the theory from these divergent series [59].

Let us discuss with more details the asymptotic expansion of a function and Borel summability. Suppose a function $f(z)$ defined in the complex plane for large z. The formal series $\sum_{n=0}^{\infty} a_n z^{-n}$, which need not converge for any value of z, is called the asymptotic expansion or asymptotic representation of the function $f(z)$ if defining

$$i)\ S_N(z) = \sum_{n=1}^{N} a_n z^{-n} \tag{16}$$

$$ii)\ R_N(z) = z^N |f(z) - S_N(z)|, \tag{17}$$

we have $\lim_{|z|\to\infty} R_N(z) = 0$, for every fixed N. There is a similar definition of the asymptotic expansion of a function near zero, involving series of the kind $\sum_{n=0}^{\infty} a_n z^n$.

From the above definition there are two main questions. The first is the question whether a function under consideration possesses an asymptotic expansion, which we call the expansion problem. There is also the question of how the function is to be found, which is represented by a given asymptotic expansion, that we call the summation problem. or we can show that the same asymptotic representation. Now, suppose a function $f(z)$ which has the asymptotic expansion in

a region of the complex plane, defined by a divergent series. Thus we have

$$f(z) \sim \sum_{k=0}^{\infty} f_k \, z^k. \tag{18}$$

The Borel transform of $f(z)$, called $B_f(z)$ is defined as

$$B_f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f_k \, z^k. \tag{19}$$

The key point if that the Borel transform may converge even if the series is divergent. The Borel resummation of the series is obtained applying the inverse Borel transform on $B_f(z)$, given by

$$f(z) = \int_0^{\infty} \exp(-t) \, B_f(zt) \, dt. \tag{20}$$

This construct is an indispensable tool to recover a function from its asymptotic expansion in quantum field theory. A pedagogical discussion of the application of the Borel transform in perturbation theory can be found in Ref. [60]. It is not difficult to repeat this construction for the n-point Schwinger function of any renormalizable theory. If the perturbative series of the 2n-point renormalized Schwinger functions does not converge, as these series must be an asymptotic expansion for the solutions of our theory, the Borel resummation method can be used to recover the solutions. In the standard perturbative expansion we express the 2n-point renormalized Schwinger functions as the following power series for the renormalized coupling constant g:

$$G_{2n}(g; \, x_1, x_2, .., x_{2n}) \sim \sum_{k=0}^{\infty} g^k \, G_{2n}^{(k)}(x_1, x_2, .., x_{2n}). \tag{21}$$

Let us define the Borel transform of the n-point Schwinger function by

$$G_{2n}(\tau; \, x_1, x_2, .., x_{2n}) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \, G_{2n}^{(k)}(x_1, x_2, .., x_{2n}), \tag{22}$$

and it is clear that from the inverse Borel transform we have

$$G_{2n}(g; \, x_1, x_2, .., x_{2n}) = \frac{1}{g} \int_0^{\infty} d\tau \, \exp(-\frac{\tau}{g}) \, G_{2n}(\tau; \, x_1, x_2, .., x_{2n}). \tag{23}$$

The series which defines $G_{2n}(\tau; \, x_1, x_2, .., x_{2n})$ in Eq. (22) has much better convergence properties than the original series of $G_{2n}(g; \, x_1, x_2, .., x_{2n})$ in Eq.(21). Eckmann et al. [61] obtained the

Schwinger functions of the $(\lambda\varphi^4)_2$ model from the divergent perturbative series using Borel re-summation method. The extension of this result to $(\lambda\varphi^4)_3$ model was obtained by Magnen and Seneor [62].

Note that there are some situation where the Borel resummation method can not be implemented. This happens, for example, when the Borel transform has singularities in the real line. These singularities are related to non-perturbative effects which are not apparent in the weak-coupling perturbative expansion [63]. If there are no singularities in the positive axis of the Borel transform of the perturbative series, the Borel resummation method is a powerful way to extract results from a divergent series. See also the Ref. [64] and Ref. [65].

3 The strong-coupling perturbative expansion for the scalar $(\lambda\varphi^4)_d$ model

From now one, we are studying the strong-coupling perturbative expansion in field theory. This perturbative expansion may be used in the strong-coupling regime of a model, as for example the ultraviolet limit of a non-asymptoticaly free model. In this situation, or in the case of non-renormalizable theories, is interesting to investigate an alternative perturbation expansions. Before continue, we would like to call the attention of the reader that a alternative perturbative program for dealing with non-renormalizable theories, has been developed by Klauder and others [67] [68] [69] [70]. Klauder proposed a non-canonical formulation for the quantization of $(\lambda\varphi^p)_4$ model $(p > 4)$ using an non-translational invariant functional measure. To support this approach he observes that there are many situations where an infinitesimal perturbation causes a discontinuous change in the eigenfunctions and eigenvalues associated with an Hamiltonian system [71] [72]. For example in the $(\lambda\varphi^p)_4$ model, if $p \leq 4$ the theory is renormalizable and field configurations which have a finite free action also give a finite contribution from the interaction term. For $p > 4$ this does not happens: the free field situation can not be obtained when $\lambda \to 0^+$ and this limit is the pseudo-free solution. In this paper we decided to follow a more conventional treatment. We use an alternative perturbative expansion, but we are assuming that the measure in the functional integral is translational invariant instead of using a scale covariant measure.

The lesson that we have from these discussions is that if someone decide to perform a perturbative expansion of a strongly-coupled theory, then a resummation of the weak-coupling perturbative series to obtain non-perturbative results is necessary. An alternative procedure is not use the the conventional perturbation theory around the Gaussian-free theory. Consequently, we now turn to the alternative expansion that has been called in the literature the strong-coupling perturbative expansion. The basic idea of this approach is to treat off-diagonal terms of the Gaussian factor as a perturbation about the remaining diagonal terms in the integral. Although we are studying only scalar models, the extension to higher spin fields is straightforward. See for example the discussion given in Ref. [73].

Let us suppose a compact Euclidean space with or withouth a boundary. An equivalent possibility is to work in an unbounded Euclidean space but assume that the functional integral is taken over field configurations that vanish at large Euclidean distances. Let us suppose that there exists an eliptic, semi-positive, and self-adjoint differential operator D acting on scalar functions on the Euclidean space. The usual examples are $D = (-\Delta)$ and $D = (-\Delta + m_0^2)$. The kernel $K(m_0; x - y)$ is given by $K(m_0; x - y) = D\,\delta^d(x - y)$. Let us study first the self-interacting $(\lambda\varphi^4)_d$ model, where the dimension of the order parameter is one. Thus we have:

$$\mathcal{L}_{\mathcal{I}}(g_0; \varphi) = \frac{g_0}{4!}\varphi^4(x). \tag{24}$$

Treating the off-diagonal terms of the Gaussian factor as a perturbation about the remaining terms in the integral, we get a formal expression for the generating functional of complete Schwinger functions $Z(h)$:

$$Z(h) = \exp\left(-\frac{1}{2}\int d^d x \int d^d y \frac{\delta}{\delta h(x)} K(m_0; x - y)\frac{\delta}{\delta h(y)}\right) Q_0(h), \tag{25}$$

where the ultra-local generating functional $Q_0(h)$ is defined by

$$Q_0(h) = \mathcal{N} \int [d\varphi] \exp\left(\int d^d x \left(-\frac{g_0}{4!}\varphi^4(x) + h(x)\varphi(x)\right)\right). \tag{26}$$

The factor $\mathcal{N}$ is a normalization that can be found using that $Q_0(h)|_{h=0} = 1$. Observe that all the non-derivative terms in the original action appears in the functional integral that defines $Q_0(h)$. As we discussed in section II, in the $(\lambda\varphi^4)_d$ model, the kernel is defined by Eq.(7): $K(m_0; x - y) =$

$(-\Delta + m_0^2)\,\delta^d(x-y)$. At this point it is convenient to consider $h(x)$ to be complex. Consequently $h(x) = \text{Re}(h) + i\text{Im}(h)$ (we are concerned with the case $\text{Re}(h) = 0$). It should be noted that although the functional integral $Q_0(h)$ is not a product of Gaussian integrals, it can be view formaly as an infinite product of ordinary integrals, one for each point of the d-dimensional Euclidean space. The fundamental problem of the strong-coupling expansion, is how to construct non-Gaussian measures to define the Schwinger functional. It is important to point out that the solution of this problem would allow us to deal with non-renormalizable models in the weak-coupling expansion framework.

The expantion of the exponential term on Eq.(25) a formal expansion of $Z(h)$ as a perturbative series in the following form:

$$Z(h) = \sum_{i=0}^{\infty} Z^{(i)}(h), \tag{27}$$

where the two first terms of the perturbative series are respectively the ultra-local generating functional $Q_0(h)$ and $Z^{(1)}(h)$ defined by

$$Z^{(1)}(h) = \left(-\frac{1}{2}\int d^d x \int d^d y\, \frac{\delta}{\delta h(x)} K(m_0; x-y)\frac{\delta}{\delta h(y)}\right) Q_0(h). \tag{28}$$

The main difference from the the standard perturbative expansion is that we have an expansion of the generating functional of complete Schwinger functions in inverse powers of the coupling constant. We are developing our perturbative expansion around the static ultra-local functional $Q_0(h)$ [66] [74] [75] [76]. Fields defined in different points of the Euclidean space are decoupled in the ultra-local approximation since the gradient term is dropped. The above formal representation of $Z(h)$, defined by the unrenormalized perturbative series can be truncated in the order of the approximation. For example, if $Z(h) \equiv Q_0(h)$ we call it the ultra-local approximation or the zero-order approximation. The next order approximation where we link every two points in the Euclidean space we call it the first-order approximation since $Z(h) \equiv Z^{(0)}(h) + Z^{(1)}(h)$.

As we stressed, although the strong-coupling expansion is very inconvenient for pratical calculations in the continuum R^d Euclidean space, it is very natural in the lattice. The technical problems that we have to deal in the continuum Euclidean space are the following: first we have to define non-Gaussian functional measures; second we have to to regularize and renormalize the Schwinger functions obtained from the generating functional, going beyond the ultra-local approximation. We would like to stress that we will not use a lattice structure of the Euclidean space as

a regulator to implement the renormalization program. Instead, we use the lattice structure only to define what we mean by the ultra-local generating functional $Q_0(h)$.

In this paper we are not interested to regularized the complete series of the strong-coupling perturbative expansion. We are interested first, in the analytic structure of the Schwinger functional in the complex coupling constant plane. We will show that the the zero-dimensional generating function of the $\lambda\varphi^4$ model has a branch point singularity. This kind of singularity in the complex coupling constant plane has already been found in the ultra-local quantum electrodynamics [77]. Our second interest is to study scalar models where the use of the strong-coupling expansion is imperative. Finally, we briefly discuss how it is possible use a analytic regularization procedure, in order to obtain a renormalized Schwinger functional, going beyond the ultra-local approximation.

Using the fact that the functional integral which defines $Z(h)$ is invariant with respect to the choice of the quadratic part, let us consider a slightly modification of the strong-coupling expansion. We will split the quadratic part in the functional integral proportional to the mass squared in two parts: the off-diagonal terms of the Gaussian factor and the ultra-local generating functional. The new generating functional of the complete Schwinger functions will be defined by the following functional integral

$$Z(h) = \exp\left(-\frac{1}{2}\int d^d x \int d^d y \frac{\delta}{\delta h(x)} K(m_0, \sigma; x - y) \frac{\delta}{\delta h(y)}\right) Q_0(\sigma; h), \qquad (29)$$

where $Q_0(\sigma; h)$, the new ultra-local functional integral, is given by

$$Q_0(\sigma; h) = \mathcal{N} \int [d\varphi] \exp\left(\int d^d x \left(-\frac{1}{2}\sigma \, m_0^2 \, \varphi^2(x) - \frac{g_0}{4!}\varphi^4(x), +h(x)\varphi(x)\right)\right). \qquad (30)$$

and the new kernel $K(m_0, \sigma; x - y)$ is defined by

$$K(m_0, \sigma; x - y) = \left(-\Delta + (1 - \sigma)m_0^2\right)\delta^d(x - y), \qquad (31)$$

where σ is a complex parameter defined in the region $0 \leq Re(\sigma) \leq 1$. The choice of a suitable σ will simplify our calculations, and avoid infrared divergences.

In the next section we study the analytic structure of the ultra-local $(\lambda\varphi^4)_d$ theory. The divergences that appear in the formal representation for the Schwinger functional $Z(h)$ are from two different kinds. The first kind is related to the infinite volume and continuum hypotesis of the

Euclidean space and this divergence can be controled by the introduction of a box and a regulator function. The second kind of divergences is related to the functional form of the non-Gaussian part of the action and appears as a divergent perturbative series. We are concerned to this second kind of divergences.

The study of the ultra-local model in different field theories will clarify the structure of the singularities for the perturbative series and also the structure of the singularities for the n-point Schwinger functions in the complex coupling constant plane. We first investigate the analytic structure of the $(\lambda \varphi^4)_d$ ultra-local model, in the complex coupling constant plane.

4   The analytic structure of the ultra-local $(\lambda\varphi^4)_d$ model.

The aim of this section is to analyze the analytic structure of the zero-dimensional $\lambda\varphi^4$ model in the complex coupling constant g_0 plane. As we discussed before, the first term of the strong-coupling expansion of $Z(h)$ is exactly the ultra-local model [66] [74] [75] [78] [79], also called the static independent value model. Since we can interpret the ultra-local model as an infinite product of ordinary integrals, let us introduce a Euclidean lattice and analyse the generating function defined in each point of the Euclidean lattice given by

$$z(m_0, g_0; h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi \, \exp\left(-\frac{1}{2}m_0^2\,\varphi^2 - \frac{g_0}{4!}\varphi^4 + h\varphi\right), \tag{32}$$

where for simplicity we are assuming $\sigma = 1$. The generating function in the absence of external source is defined by $z(m_0, g_0; h)|_{h=0} \equiv z_0(m_0, g_0)$. Consequently, our aim is to analyse the following integral with a quartic probability distribution in which the zero-dimensional partition function $z_0(m_0, g_0)$ given by

$$z_0(m_0, g_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi \, \exp\left(-\frac{1}{2}m_0^2\,\varphi^2 - \frac{g_0}{4!}\varphi^4\right). \tag{33}$$

Note that this integral is well defined for Re $g_0 \geq 0$. As the exponential power series is convergent everywhere, we may write

$$z_0(m_0, g_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi \, \sum_{k=0}^{\infty} \exp\left(-\frac{1}{2}m_0^2\,\varphi^2\right)(-g_0)^k \frac{\varphi^{4k}}{4!^k k!}. \tag{34}$$

The above series is not uniformly convergent. Therefore it should not interchange the integration and the summation. Even thought, we perform this interchange, integrate term by term, and we get a formal series $z^{(1)}(m_0, g_0)$. We interpret this formal series as an asymptotic expansion for $z_0(m_0, g_0)$. Thus we have $z_0(m_0, g_0) \sim z^{(1)}(m_0, g_0)$ and choosing $m_0^2 = 1$ it is not difficult to show that

$$z^{(1)}(m_0, g_0)|_{m_0^2 = 1} = \sum_{k=0}^{\infty} (-g_0)^k c_k, \tag{35}$$

where the coefficients c_k are given by $c_k = \frac{(4k-1)!!}{(4!)^k k!}$. The formal series $z^{(1)}(m_0, g_0)$ is divergent for any non-nule value of g_0. That is the reason we interpret it as the asymptotic expansion for $z_0(m_0, g_0)$. The asymptotic expansion for the zero-dimensional partition function given by $z^{(1)}(m_0, g_0)$ has the contribution from the vacuum diagrams [80] [81], and each coefficient c_k is given by the sum of symmetry factors over all diagrams of order k.

It is not necessary to go so far. In this particular model we will take a short cut. The integral given by Eq.(33) can be solved exactly for Re $g_0 \geq 0$, yielding

$$z_0(m_0, g_0) = (\frac{3}{2g_0})^{\frac{3}{4}} m_0^2 \Psi \left(\frac{3}{4}, \frac{3}{2}; \frac{3m_0^4}{2g_0} \right), \tag{36}$$

where $\Psi(a, c; z)$ is the confluent hypergeometric function of second kind [82] [83], and we are using the principal branch of this function. Since we are interested to study the analytic structure of $z_0(m_0, g_0)$ in the coupling constant complex plane at $g_0 = 0$, we must investigate the analytic structure of the $\Psi(a, c; z)$ at $z = \infty$. We are following the discussion developed in Ref.[84]. The confluent hypergeometric function of second kind $\Psi(a, c; z)$ is a many valued analytic function of z, with a usual branch cut for $|\arg z| = \pi$, and a singularity at $z = 0$. Therefore $z_0(m_0, g_0)$ can be defined as a multivalued analytic function on the complex g_0 plane, with a branch cut for $|\arg g_0| = \pi$ and a singularity at $g_0 = 0$. So we have to consider its principal branch in the plane cut along the negative real axis. The analytic continuation corresponds to the definition for $z_0(m_0, g_0)$ in the whole coupling constant complex plane except for a branch cut for $|\arg z| = \pi$.

For non-real of the coupling constant the resulting a theory probably violates the Osterwalder-Schrader axioms, which ensure that it is possible to obtain the Wightman functions from the Schwinger functions defined in Euclidean space using the Osterwalder-Schraeder reconstruction theorem. So we are using the analytic extension as a theoretical tool. As we discussed, the

generating function in the absence of external sources was defined by $z_0(m_0, g_0)$ and it is called the zero-dimensional partition function. This situation defines a genuine stocastic process, since the Schwinger functions of the theory are the moments of a non-Gaussian probability distribution. In the case of the analytic extended zero-dimensional generating function we are loosing the positive probability interpretation.

To generate the Schwinger functions we introduce sources in the model. Thus we have that the zero-dimensional generating function $z(m_0, g_0; h)$ is given by

$$z(m_0, g_0; h) = \sqrt{\frac{2}{\pi}} \int_0^\infty d\varphi \, \exp\left(-\frac{1}{2} m_0^2 \varphi^2 - \frac{g_0}{4!} \varphi^4\right) \cosh(h\varphi). \tag{37}$$

As we discussed, it is possible to find $z(m_0, g_0; h)|_{h=0}$ in a closed form. Nevertheless it is not possible to express $z(m_0, g_0; h)$ in terms of known functions. If we try to expand $\exp(h\varphi)$ in power series and, in order to solve the resulting integrals, we interchange the summation and the integration, we have problems because the power series is uniformly convergent only if $|h\varphi| < 1$. The result of this operation is $z^{(1)}(m_0, g_0; h)$ which we interpret as the asymptotic expansion of $z(m_0, g_0; h)$. We may write $z^{(1)}(m_0, g_0; h) \sim z(m_0, g_0; h)$. Thus we have

$$z^{(1)}(m_0, g_0; h) = \sum_{k=0}^\infty h^{2k} f_k(m_0, g_0), \tag{38}$$

where the coefficients f_k are given by

$$f_k(m_0, g_0) = \frac{(-1)^k}{\sqrt{2\pi}} \frac{2^{k+1}}{2k!} \left(\frac{\partial}{\partial m_0^2}\right)^k z_0(m_0, g_0). \tag{39}$$

Recall that $z_0(m_0, g_0)$ is the generating function in the absence of sources. Using Eq.(36) we evaluate the partial derivatives of $z_0(m_0, g_0)$ in the above formula. After some algebra, we have the asymptotic representation for $z(m_0, g_0; h)$ in terms of derivatives of the confluent hypergeometric function of second kind,

$$z^{(1)}(m_0, g_0; h) = \left(\frac{3}{2g_0}\right)^{\frac{3}{4}} \left(\sqrt{\frac{2}{\pi}} m_0^2 \Psi\left(\frac{3}{4}, \frac{3}{2}; \frac{3m_0^4}{2g_0}\right) + \sum_{k=1}^\infty h^{2k} c_k \left(\frac{\partial}{\partial m_0^2}\right)^k \Psi\left(\frac{3}{4}, \frac{3}{2}; \frac{3m_0^4}{2g_0}\right)\right) \tag{40}$$

where the coefficients c_k are given by $c_k = \frac{(-1)^k}{\sqrt{2\pi}} \frac{2^{k+1}}{2k!}$. Let us study the singularities of $z^{(1)}(m_0, g_0; h)$ in the complex coupling constant for $0 < |g_0| < \infty$. The derivatives of the confluent hypergeometric

functions of second kind are given by

$$\frac{d^n}{dz^n}\Psi(\alpha,\gamma;z) = (-1)^n(\alpha)_n\Psi(\alpha+n,\gamma+n;z), \tag{41}$$

where the coefficients $(\alpha)_k$ are defined by

$$(\alpha)_0 = 1, ... \ (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)...(\alpha+k-1), \tag{42}$$

for $k = 1, 2, ...$ Therefore, again we note that in the series representation for $z^{(1)}(m_0, g_0; h)$ we found a branch points at $g_0 = 0$, $g_0 = \infty$ and a branch cut at arg $g_0 = \pi$.

Now we turn to the second question that we have raised: in which circunstances the strong-coupling expansion must be used? To clarify this problem, in the next section we perform an perturbative expansion in two toy-models where the use of the strong-coupling expansion is imperative.

5 The ultra-local approximation in scalar models.

In the infinite cut-off limit of an infrared free theory, the usual perturbative expansion where we assume that the non-Gaussian contribution is a perturbation of the corresponding free theory seems to be meaningless, and a alternative scheme must be used. The strong-coupling expansion is alternative perturbative expansion that is suitable for treat this situation. It is remarkable that in the strong coupling expansion different scalar theories can be treated in the same way, since we factor out the free part of the Lagrangian density and evaluate the remaining non-Gaussian contribution in a closed form. From this discussion we see that this unusual expansion can be performed for any polynomial or non-polynomial interaction $V(g_i; \varphi)$, where $g_i, i = 1, 2, ..n$ are the coupling constants of the model. In a different context, for the study of non-polynomial scalar models at finite temperature in the one-loop approximation, see for instance Ref. [85].

Going back, the formal representation for the generating functional of complete Schwinger functions $Z(h)$ using the strong-coupling expansion is given by

$$Z(h) = \left(1 - \frac{1}{2}\int d^dx \int d^dy \frac{\delta}{\delta h(x)}K(m_0,\sigma;x-y)\frac{\delta}{\delta h(y)} + ...\right)Q_0(\sigma;h), \tag{43}$$

where the ultra-local generating functional $Q_0(\sigma; h)$ is defined by the following functional integral:

$$Q_0(\sigma; h) = \mathcal{N} \int [d\varphi] \, \exp \left(\int d^d x \left(-\frac{1}{2} \sigma \, m_0^2 \, \varphi^2(x) - V(g_i; \varphi) + h(x)\varphi(x) \right) \right), \tag{44}$$

and $\mathcal{N}$ is the normalization factor. Let us study the ultra-local generating functional $Q_0(\sigma; h)$ in details. This generating functional is a mean zero Gaussian functional integral and using the fact that the fields defined in each point of the Euclidean space are statistically independent we are able to represent $Q_0(\sigma; h)$ as

$$Q_0(\sigma; h) = \exp \left(- \int d^d x \, L(\sigma; h(x)) \right), \tag{45}$$

where $L(\sigma; h(x))$ is some function. The formulas given by Eq.(44) and Eq.(45) are fundamental for our study. To proceed, let us see how it is possible to extract some informations choosing particulars $V(g_i; \varphi)$. We limit ourselves to models with only one component. The generalization of our investigations to models with more than one component does not present any difficulty. For a discussion of the strong-coupling expansion in the $O(N)$ model, see for instance [42].

The ideas of the preceding sections will be ilustrated in two models where the usual perturbative expansion in the coupling constant can not be performed. The first one is given by interaction Lagrangian

$$\mathcal{L}_{\mathcal{II}}(\beta, \gamma; \varphi) = \beta \varphi^p(x) + \gamma \varphi^{-p}(x). \tag{46}$$

where β and γ are bare parameters. The second model that we would like to discuss, which we call the sinh-Gordon model, is defined by the following interaction Lagrangian:

$$\mathcal{L}_{\mathcal{III}}(\beta, \gamma; \varphi) = \beta \left(\cosh \gamma\varphi(x) - 1 \right), \tag{47}$$

where β and γ are also bare parameters. Let us start studying the model defined by Eq.(46). Note that in this model we have a suppression of the configurations fluctuations around $\varphi = 0$. For p even the model has two minima. It is not difficult to show that the model has a power series representation. The equilibrium values are given by $\varphi_0 = (\frac{\gamma}{\beta})^{\frac{1}{2p}}$ and $\varphi_0 = -(\frac{\gamma}{\beta})^{\frac{1}{2p}}$. Let us choose the case where $\varphi_0 > 0$ and define a new field $\phi(x) = (\varphi(x) - \varphi_0)$. Using the binomial expansion and its generalization

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad |x| < 1, \tag{48}$$

where the coefficients of the expansion are given by

$$\binom{\alpha}{0} = 1, \ \binom{\alpha}{k} = \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}, \ for \ k \geq 1, \tag{49}$$

we have

$$\mathcal{L}_{\mathcal{II}}(\gamma, \beta; \ \phi) = \sum_{k=0}^{p} c(p,k)\,\phi(x)^k + \gamma\,\varphi_0^{-p-k} \sum_{k=p+1}^{\infty} \binom{-p}{k}\,\phi(x)^k. \tag{50}$$

The coefficients $c(p,k)$ are given by

$$c(p,k) = \left(\beta\,\varphi_0^{-p-k}\,\binom{p}{k} + \gamma\,\varphi_0^{-p-k}\,\binom{-p}{k}\right). \tag{51}$$

Note that the generalization of the binomial series is valid for any complex exponent α. In other words, the power series in Eq.(48) is convergent everywhere in α. complex plane. The zero-dimensional generating functional is given by

$$z_2(\beta, \gamma; \ h) = \sqrt{\frac{2}{\pi}} \int_0^\infty d\varphi \ \exp\left(-\beta\varphi^p - \gamma\varphi^{-p}\right) \cosh(h\varphi). \tag{52}$$

It is not possible to express $z_2(\beta, \gamma; \ h)$ in terms of known functions. First let us express the zero-dimensional generating function $z_2(\beta, \gamma; \ h)$ in the absence of sources in a closed form. Than, expand $\cosh(h\varphi)$ in power series and, in order to solve the resulting integrals, interchange the summation and the integration. Again $z^{(2)}(\beta, \gamma; \ h)$ that we obtained after use the power series expansion is the asymptotic expansion of $z_2(\beta, \gamma; \ h)$ and we write that $z^{(2)}(\beta, \gamma; \ h) \sim z_2(\beta, \gamma; \ h)$. It is not difficult to find $z_2(\beta, \gamma; \ h)|_{h=0}$. Using the identity

$$\int_0^\infty dx \ x^{\nu-1} \exp\left(-\beta\,x^p - \gamma\,x^{-p}\right) = \frac{2}{p}(\frac{\gamma}{\beta})^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}\left(2\sqrt{\beta\gamma}\right), \tag{53}$$

that is valid for $Re\,\beta > 0$ and $Re\,\gamma > 0$, and where $K_\nu(z)$ is the modified Bessel function of third kind, we have that the zero-dimensional generating function $z_2(\beta, \gamma; \ h)$ in the absence of sources is given by

$$z_2(\beta, \gamma; \ h)|_{h=0} = \frac{1}{p}\sqrt{\frac{8}{\pi}}(\frac{\gamma}{\beta})^{\frac{1}{2p}} K_{\frac{1}{p}}\left(2\sqrt{\beta\gamma}\right). \tag{54}$$

Using that $z^{(2)}(\beta, \gamma; \ h) \sim z_2(\beta, \gamma; \ h)$ we have

$$z^2(\gamma, \beta, h) = \sum_{k=0}^{\infty} h^{2k} c(p,k)(\frac{\gamma}{\beta})^{\frac{2k+1}{p}} K_{\frac{2k+1}{p}}\left(2\sqrt{\beta\gamma}\right), \tag{55}$$

where the coefficients $c(p, k)$ are given by

$$c(p, k) = \frac{1}{p}\sqrt{\frac{8}{\pi}}\frac{1}{2k!}. \tag{56}$$

Let us discuss now the sinh-Gordon model. This model is also non-renormalizable in the weak-coupling-perturbative expansion, where β and γ are the coupling constants and choosing $\sigma = 0$. It is clear that in the zero-dimensional generating function $z_3(\beta, \gamma; h)$ in the absence of sources can be found in a closed form and it is given by:

$$z_3(\beta, \gamma; h)|_{h=0} = \frac{2e^{\beta}}{\gamma}K_0(\beta). \tag{57}$$

It is interesting that for this kind of theory, even in the presence of sources, we can find a closed form for the zero-dimensional generating fuction $z_3(\beta, \gamma; h)$,

$$z_3(\beta, \gamma; h) = \frac{2e^{\beta}}{\gamma}K_{\frac{h}{\gamma}}(\beta), \tag{58}$$

where again $K_{\nu}(z)$ is the modified Bessel function of third kind. Using Eq.(45), Eq.(57) and Eq.(58) we have that the ultra-local generating functional for the sinh-Gordon model is given by

$$Q_0(h) = \mathcal{N}\exp\left(\frac{1}{a^d}\int d^dx \ln\left(\frac{2e^{\beta}}{\gamma}K_{\frac{h}{\gamma}}(\beta)\right)\right), \tag{59}$$

where the normalization N can be found using that $Q_0(h)|_{h=0} = 1$. Note that the lattice space a appears in the above equation, but in the folowing we shall neglect the lattice structure. In the next section we sketch how it is possible to obtain a renormalized Schwinger functional, going beyond the ultra-local approximation.

6 Going beyond the ultra-local approximation in scalar models.

In this section we are interested to compute global quantities as for example the free energy or the pressure of the vacuum. The picture that emerges from the discussion is the following: in the

strong-coupling perturbative expansion we reduce the problem of the singularities of the Schwinger functional into two parts. The first one is how to define the ultra-local generating functional and the second one is to regularize and renormalize the other terms of the perturbative expansion, which came from the coupling between distinct points and are giving by the non-diagonal part of the kernel. Let us assume that a scalar field with some generic interaction Lagrangian is defined in a compact region of the Euclidean space. Thus it is clear that the zeta function regularization can be used to control the divergences of the kernel $K(m_0; x - y)$ integrated over the volume [86].

In the infinite volume limit to complete our discussion we have to sketch the formalism that can be used to obtain a regularized expression for $Z^{(1)}(h)$, going beyond the ultra-local approximation. We are using two different regularization procedures, and it is possible to identify the divergent contribution in each regularized expression and a renormalization procedure is implemented with an appropriate subtraction of the singular contribution. We would like to stress that such kind of study in another context was performed by Svaiter and Svaiter [87]. These authors developed a method to unify two unrelated regularization methods frequently employed to obtain the renormalized zero-point energy of quantum fields. Introducing a mixed cut-off function and studying the analytic properties of the regularized energy as a function of the two cut-off parameters it was possible to not only relate the usual cut-off method and the analytic regularization method, but also unify both methods.

Let us use the ideas discussed above introducing a exponential cut-off and also an algebraic cut-off to regularize the kernel $K(m_0, x - y)$. For simplicity let us assume that we have a constant external source, i.e., $h(x) = h = cte$ and also $\sigma = 0$. The second term of the perturbative series given by Eq.(28) becomes

$$Z^{(1)}(h) = -\frac{1}{2} \frac{\partial^2}{\partial h^2} \int d^d x \ \int d^d y \, K(m_0; x - y) \, Q_0(h). \tag{60}$$

The singularities of the above expression are comming from different terms. First the second derivative with respect to the source of the ultra-local generating functional is singular in the continuum limt. Since we have been discussed how to deal with the singularities of the ultra-local generating functional, let us study only the divergences comming from the kernel $K(m_0, x - y)$ integrated over the Euclidean volume where the scalar field has been defined. In the infinite

volume limit, the Fourier representation for the kernel $K(m_0; x - y)$ is given by

$$K(m_0; x - y) = \frac{1}{(2\pi)^d} \int d^d q \, (q^2 + m_0^2) \exp(iq(x - y)).$$ (61)

To evaluate the behavior of the kernel $K(m_0; x - y)$ for $|x - y|$ small and large, as we discussed let us introduce two different regulators. The divergent expression given by Eq.(61) can be regularized using for example a exponential cut-off function $f_1(m_0, \eta; q)$ defined by

$$f_1(m_0, \eta; q) = \exp\left(-\eta\left(q^2 + m_0^2\right)\right), \quad \text{Re}(\eta) > 0.$$ (62)

Another possibility is to use an analytic regularization procedure introducing an algebraic cut-off function $f_2(m_0.\rho; q)$ defined by

$$f_2(m_0, \rho; q) = (q^2 + m_0^2)^\rho, \quad \text{Re}(\rho) < -\frac{d}{2} - 1.$$ (63)

It is clear that the analytic regularization that we are using is similar to the analytic and dimensional regularization used to control divergences of the Feynman diagrams in the weak coupling expansion [88]. Explicit and exact integrations can be performed in both cases.

In order of carried out this program, let us study first the exponential cut-off method. The regularized kernel $K(m_0, \eta; x - y)$ is defined by

$$K(m_0, \eta; x - y) = \frac{1}{(2\pi)^d} \int d^d q \, \exp(iq(x - y)) \, (q^2 + m_0^2) \exp\left(-\eta(q^2 + m_0^2)\right).$$ (64)

It is clear that we can write the regularized kernel $K(m_0, \eta; x - y)$ as

$$K(m_0, \eta; x - y) = -\frac{1}{(2\pi)^d} \frac{\partial}{\partial \eta} \int d^d q \, \exp(iq(x - y)) \exp\left(-\eta(q^2 + m_0^2)\right).$$ (65)

Since the integral in Eq.(65) is Gaussian, can be performed and we obtain the following expression for the regularized kernel:

$$K(m_0, \eta; x - y) = -\frac{1}{(2\sqrt{\pi})^d} \frac{\partial}{\partial \eta} \left(\eta^{-\frac{d}{2}} \exp(-\eta \, m_0^2 - \frac{1}{4\eta}(x - y)^2)\right).$$ (66)

Thus we have that the regularized kernel $K(m_0, \eta; x - y)$ can be expressed as

$$K(m_0, \eta; x - y) =$$
$$-\frac{1}{(2\sqrt{\pi})^d} \exp\left(-\eta \, m_0^2 - \frac{1}{4\eta}(x - y)^2\right) \left(-\frac{1}{4\eta^{\frac{d}{2}+2}}(x - y)^2 + \frac{d}{2\eta^{\frac{d}{2}+1}} + \frac{m_0^2}{\eta^{\frac{d}{2}}}\right).$$ (67)

Note that the negative powers portion of the Laurent series expansion of $K(m_0, \eta; x - y)$ around $\eta = 0$ has an infinite number of terms and the regularized expression has an essential singularity at $\sigma = 0$. Thus, let use an alternative method, i.e. the analytic regularization procedure, that we call an algebraic cut-off. The same idea was presented by Kovesi-Domokos [40] in the regularization of the strong-coupling perturbative expansion.

Using the algebric cut-off function $f_2(m_0, \rho; q)$ the regularized kernel $K(m_0, \rho; x - y)$ is defined by

$$K(m_0, \rho; x - y) = \frac{1}{(2\pi)^d} \int d^d q \, \exp\left(iq(x - y)\right) \left(q^2 + m_0^2\right)^{1 + \rho}. \tag{68}$$

The regularized kernel $K(m_0, \rho; x - y)$ is convergent and analytic in the complex ρ plane for $\operatorname{Re}(\rho) < -\frac{d}{2} - 1$. As in any cut-off method we have to take the limit $\rho \to 0$, starting from $\operatorname{Re}(\rho) < -\frac{d}{2} - 1$. To perform the d-dimensional integration let us work in a d-dimensional polar coordinate system. Defining $|x - y| = r$ and $q = (q_1^2 + q_2^2 + .. + q_d^2)^{\frac{1}{2}}$ it is easy to show that the regularized kernel can be expressed in the following way:

$$K(m_0, \rho; r) = \frac{1}{(2\pi)^d r^{\frac{d}{2} - 1}} \int_0^\infty dq \, q^{\frac{d}{2}} (q^2 + m_0^2)^{1 + \rho} J_{\frac{d}{2} - 1}(q \, r), \tag{69}$$

where $J_\nu(z)$ is the Bessel function of first kind of order ν. Let us analyse the cases $r \neq 0$ and the case where $r = 0$ separately to make our discussion more precise:

i) the case $r = 0$: the case where $r = 0$ is trivial. We have for odd d that the kernel is given by $K(m_0, \rho; x \approx y)|_{\rho=0} = 0$ and for even d it is trivial to show that

$$K(m_0, \rho; x \approx y) = \frac{1}{(2\sqrt{\pi})^d} (m_0^2)^{\frac{d}{2} + \rho + 1}. \tag{70}$$

ii) the case $r \neq 0$: the more interesting case, where $r \neq 0$ can be solved evaluating the integral $I(\mu, \nu; a, b)$ defined by

$$I(\mu, \nu; a, b) = \int_0^\infty dx \, \frac{x^{\nu + 1}}{(x^2 + a^2)^{\mu + 1}} J_\nu(bx), \quad a > 0, \ b > 0. \tag{71}$$

To evaluate this integral first let us start from an integral representation for the gamma function $\Gamma(z)$, using the identity

$$\frac{1}{(x^2 + a^2)^{\mu + 1}} = \frac{1}{\Gamma(\mu + 1)} \int_0^\infty dt \, t^\mu \exp\left(-t(x^2 + a^2)\right), \tag{72}$$

which is valid for $\text{Re}(\mu) > -1$ and to guaranty absolute convergence of the double integral that we are evaluating we have to assume that the parameters μ and ν are defined in the region $-1 < \text{Re}(\nu) < 2\,\text{Re}(\mu) + \frac{1}{2}$. Then, using an integral representation for the Macdonald's function and also the identity given by

$$\int_0^\infty dx\, x^{\nu+1} \exp(-a^2 x^2)\, J_\nu(bx) = \frac{b^\nu}{(2a^2)^{\nu+1}} \exp\left(-\frac{b^2}{4a^2}\right), \;\; a > 0, \;\; b > 0, \tag{73}$$

it is possible to show that $I(\mu, \nu;\, a, b)$ is given by

$$I(\mu, \nu;\, a, b) = \frac{a^{\nu-\mu} b^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab), \tag{74}$$

which is valid for $-1 < \text{Re}(\nu) < 2\,\text{Re}(\mu) + \frac{1}{2}$. Using the principle of analytic continuation we have that the regularized kernel is given by

$$K(m_0, \rho;\, r) = \frac{1}{(2\pi)^d} \left(\frac{m_0}{r}\right)^{\frac{d}{2}+\rho+1} \Gamma(-1-\rho)^{-1} K_{\frac{d}{2}+\rho+1}(m_0\, r). \tag{75}$$

We obtained that in the limit where $\rho \to 0$ only the $r = 0$ case gives contribution to $Z^{(1)}(h)$, since the Gamma function $\Gamma(z)$ has simple poles at $z = 0, -1, -2...$ and $\frac{1}{\Gamma(z)}$ is an entire function of z. From our discussions we see that the divergence that appears in the strong-coupling perturbative expansion in the first-order approximation is proportional to the volume of the domain where we defined the fields. Working in a bounded Euclidean volume, it is interesting to speculate that since the free energy is proportional to the volume, the high orders terms in the perturbative expansion defined by Eq.(27) does not contribute to the free energy. Consequently in the high-order approximation, the introduction of appropriate counterterms makes the free energy proportional to the volume of the Euclidean space. The necessity of such renormalization procedure is still unclear for us.

7 Conclusions

In this paper, we first discuss the usual weak-coupling perturbative expansion and the problems presented by this perturbative expansion. The weak-coupling Feynman-Dyson perturbative

expansion with the respective Feynman diagrams is a general method to calculate the Green's functions of a renormalizable model in field theory. Since it is possible to express the S matrix elements in terms of the vacuum expectation values of products of the field operator, in particle physics it seems natural to perform in the majority of situations the weak-coupling perturbative expansion. Also, when studying for example the $(\lambda\varphi^4)_d$ model, in a four dimensional Euclidean space, since the free field Gaussian model gives a correct description of the critical regime when $d > 4$, it is natural to perform a perturbative expansion with respect to the anharmonic terms of the Lagrangian density. Nevertheless, as we discussed there are many situations where the usual perturbative expansion can no be used. Consequently, we discuss a non-standard perturbative approach that have been called in the literature the strong-coupling expansion.

From the ultra-local generating functional in different models we obtained non-perturbative results. We first analyse the singularities of the strong-coupling perturbative expansion in the $(\lambda\varphi^4)_d$ model defined in a d-dimensional Euclidean space. From the discussions we can see that the divergences which occur in any scalar model in the strong coupling expansion fall into two distinct classes. The first class is related to the infinite volume-continuum hypotesis for the physical Euclidean space. The second one is related to the functional form of the interaction action. We showed that there is a branch point singularity in the complex coupling constant plane in the $\lambda\varphi^4$ model. Second, we discuss the ultra-local generating functional in two field theory toy-models defined by the following interaction Lagrangians: $\mathcal{L}_{\mathcal{II}}(\beta, \gamma; \varphi) = \beta\varphi^p(x) + \gamma\varphi^{-p}(x)$, and the sinh-Gordon model $(\mathcal{L}_{\mathcal{III}}(g_1, g_2; \varphi) = g_1(\cosh(g_2\,\varphi) - 1))$.

Finally, performing our expansion in bounded Euclidean volume, we sketch how it is possible to obtain a renormalized generating functional for all Schwinger functions, going beyond the ultra-local approximation. Note that the picture that emerges from the discussion is the following: in the strong-coupling perturbative expansion we split the problem of defining the Schwinger functional in two parts: the first is how to define precisely the ultra-local generating functional. Here is imperative the use of a lattice appoximation to give a mathematical meaning to the non-Gaussian functional (actually is not easy to recover the continuum limit). The second part is to go beyond the ultra-local approximation and taking into account the perturbation part. This problem can be controled using an analytic regularization in the continuum. Besides these technical problems we still have the problem of obtaning the Schwinger functions from this approach. By summing over all the terms of the perturbative expansion these difficulties might be solved. We believe that

the strong-coupling perturbative expansion is not adequate to obtain the Schwinger functions of the models. In the strong-coupling perturbative expansion the free energy, or any quantity that can be derived from the free energy can be obtained. An interesting application is to calculate the free energy associated with self-interacting fields going beyond the one-loop level.

The renormalized vacuum energy of free quantum fields has been derived using different methods, as for example the zeta-function regularization [86]. In fact, the majority of the results in the literature are at the one-loop level. Although, higher-order loop corrections seems now beyond the experimental reach, at least theoretically such corrections are of interest. In the strong-coupling perturbative expansion we split the problem of defining the Schwinger functional in two parts: the ultra-local generating functional and the perturbation part that can be controled using the heat kernel, zeta function regularization or any analytic regularization in the continuum. Consequently, a natural extension of this work in to study the renormalized zero-point energy of interacting fields confined in a finite volume. The strong-coupling perturbative expansion is an alternative method to compute the renormalized free energy or the vacuum energy associated with self-interacting fields, going beyond the one-loop level. This will be a subject of future investigations.

8   Acknowlegements

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