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CRITICAL FRONTIER OF ANISOTROPIC PLANAR POTTS FERROMAGNETS: A NEW CONJUTURE

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ABSTRACT

The critical frontier of the nearest-neighbour q-state Potts ferromagnet in the fully anisotropic 3-12 lattice is conjectured through a star-triangle transformation. It recovers all the available exact results concerning particular cases, namely: (i) anisotropic square lattice for all q; (ii) anisotropic triangular and honeycomb lattices for all q; (iii) anisotropic Kagomé and diced lattices for q=2; (iv) isotropic 3-12 and Asanoha lattices for q=2. It provides proposals for several other planar lattices, in particular for the anisotropic Kagomé (and diced) one for $q\neq 2$, where it slightly differs from the Wu 1979 conjecture (which also satisfies the cases (i) and (iii)). The bond percolation critical probabilities on the 3-12 and Kagomé lattices are determined to be respectively $p_c = 0.739830...$ and $p_c = 0.522372...$

The critical frontier (CF) associated with the Ising ferromagnet is presently known for a great number of isotropic and anisotropic planar lattices (Domb 1960, Syozi 1972 and references therein). It is clearly desirable to extend this knowledge to the q-state Potts (1952) ferromagnet (whose Hamiltonian is given by $\mathcal{H} = -q \sum_{i,j} J_{ij} \delta_{\sigma_i, \sigma_j}$ where $J_{ij}>0$, $\sigma_i=1,2,\ldots,q$, $\forall i$, and (i,j) run over all couples of sites compatible with the strict planarity of the lattice). Only the CF associated with anisotropic square, triangular and honeycomb lattices are up to now well established (Potts 1952, Baxter 1973, Kim and Joseph 1975, Baxter et al 1978, Burkhardt and Southern 1978, Hintermann et al 1978) for all values of q, including $q=1$ which, through the well known Kasteleyn and Fortuin 1969 isomorphism, corresponds to bond percolation (the $q\to 0$ limit is physically interesting as well, as it corresponds to tree-like percolation and random resistor network; see Stephen 1976 and Lubensky 1978). Furthermore Wu(1979) worked out, on conjectural grounds, a proposal for the anisotropic Kagomé lattice (and its dual, namely the diced lattice). As far as we know no other exact proposals exist for any other planar lattice (we restrict ourselves to two -body Potts interactions).

Herein we intend to conjecture, by using a star-triangle transformation, the CF associated with the fully anisotropic 3-12 lattice (and its dual, namely the Asanoha or Hemp-Leaf lattice) represented in Fig. 1.a. Our proposal recovers the exact results corresponding to the anisotropic square, triangular and honeycomb lattices for all values of q and, for $q=2$ (spin - $\frac{1}{2}$ Ising model), those associated with the anisotropic Kagomé and diced lattices (Kano and Naya 1953, Domb 1960 and Syozi 1972) as well as with the isotropic 3-12 and Asanoha lattices (Syozi 1972). It provides also, as particular cases and for all values of q, conjectural CF's for several other lattices. However a rather surprising fact is verified: the present Kagomé lattice proposal (slightly) differs from Wu's 1979 one (excepting for $q = 2$ and the asymptotic behaviour in the limit $q \to 0$); for let us say the isotropic $q = 1$ case (bond percolation model) we obtain for the critical probability $p_c = 0.522372\ldots$ whereas Wu conjectures $p_c = 0.524430\ldots$ (0.4% discrepancy). If one takes into account the apparent simplicity of both conjectural arguments, this fact was a priori unexpected; let us anticipate that, at our present stage of knowledge, the problem remains open.

Let us now associate with any arbitrary lattice bond (with coupling constant J_r) a convenient variable (hereafter referred to as thermal transmissivity; Tsallis and Levy 1980 and 1981, Levy et al 1980, Tsallis 1981) introduced through

$$t_r \equiv \frac{1-e^{-qJ_r/k_BT}}{1+(q-1)e^{-qJ_r/k_BT}} \tag{1}$$

Two bonds (with transmissivities t_1 and t_2) in a series or parallel array yield an equivalent transmissivity respectively given by

$$t_s = t_1\, t_2 \tag{2}$$

$$t_p = \frac{t_1+t_2+(q-2)t_1t_2}{1+(q-1)t_1t_2} \tag{3}$$

hence

$$t_p^D = t_1^D \, t_2^D \qquad (3')$$

where we have introduced the dual transmissivity (see Tsallis and Levy 1981 and references therein)

$$t_r^D \equiv \frac{1-t_r}{1+(q-1)t_r} \quad (r = 1,2,p) \qquad (4)$$

(for details on Potts duality see Wu 1977, Alcaraz and Tsallis 1982 and references therein). The well established CF's can be written as follows:

$$(q-1)t_1t_2 + t_1 + t_2 = 1 \qquad \text{(square)} \qquad (5)$$

therefore, if $t_1 = t_2 \equiv t$,

$$t = \frac{1}{\sqrt{q} + 1} \; ; \qquad (5')$$

$$(q-2)t_1t_2t_3 + t_1t_2 + t_1t_3 + t_2t_3 = 1 \quad \text{(honeycomb)} \qquad (6)$$

therefore, if $t_1=t_2=t_3 \equiv t$,

$$t=h(q) \equiv \begin{cases} \dfrac{1}{2\cos\left[\frac{1}{3}\arccos\left(\frac{q}{2}-1\right)\right]} & \text{if } q \leq 4 \\ \dfrac{1}{\left[\frac{q}{2}-1+\sqrt{(\frac{q}{2}-1)^2-1}\right]^{1/3}+\left[\frac{q}{2}-1-\sqrt{(\frac{q}{2}-1)^2-1}\right]^{1/3}} & \text{if } q \geq 4 \end{cases} \qquad (6')$$

$$(q^2-3q+1)t_1t_2t_3+(q-1)(t_1t_2+t_1t_3+t_2t_3)+t_1+t_2+t_3=1 \quad \text{(triangular)} \quad (7)$$

therefore, if $t_1=t_2=t_3 \equiv t$,

$$t = \frac{1-h(q)}{1+(q-1)h(q)} \tag{7'}$$

Remark that $t_3=1$ in Eq.(6) or $t_3=0$ in Eq.(7) recover Eq.(5) and that $t_r \leftrightarrow t_r^D$ (r=1,2,3) transform Eq.(6) into Eq.(7) and vice-versa; note also that the discriminant associated with Eq.(6') vanishes at q=4, which is precisely (Baxter 1973, Straley and Fisher 1973 and Kim and Joseph 1975) the value separating, for two-dimensional systems, the first order phase transitions ($q>4$) from the higher order ones ($q \leq 4$).

Let us now consider the anisotropic 3-12 lattice reproduced in Fig. 1.a and perform on each single triangle the two-rooted graphs star-triangle transformation indicated in Fig. 2; this transformation implies (through Eqs. (2) and (3); see also Syozi 1972, Tsallis and Levy 1981 and de Magalhães et al 1982)

$$f(u_1;u_2,u_3) = v_2\, v_3 \tag{8}$$

$$f(u_2;u_1,u_3) = v_1\, v_3 \tag{9}$$

$$f(u_3;u_1,u_2) = v_1\, v_2 \tag{10}$$

where

$$f(u_1;u_2,u_3) \equiv \frac{u_1+u_2u_3+(q-2)u_1u_2u_3}{1+(q-1)u_1u_2u_3} \tag{11}$$

Eqs. (8-10) immediately imply

$$v_1 = \left[f(u_2;u_1,u_3)f(u_3;u_1,u_2)/f(u_1;u_2,u_3)\right]^{1/2} \quad (12)$$

$$v_2 = \left[f(u_1;u_2,u_3)f(u_3;u_1,u_2)/f(u_2;u_1,u_3)\right]^{1/2} \quad (13)$$

$$v_3 = \left[f(u_1;u_2,u_3)f(u_2;u_1,u_3)/f(u_3;u_1,u_2)\right]^{1/2} \quad (14)$$

Through this transformation the 3-12 lattice becomes a honeycomb one with transmissivities $t_1v_1^2$, $t_2v_2^2$ and $t_3v_3^2$ (where we have used again Eq.(2)); consequently, through Eq.(6), its CF is given by

$$(q-2)t_1t_2t_3v_1^2v_2^2v_3^2+t_1t_2v_1^2v_2^2+t_1t_3v_1^2v_3^2+ t_2t_3v_2^2v_3^2 = 1 \quad (15)$$

This equation (together with Eqs. (12-14) and definition (11)) univoquely determines the CF of the 3-12 lattice indicated in Fig. 1.a. Naturally the particular case $u_1=u_2=u_3=1$ (hence $v_1=v_2=v_3=1$) reproduces Eq. (6). Other interesting particular cases are the following ones:

a) $t_1=t_2=t_3 \equiv t$ and $u_1=u_2=u_3 \equiv u$ whose CF (see Fig. 3) is given by

$$t\,\frac{u+u^2+(q-2)u^3}{1+(q-1)u^3} = h(q) \quad (15')$$

which, for t=u and q=1, yields the 3-12 lattice bond percolation critical probability

$$p_c = 0.739830... \quad (15'')$$

and, for q=2, recovers the exact (isotropic) result (Syozi 1972).

b) $t_3=1$ (see Fig. 1b; noted L_1 lattice) whose CF is given by

$$(q-2)t_1t_2v_1^2v_2^2v_3^2 + t_1t_2v_1^2v_2^2 + (t_1v_1^2+t_2v_2^2)v_3^2 = 1 \qquad (16)$$

therefore, if $t_1=t_2\equiv t$ and $u_1=u_2=u_3\equiv u$ (see Fig.3),

$$(q-2)t^2\left[\frac{u+u^2+(q-2)u^3}{1+(q-1)u^3}\right]^3+(t^2+2t)\left[\frac{u+u^2+(q-2)u^3}{1+(q-1)u^3}\right]^2=1 \qquad (16')$$

c) $t_2=t_3=1$ and $t_1\equiv t$ (see Fig. 1.c; noted L_2 lattice) whose CF is given by

$$(q-2)tv_1^2v_2^2v_3^2 + tv_1^2(v_2^2+v_3^2)+v_2^2v_3^2 = 1 \qquad (17)$$

therefore, if $u_1=u_2=u_3\equiv u$ (see Fig.3),

$$(q-2)t\left[\frac{u+u^2+(q-2)u^3}{1+(q-1)u^3}\right]^3+(2t+1)\left[\frac{u+u^2+(q-2)u^3}{1+(q-1)u^3}\right]^2=1 \qquad (17')$$

d) $t_1=t_2=t_3=1$ (see Fig. 1.d; Kagomé lattice) whose CF is given by

$$(q-2)v_1^2v_2^2v_3^2+v_1^2v_2^2+v_1^2v_3^2+v_2^2v_3^2 = 1 \qquad (18)$$

therefore, if $u_1=u_2=u_3\equiv u$ (see Fig.3),

$$\frac{u+u^2+(q-2)u^3}{1+(q-1)u^3} = h(q) \qquad (18')$$

This equation yields, for q=1, the bond percolation critical probability

$$p_c \equiv u(q=1) = \frac{1}{3}\{1+4\sin\left[\frac{1}{3}\arcsin(1-\frac{27}{8}\sin\frac{\pi}{18})\right]\}=0.522372 \tag{18''}$$

Eq.(18) recovers, for q=2, the Kano and Naya 1953(see also Domb 1960 and Syozi 1972) results.

In Table 1 we have indicated the present main results for isotropic lattices (those associated with the square and honeycomb lattices have been included for comparison, as well as those calculated from the Wu 1979 Kagomé lattice conjecture).In the limit $q\to 0$ all the present planar lattices provide $t_1 \sim 1 - L\sqrt{q}$ as recently conjectured (Tsallis and de Magalhães 1981); dual lattices satisfy $L^D=1/L$. In the limit $q\to\infty$ we obtain in all the present cases $t \sim K/q^{\alpha}$; dual lattices satisfy $K^D = 1/K$ and $\alpha^D = 1-\alpha$. If we consider the convenient variable (Tsallis 1981)

$$s = \frac{\ln\left[1+(q-1)t\right]}{\ln q} \tag{19}$$

(it satisfies $s(t^D) = 1-s(t)$) we can verify (values within parenthesis in Table 1; see also Fig. 4.a) the quasi-universality (s practically independs from q for $1\leq q\leq 4$) recently proposed (Tsallis and de Magalhães 1981) for planar lattices. Another interesting variable is

$$y \equiv \frac{qt}{1-t} \sim \begin{cases} \sqrt{q}/L & \text{if } q\to 0 \\ K\sqrt{q} & \text{if } q\to\infty \end{cases} \tag{20}$$

(this is essentially that appearing in Fig. 5 of Wu 1979, and satisfies $y^D = q/y$); for a given lattice, y exhibits an almost linearity as a function of $\sqrt{q}$ (see Fig. 4b).

Let us conclude by saying that a simple two-rooted graphs star-triangle transformation has enabled the conjectural formulation of the critical frontier (Eq.(15)) associated with the fully anisotropic nearest-neighbour 3-12 lattice q-state Potts ferromagnet. A considerable amount of new results (Eqs. (15' - 18"), Figs. 3 and 4 and Table 1) follow which can be useful as reference or similar purposes. The question concerning the slight discrepancy (excepting for $q=2$ and $q \to 0$) between the present proposal for the Kagomé lattice (and its dual) and that conjectured by Wu 1979 remains open as no clear cut arguments have been found favoring one or the other. In any case it is clear that both are excellent numerical approximations even if at least one of them is necessarily wrong.

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CAPTION FOR FIGURES AND TABLE

Fig. 1 - The anisotropic 3-12 lattice (a) and relevant particular cases namely the L_1(b), L_2(c) and Kagomé (d) lattices, obtained by contracting respectively 1, 2 and 3 t-bonds; the transmissivities $\{t_r\}$ and $\{u_r\}$ ($r = 1,2,3$) are related to the Potts coupling constants through Eq.(1). The dual are indicated as well (dashed lines): Asanoha or Hemp-leaf (a), L_1^D(b), L_2^D(c) and diced (d); the performance of the transformation $t_r \to t_r^D$ and $u_r \to u_r^D$ (Eq.(4)) into Eq.(15) yields the CF associated with the anisotropic Asanoha lattice.

Fig. 2 - Two-rooted graphs star-triangle transformation which provides Eq.(8) (with definition (11)); (o) and (•) respectively represent the roots (terminal sites) and nodes (internal sites) (for details see Tsallis and Levy 1981 and references therein).

Fig. 3 - Para(P)-ferro(F)magnetic critical frontiers associated with partially "anisotropic" 3-12 (Eq.(15')), L_1 (Eq.(16')) and L_2(Eq.(17')) lattices; the various <u>isotropic</u> particular cases are indicated (by $z = \infty$ we refer to a linear chain of "bonds", each of them being an array of infinite parallel t-bonds; this structure has an infinite coordination number z). If $q = 1$ the abcissa and ordinate directly represent the corresponding bond occupancy probabilities; otherwise s(t) and s(u) are respectively related to t and u through Eq.(18). If q is nor much below 1 neither much above 4, the present CF's are, in agree

ment with Tsallis and de Magalhães 1981 proposal, quasi universal (one and the same, within the graphical width and for a given lattice, for $1 \lesssim q \lesssim 4$).

Fig. 4 - Critical point, as a function of the number of states q, associated with the present isotropic lattices (HC= =honeycomb; KG = Kagomé; SQ = square; DC = diced; TR=triangular; AS = Asanoha); we recall that $q > 4$ ($q \leq 4$) implies first (higher) order phase transition. The ordinate $s(t)$ (ordinate $y(t)$) in (a) (in (b)) is related to t through Eq. (19) (Eq. (20)) and has been chosen in order to exhibit the quasi-universality (quasi-linearity) mentioned in the caption of Fig.3 (the paper by Wu 1979). Although not obvious graphically, $\lim_{q\to 0} s(t(q)) = 1/2$ for all planar lattices; furthermore $s[q;\text{lattice}] + s[q;\text{dual lattice}] = y[q;\text{lattice}]\; y[q;\text{dual lattice}]/q = 1$.

TABLE 1 - $q = 1$, 2, 3 and 4 critical transmissivities (related to the Potts coupling constants through Eq. (1)) associated with the present lattices (the square, honeycomb and Wu 1979 Kagomé values are included for comparison). For the $q \to 0$ ($q \to \infty$) limit, where $t \sim 1-L\sqrt{q}$ ($t \sim K/q^{\alpha}$), we have indicated L (α and K); we recall that $LL^D = KK^D = \alpha + \alpha^D = 1$. All the values within parentheses are those of $s(t)$ (Eq. (19)); remark that $\lim_{q\to 1} s(t) = t \equiv p_c$ and that $\lim_{q\to\infty} s(t(q)) = 1-\alpha$. (a) Baxter et al 1978, Burkhardt and Southern 1978 and Hintermann et al 1978 and references therein; (b) Syozi 1972 and references therein.

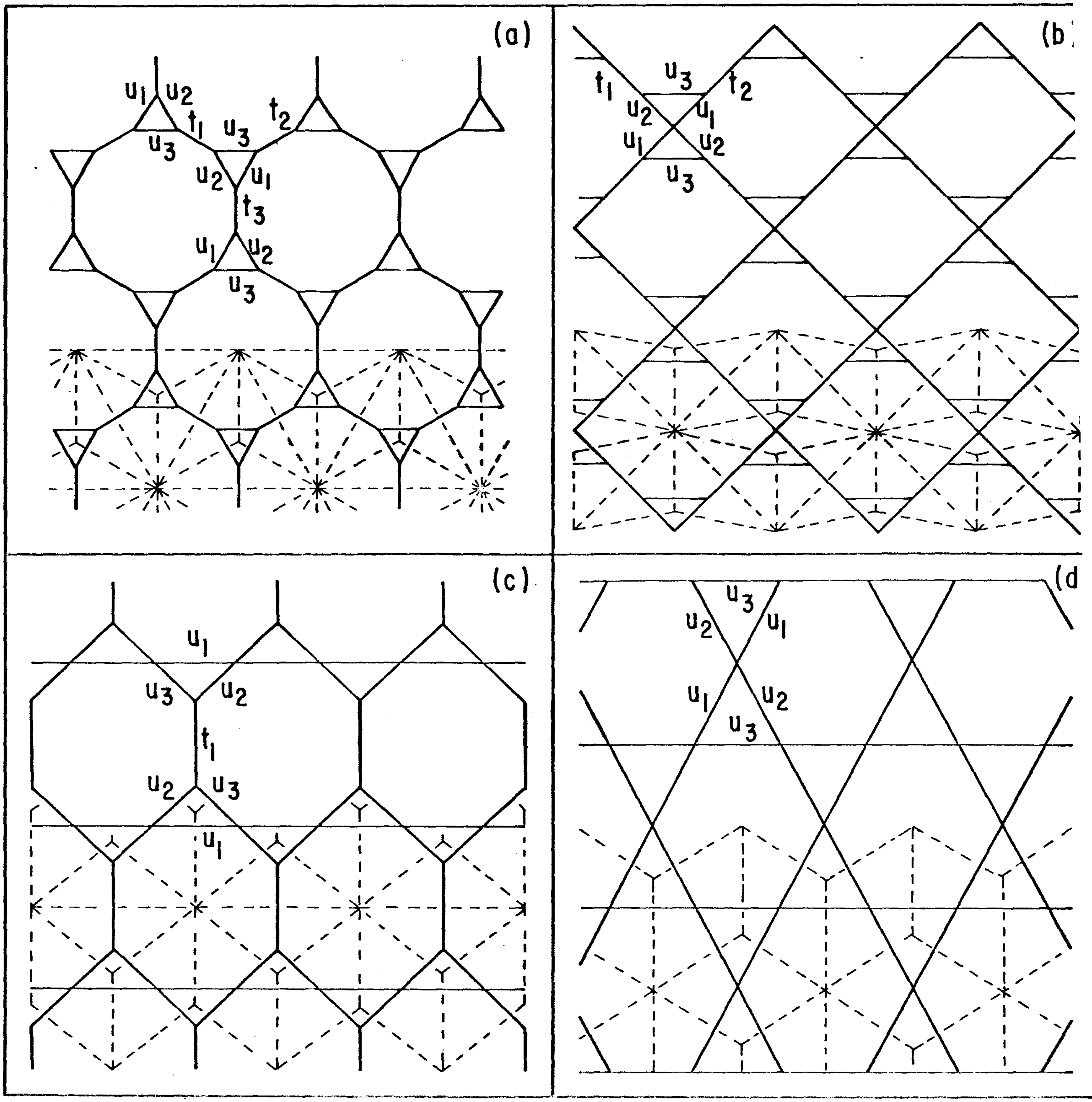
(a)
u1
u2
t1
u3
t2
u3
u2
u1
t3
u1
u2
u3
(b)
t1
u3
t2
u2
u1
u1
u2
u3
(c)
u1
u3
u2
t1
u2
u3
u1
(d)
u3
u2
u1
u1
u2
u3

FIG.1

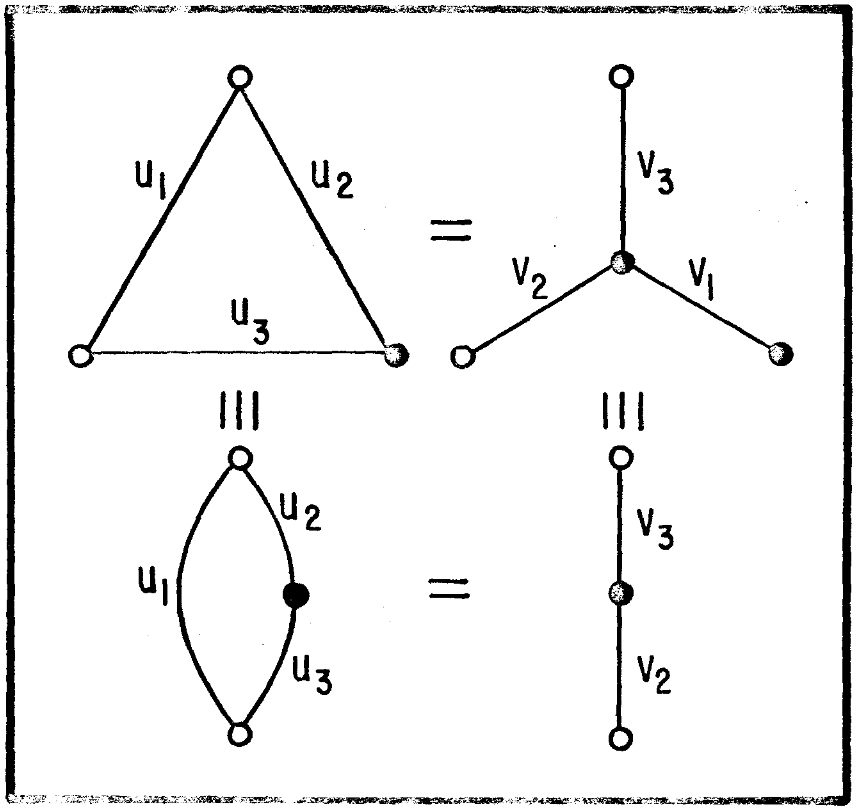

FIG. 2

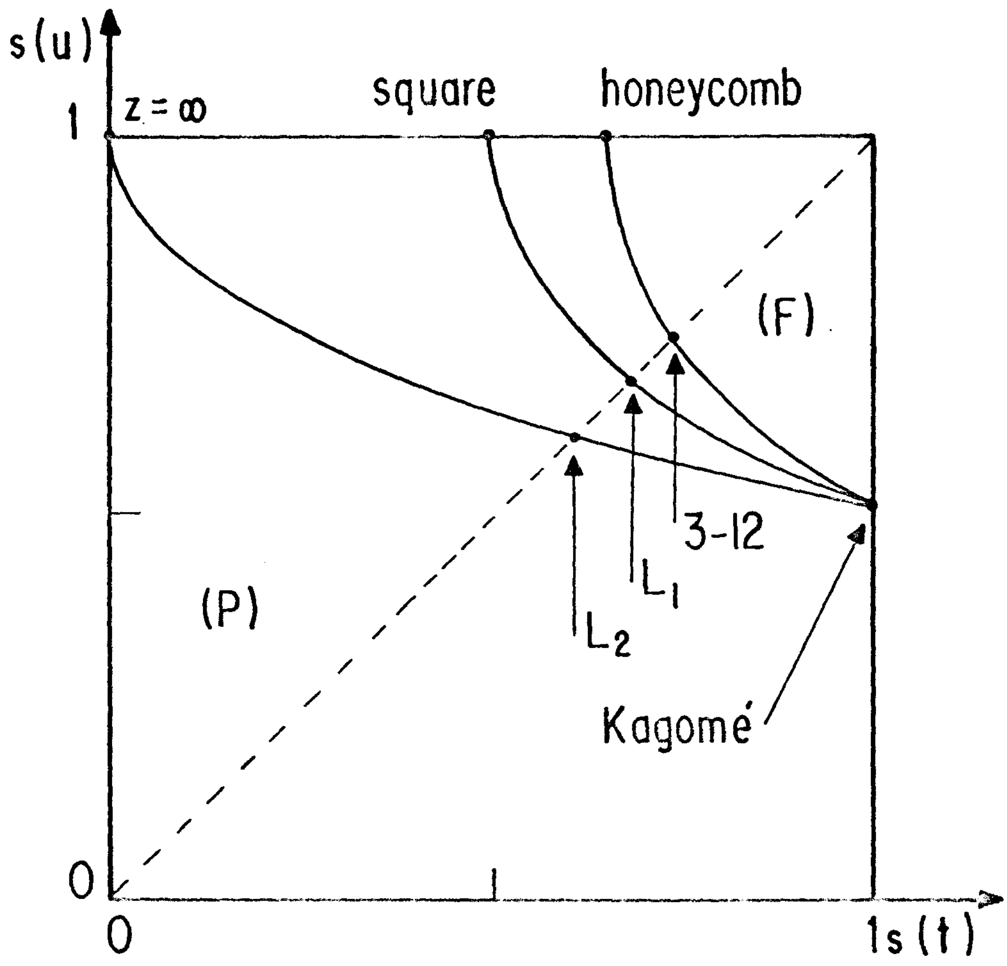

FIG.3

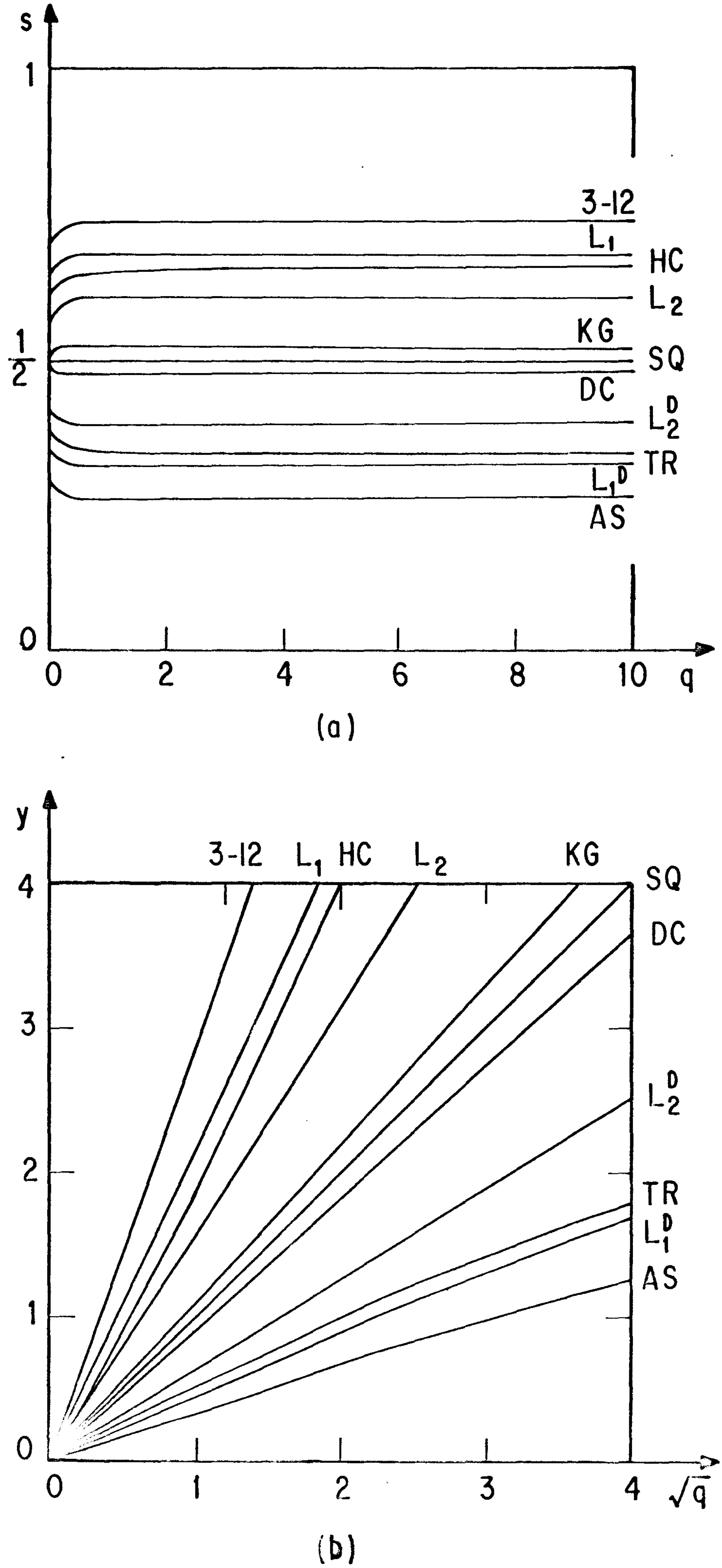
s
1
3-12
L1
HC
L2
KG
1/2
SQ
DC
L_2^D
TR
L_1^D
AS
0
0
2
4
6
8
10
q
(a)
y
3-12
L1
HC
L2
KG
4
SQ
DC
3
L_2^D
2
TR
L_1^D
AS
1
0
0
1
2
3
4
$\sqrt{q}$
(b)

FIG. 4

lattice	$q \to 0$ L	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q \to \infty$ α ; K
square[a]	1 (1/2)	1/2 (1/2)	0.41421[b] (1/2)	0.36603 (1/2)	0.33333 (1/2)	1/2; 1 (1/2)
honeycomb[a]	$1/\sqrt{3}$ (1/2)	0.65270 (0.65270)	0.57735[b] (0.65750)	0.53209 (0.65968)	1/2 (0.66096)	1/3; 1 (2/3)
3-12 (Fig. 1.a)	$\sqrt{3}/5$ (1/2)	0.73983 (0.73983)	0.67070[b] (0.74045)	0.62711 (0.73985)	0.59527 (0.73905)	1/3;1.38 (2/3)
L_1 (Fig. 1.b)	9/20 (1/2)	0.68371 (0.68371)	0.60583 (0.68332)	0.55804 (0.68228)	0.52377 (0.68125)	1/2;0.46 (1/2)
L_2 (Fig. 1.c)	3/5 (1/2)	0.60902 (0.60902)	0.52439 (0.60823)	0.47453 (0.60745)	0.43972 (0.60680)	1/2;0.60 (1/2)
Kagomē (Fig. 1.d)	$\sqrt{3}/2$ (1/2)	0.52237 (0.52237)	0.43542[b] (0.52147)	0.38645 (0.52122)	0.35321 (0.52119)	1/2;0.68 (1/2)
Kagomē (Wu 1979)	$\sqrt{3}/2$ (1/2)	0.52443 (0.52443)	0.43542[b] (0.52147)	0.38476 (0.51948)	0.35021 (0.51803)	1/2; 1 (1/2)

Table 1