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UNITARITY BOUNDS OF THE SCATTERING AMPLITUDE AND THE DIFFRACTION PEAK

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UNITARITY BOUNDS OF THE SCATTERING AMPLITUDE AND THE DIFFRACTION PEAK *

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Abstract: From unitarity alone a lower bound for the derivative of the absorptive part of the forward scattering amplitude with respect to the momentum transfer is obtained, in terms of the elastic and total cross sections. Comparison with high energy scattering experiments shows that the actual value of this derivative is rather close to the lower bound, which provides some information on the partial wave distribution. Our result can also be used to obtain consistancy requirements on theoretical models. If Regge behaviour is assumed for high energy scattering namely $F(s,t) \simeq f(t)\ s^{\alpha(t)}$, then one can show that either $\alpha'(0) \geqslant \epsilon > 0$ or $\alpha(t) \equiv$ const.

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INTRODUCTION

Two qualitative features of high energy scattering have been known for some time: i) At a given energy the total cross section and the width of the diffraction peak may not assume arbitrary values; The larger the total cross section the greater is the minimum number of partial waves required to build it up, which means a larger "radius" of the scattering object and consequently a narrower diffraction peak. An expression of such a relationship in the form of an inequality was given in a previous paper [1,2]. ii) For a given total cross section the width of the diffraction peak increases as one increases the total elastic cross section [1,2].

A rough estimate of the width of the diffraction peak is indeed easily obtained from:

$$\sigma_{el.} = \frac{2\pi}{s} \int_{-4k^2}^{0} |f(s,t)|^2 \frac{dt}{2k^2} \approx \frac{2\pi}{s} |f(s,0)|^2 \frac{\Delta}{2k^2} \geqslant \frac{\Delta}{4} \frac{\sigma_{tot}^2}{4\pi} \qquad (1)$$

which gives

$$\frac{1}{\Delta} \lesssim \frac{1}{4} \frac{\sigma_{tot}}{4\pi} \left(\frac{\sigma_{tot}}{\sigma_{el.}} \right) \qquad (2)$$

In section 1, we give a precise meaning to such a relationship by calculating a lower bound for the derivative of the scattering amplitude in the forward direction. We have found the following inequality:

$$\frac{d}{dt} \ln A(s,t) > \frac{1}{9} \left[\frac{\sigma_{tot}}{4\pi} \frac{\sigma_{tot}}{\sigma_{el.}} - \frac{1}{k^2} \right] \qquad (3)$$

In section 2 and 3 we have applied this result in connection with high energy scattering experiments and theoretical predictions.

1. Derivation of inequality 3.

Let us consider the absorptive part of the scattering amplitude:

$$A(s,t) = \frac{\sqrt{s}}{k} \sum (2\ell+1)\, a_\ell(s)\, P_\ell\left(1 + \frac{t}{2k^2}\right) \qquad (4)$$

where $a_\ell(s)$ is the imaginary part of the partial wave amplitude $f_\ell(s)$, k is the momentum in the center of mass system, s is the square of the energy and -t is the square of the momentum transfer. The requirement of unitarity imposes the following restriction on the partial wave amplitudes $a_\ell(s)$:

$$0 \leqslant a_\ell(s) \leqslant 1 \qquad (5)$$

The total cross section is given by:

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum (2\ell+1)\, a_\ell(s) \qquad (6)$$

and the total elastic cross section is given by:

$$\sigma_{el.} = \frac{4\pi}{k^2} \sum (2\ell+1)|f_\ell(s)|^2 \qquad (7)$$

which evidently satisfies the inequality:

$$\sigma_{el.} \geqslant \sigma_{el.\ im.} = \frac{4\pi}{k^2} \sum (2\ell+1)\, a_\ell(s)^2 \qquad (8)$$

The derivative of A(s,t) in the forward direction is:

$$\frac{d}{dt} A(s,t)\Big|_{t=0} = \frac{1}{2k^2} \frac{\sqrt{s}}{k} \sum (2\ell+1) \frac{\ell(\ell+1)}{2}\, a_\ell(s) \qquad (9)$$

Now one can obtain in a straightforward calculation an extremum of (9) when σ_{tot} and $\sigma_{el.im.}$ are held fix. Using the method of Lagrange multipliers we readily get that (9) is an extremum when $a_\ell(s)$ is of the form:

$$a_\ell(s) = \alpha - \beta\ell(\ell+1) \qquad (10)$$

whenever (5) is satisfied. We have, thus, to consider two cases:

a) $\alpha > 1 \left(\sigma_{el.im} > \frac{2}{3}\,\sigma_{tot}\right)$

Then an extremum of (9) is obtained for:

$$\begin{aligned} a_\ell(s) &= 1 & \ell &< L_o \\ a_\ell(s) &= \alpha - \beta\ell(\ell+1) & L_o &< \ell < L_1 \\ a_\ell(s) &= 0 & \ell &> L_1 \end{aligned} \qquad (11)$$

where L_o is the smallest integer for which $[\alpha - \beta\ell(\ell+1)] < 1$ and L_1 is the largest integer for which $[\alpha - \beta\ell(\ell+1)] > 0$. As a first approximation in our calculations we replace sums by integrals* and readily obtain:

$$\frac{d}{dt}\,\ell n\; A(s,t)|_{t=0} > \frac{1}{8}\,\frac{\sigma_{tot}}{4\pi}\left[1 + 3\left(1 - \frac{\sigma_{el.im.}}{\sigma_{tot}}\right)^2\right] \qquad (12)$$

which is in agreement with the result of ref. 1 to order $O\left(\frac{1}{k^2}\right)$. Now one can verify that this case corresponds to small inelasticity. Actually one obtains that whenever $\alpha > 1$, $\sigma_{el.im.} > \frac{2}{3}\,\sigma_{tot}$. Since in the high energy region (10 - 30 Bev) all elementary particle scattering cross sections turn out to be such that $\sigma_{el.im.} < \frac{2}{3}\,\sigma_{tot}$ we shall not proceed to give a more accurate

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* It turns out that the relative error committed in doing so is of the order $\frac{c}{k^2}$ and becomes negligible in the high energy region.

bound than (10), for this case.

b) $\alpha < 1\left(\sigma_{el.im.} < \frac{2}{3}\,\sigma_{tot}\right)$.

This case corresponds to higher inelasticity. The partial wave distribution leading to a minimum is given by

$$\begin{aligned} a_\ell(s) &= \alpha - \beta\ell(\ell+1) \qquad && \ell < L_1 \\ a_\ell(s) &= 0 && \ell > L_1 \end{aligned} \tag{13}$$

The condition $a_\ell(s) < 1$ is automatically satisfied and the only restriction imposed by unitarity is $a_\ell(s) > 0$. The exact result for the minimum is:

$$\frac{d}{dt}\,\ell n\, A(s,t)\big|_{t=0} > \frac{1}{9}\left[\frac{\sigma_{tot}}{4\pi}\;\frac{\sigma_{tot} + \sqrt{\sigma_{tot}^{2} + 12\pi\,\sigma_{el.im.}/k^2}}{2\,\sigma_{el.im.}} - \frac{3}{2k^2}\right] \tag{14}$$

The right hand side is positive for $\frac{\sigma_{tot}}{4\pi}\left(\frac{\sigma_{tot}}{\sigma_{el.}}\right) > \frac{1}{k^2}$. Actually one can verify that the expression in brackets is only slightly larger than $\left[\frac{\sigma_{tot}}{4\pi}\left(\frac{\sigma_{tot}}{\sigma_{el.}}\right) - \frac{1}{k^2}\right]$ over the entire range of energies for which this expression is positive. We shall henceforth use the simpler and more convenient inequality.

$$\frac{d}{dt}\,\ell n\, A(s,t)\big|_{t=0} > \frac{1}{9}\left(\frac{\sigma_{tot}}{4\pi}\;\frac{\sigma_{tot}}{\sigma_{el.im.}} - \frac{1}{k^2}\right) \tag{15}$$

One can make use of this inequality in two ways:

i) by direct comparison with experiment.

ii) to check the internal consistency of theoretical models.

2. COMPARISON WITH EXPERIMENT.

In order to confront (15) with experimental results we first remark that since $\sigma_{el.} > \sigma_{el.im.}$ that inequality holds even more so if one replaces $\sigma_{el.im.}$ by $\sigma_{el.}$ on the right hand side. Then the right hand side may be experimentally determined. The left hand side is not directly accessible to experimental determination. However let us assume that in the forward direction the product Re $f(s,t) \frac{d}{dt}$ Re $f(s,t)$ is negligible as compared with Im $f(s,t) \frac{d}{dt}$ Im $f(s,t)$ and in addition that at high energies the interaction becomes spin-independent. Then the left hand side will be approximately equal to $\frac{1}{2} \frac{d}{dt} \ln \left(\frac{d\sigma}{d\Omega}\right)$ which is a measurable quantity. Now using the results of Foley and others [3] for small angle scattering and the interpolation curves they propose, one finds that in the whole range of energies above 7 Bev the ratio

$$R = \frac{1}{2} \frac{d}{dt} \ln \left(\frac{d\sigma}{d\Omega}\right) \Big/ \frac{1}{9} \left[\frac{\sigma_{tot}}{4\pi} \left(\frac{\sigma_{tot}}{\sigma_{el.}} \right) - \frac{1}{k^2} \right] \qquad (16)$$

is remarkably close to one. For pp, $\pi^{\pm}$ p, $K^{\pm}$ p scattering this ratio lies in the range 1.4 - 1.5 while for $p\bar{p}$ scattering one gets $R \simeq 1.1$. This value comes about because in $p\bar{p}$ scattering the best fit for the momentum transfer distribution of the form exp. $(a + bt + ct^2)$ was obtained for $c = 0$, that is with a pure exponential.

Now the closer the ratio to the value one, the stronger the restriction on the partial wave distribution which must approach the parabolic distribution given by (13). To show how sensitive this ratio is to the partial wave distribution we give a few examples: for a rectangular distribution (a_ℓ = const. for $\ell > L$, $a_\ell = 0$ for $\ell > L$) R = 2.25; for an exponential a_ℓ = const. exp. $(-\alpha\ell)$ one obtains R = 1.68 and for a gaussian a_ℓ = const. exp. $[-\alpha\ell(\ell+1)]$ one finds R = 1.1. The first example would be in disagreement with the experimental value for all processes and the second one would be inconsistent with the data for $p\bar{p}$ scattering. One must emphasize, however, that such an analysis can only give the general behaviour of the ℓ dependence of the partial waves. One cannot for instance rule out an exponential tail in $p\bar{p}$ scattering, as required by analyticity. However, one expects that this tail does not give any sizeable contribution for the scattering amplitude. These considerations are of course valid only in so far as the two assumptions made, namely the smallness of Re $f(s,0)\frac{d}{dt}$ Re $f(s,t)|_{t=0}$ and spin independence hold true. K. J. Foley[3] and others have checked the smallness of Re $f(s,0)$ with respect to Im $f(s,0)$ and spin independence by extrapolating the elastic differential cross section to zero momentum transfer and comparing with the optical limit.

$$\frac{d\sigma_{el.}}{d\Omega}\bigg|_{t=0} \geq k^2\left(\frac{\sigma_{tot}}{4\pi}\right)^2 \qquad (17)$$

The equality is verified only if the amplitude is purely absorptive and spin independent. They have found that the equality is at least very nearly satisfied in pp and $p\bar{p}$ scattering. On the other hand the experimental determination of the derivative of the real part with respect to t is very dif ficult although not impossible in principle. One could, thus also take the view that the value of R close to one would indicate a large value for $\frac{d}{dt}$ Re $f(s,t)|_{t=0}$ with sign opposite to that of Re $f(s,0)$.

3. A THEORETICAL CONSEQUENCE

We shall now consider a theoretical implication of inequality (12). If the high energy scattering amplitude has the Regge behaviour:

$$A(s,t) \simeq \beta(t)\left(\frac{s}{s_o}\right)^{\alpha(t)} \tag{18}$$

where $\alpha(t)$ is analytic in the neighbourhood of the physical region then either $\alpha(t)$ is a constant or $\alpha'(0) \geqslant \epsilon > 0$. An elegant proof of this assertion was independently given by Sugawara [4] and Yamamoto [5] [6]. We want to show that this result also follows as a natural consequence of inequality (12).

First we notice that if $A(s,t)$ is asymptotically given by (18) and if the contribution of large momentum transfer ($t < -T$) to the elastic imaginary cross section can be neglected,* then the ratio $\sigma_{el.im.}/(\sigma_{tot})^2$ approaches zero as $s \to \infty$**, provided that $\alpha(t)$ is not a constant. Indeed, under these hypotheses, one can write:

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* In the case of identical particles one should add forward and backward contributions.

** We must point out that this hypothesis is not required in the proof presented in Reference (4) and (5).

$$\frac{\sigma_{el.im.}}{\sigma_{tot}^2} \simeq \frac{1}{16\pi^2} \int_{-T}^{0} \left(\frac{\beta(t)}{\beta(0)}\right)^2 \left(\frac{s}{s_o}\right)^{2\,[\alpha(t)-\alpha(0)]} dt \qquad (19)$$

Since $\alpha(t)$ is analytic in the neighbourhood of the physical region and by unitarity $\alpha(t) \leqslant \alpha(0)$ then the right hand side of (19) vanishes at s going to infinity. Now taking (18) into (14) one obtains:

$$\frac{\beta'(0)}{\beta(0)} + \alpha'(0)\, \ln s \geqslant \frac{1}{9} \left[\frac{\sigma_{tot}}{4\pi} \left(\frac{\sigma_{tot}}{\sigma_{el.im.}} - \frac{1}{k^2}\right)\right] \qquad (20)$$

Hence if $\alpha(t)$ is not a constant the right hand side of (20) tends to infinity which implies that

$$\alpha'(0) \geqslant \epsilon > 0$$

However if $\alpha(t)$ is constant both sides of (20) remain finite and there would be no violation of the inequality implied by unitarity in the S-channel.

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