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COUPLED INTEGRAL EQUATIONS OF THE OMNÈS-MUSKHELISHVILI TYPE

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In this note the Omnès method [1] of solution of certain singular integral equations is extended to the case of coupled equations. This problem arises in connection with many-channel γ-reactions. The extension proposed here is the analogue of Bjorken's generalization of the N/D method [2].

Let us consider a many-channel reaction:

$$\begin{aligned} a + b \longrightarrow\ & c_1 + d_1 \\ & - - - - \\ & c_n + d_n \end{aligned} \qquad (1)$$

and denote by f_i the matrix elements for these transitions.

Let us assume that the elastic channel is negligible as compared with the inelastic ones. The final state interaction is thus restricted to transitions T_{ji} from a state $(c_i\ d_i)$ to a state $(c_j\ d_j)$. In this approximation unitary gives:

$$\text{Im. } f_i = \Sigma\ T^*_{ji}\ \rho_j\ f_j$$

or in matrix notation:

$$\text{Im. } f = T^\dagger\ \rho\ f \tag{2}$$

where ρ is the phase space factor. Invariance of the interactions under time reversal implies that T_{ji} is symmetric.

Now assuming that the final state interaction is known, we want to calculate the amplitudes f_i using the partial wave equations obtained from the Mandelstam representation.

They are of the form:

$$f(\omega) = f_o\ (\omega) + \frac{1}{\pi} \int_{\omega_o}^{\infty} \text{Im } f(\omega') \frac{d\omega'}{\omega' - \omega} = f_o(\omega) + F(\omega) \tag{3}$$

where the source function $f_o(\omega)$ is real in the physical region and assumed known. In general one can write $T(\omega) = N(\omega)\ D(\omega)^{-1}$ where $N(\omega)$ is real in the physical region and $D(\omega)$ is an analytic function of ω with a branch cut on the real axis in the interval (μ_o, ∞). From unitarity one obtains:

$$\mathrm{Im}\, D = -\rho N \tag{4}$$

We then have:

$$F_+ - F_- = 2i\, \mathrm{Im}\, f_+ = 2i\, T^\dagger \rho\, f_+ = 2i\, T^\dagger \rho (f_o + F_+)$$

or:

$$(1 - 2i\, T^\dagger \rho)\, F_+ - F_- = 2i\, T^\dagger \rho\, f_o$$

Multiplying on the left by D^+ and using (4) one obtains:

$$D^T F_+ - D^\dagger F_- = 2iD^\dagger T^\dagger \rho\, f_o = 2i N^T \rho\, f_o = -2i\, \mathrm{Im}\, D^T f_o \tag{5}$$

where from one deduces:

$$F(\omega) = -D(\omega)^{-1T} \frac{1}{\pi} \int_{\omega_o}^{\infty} \frac{d\omega'}{\omega' - \omega}\, \mathrm{Im}\, D(\omega')^T f_o(\omega') \tag{6}$$

Taking this result into (3) one obtains a solution for $f(\omega)$. If $f_o(\omega)$ is an analytic function of the form:

$$f_o(\omega) = \frac{1}{\pi} \int_\Gamma \frac{d\omega'}{\omega' - \omega}\, \alpha(\omega')$$

then one can obtain directly a solution for $f(\omega)$ of the form:

$$f(\omega) = D(\omega)^{-1T} \frac{1}{\pi} \int_\Gamma D(\omega')^T \alpha(\omega') \frac{d\omega'}{\omega' - \omega} \tag{7}$$

The question of uniqueness of (6) and (7) and of substractions arises as for the single equation and can be discussed in much the same way.

The problem of determining $D(\omega)$ from the knowledge of the S-matrix by means of

$$S(\omega) = D(\omega)^* D(\omega)^{-1} \tag{8}$$

was solved by Newton and Jost [3] for non-relativistic potential scattering. It is by no means as simple as in the one channel problem where [4]

$$D(\omega) = \exp \frac{-1}{\pi} \int_{\omega_0}^{\infty} (\delta(\omega') - \delta(\omega_0)) \frac{d\omega'}{\omega' - \omega} .$$

Instead of quadratures one is led to solve a Fredholm equation.

In the field theoretical case $D(\omega)$ has the properties of the Jost functions, exeption made of the behaviour at infinity.

Let us suppose that the eigenphases violate Levinson's theorem giving:

$$\delta(\omega_0) - \delta(\infty) = (n - \alpha)\,\pi \tag{9}$$

which implies that $S(\infty) = e^{2i\pi\alpha}$. We write the S-matrix in the form:

$$S = R\, e^{2i\delta}\, R^{-1} \tag{10}$$

where R is real and unitary and δ (the eigenphase matrix) is diagonal. The removal of the bound state poles can be obtained by setting:

$$D(\omega) = R_\infty\, E(\omega)^{-1}\, \Pi(\omega) \Delta(\omega)\, R_\infty^{-1} \tag{11}$$

where

$$E(\omega) = \exp \frac{1}{\pi} \int_{\omega_0}^{\infty} \frac{\omega - \omega_1}{\omega' - \omega_1} \frac{d\omega'}{\omega' - \omega - i\epsilon} (\delta(\omega') - \delta(\omega_0)) \tag{12}$$

and $\Pi(\omega)$ is a real diagonal matrix whose elements are of the form $\lambda_i \prod_{j=1}^{n_i} (\bar{\omega}_j - \omega)$ where $\bar{\omega}_j$ are the bound state energies in the channel i. One can verify that for large ω, $E(\omega)$ behaves like $\lambda_i (e^{-i\pi}\omega)^{(n_i - \alpha_i)}$.

Taking (11) into (8) one obtains:

$$\Delta(\omega)^* \Delta(\omega)^{-1} = \Pi(\omega)^{-1} E(\omega)^* R_\infty^{-1} \; S(\omega) \; R_\infty E(\omega)^{-1} \Pi(\omega) = M(\omega) \tag{13}$$

The matrix $M(\omega)$ is no longer unitary and symmetric but satisfies the conditions (5.4) of ref. (3).

Now in order to obtain a relation (9), $D(\omega)$ must behave at infinity [5] like $R_\infty \mu (e^{-i\pi}\omega)^{\alpha} R_\infty^{-1}$ where μ is a real, constant diagonal matrix. Since $D(\omega)$ is determined but for a real, constant non-singular matrix multiplied on its right, one can choose $\mu = 1$. Therefore it follows from (11) that $\Delta(\infty) = 1$. Then the method of Newton and Jost can be applied to the determination of $\Delta(\omega)$. It is important to notice that the transformation (11) depends exclusively on the knowledge of the behaviour of the S-matrix and not of the functions $D(\omega)$.

An investigation of the photodisintegration of the deuteron along these lines is now in progress. The different channels are states of different helicities of the triplet neutron-proton system.

An interesting question occurs for J = 1. Following Goldberger et al. [6] one may atribute the deuteron bond state to a zero of $D(\omega)$ in nucleon-nucleon scattering. Then one need not introduce the deuteron intermediate state from the beginning The deuteron pole will come out in the solution (6) or (7) and in principle one is able to determine the residues which are directly related to the deuteron magnetic and quadrupole moments.

I would like to thank Dr. H. M. Nussenzveig for a helpful discussion.

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REFERENCES

1 - R. OMNÈS, Nuovo Cimento 8, 316 (1958).

2 - J. D. B[illegible]KEN, Phys. Rev. Letters 4, 473 (1960).

3 - R. G. NEWTON and R. JOST, Nuovo Cimento 1, 590 (1955).

4 - We have to subtract the phase at the origin, $\delta(\omega_0)=n\pi$, in order to avoid a pole of $D(\omega)$ at $\omega = \omega_0$.

5 - $D(\omega)$ may have, in addition, a logarithmic singularity but then $E(\omega)^{-1}$ has a singularity of the same kind and the conclusion that $\Lambda(\infty) = 1$ still holds.

6 - M. L. GOLDBERGER, M. T. GRISARU, S. W. MAC-DOWELL, D. Y. WONG. Theory of Low Energy Nucleon-Nucleon Scattering (To be published).

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