ISSN 0029-3865

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
Rio de Janeiro

Notas de Física

CBPF-NF-013/09
February 2010

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Ministério da Ciência e Tecnologia
BRASIL UM PAÍS DE TODOS
GOVERNO FEDERAL

Twist Deformation of Rotationally Invariant Quantum Mechanics

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February 5, 2010

Abstract

Non-commutative Quantum Mechanics in $3D$ is investigated in the framework of the abelian Drinfeld twist which deforms a given Hopf algebra while preserving its Hopf algebra structure.

Composite operators (of coordinates and momenta) entering the Hamiltonian have to be reinterpreted as primitive elements of a dynamical Lie algebra which could be either finite (for the harmonic oscillator) or infinite (in the general case). The deformed brackets of the deformed angular momenta close the $so(3)$ algebra. On the other hand, undeformed rotationally invariant operators can become, under deformation, anomalous (the anomaly vanishes when the deformation parameter goes to zero). The deformed operators, Taylor-expanded in the deformation parameter, can be selected to minimize the anomaly. We present the deformations (and their anomalies) of undeformed rotationally-invariant operators corresponding to the harmonic oscillator (quadratic potential), the anharmonic oscillator (quartic potential) and the Coulomb potential.

CBPF-NF-013/09

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1 Introduction

In a previous work [1] it was shown that the Wigner's Quantization [2], unlike the ordinary quantization based on creation and annihilation operators acting on a Fock vacuum, is compatible with a Hopf algebra structure of its Universal Enveloping (graded)-Lie algebra; it can therefore be regarded as the natural framework to investigate Hopf-algebra preserving, twist-deformations of quantum mechanical systems*. Due to the fact that the ordinary quantization is recovered for a special choice of the Wigner's vacuum energy it is quite important to understand whether and under which prescription a Hopf algebra structure can be implemented for the ordinary quantization (creation and annihilation operators) as well. This is the viewpoint we are adopting in this paper. Essentially, the first of our results here can be stated as follows: composite operators entering the Hamiltonian and made with Heisenberg algebra operators (coordinates, momenta, the constant $\hbar$) have to be treated as primitive elements (generators) of a dynamical Lie algebra. Their "composite" nature has to be disregarded and only their commutation relations with respect to the other primitive elements (generators) of the dynamical Lie algebra have to be retained. Within this framework the Universal Enveloping Algebra of the dynamical Lie algebra is endowed with a Hopf algebra structure.

The next topic consists in applying an abelian Drinfeld twist which deforms the Universal Enveloping Algebra while preserving its Hopf algebra structure. Since the twist is defined in terms of the three momenta p_i, for consistency these generators have to be counted among the primitive elements of the dynamical Lie algebra. A deformed Universal Enveloping Algebra expressed in terms of the twist-deformed primitive elements and their twist-deformed brackets follows from this construction. The next point consists in investigating the behavior of $3D$ non-relativistic quantum mechanical systems which are originally (i.e., in the undeformed case) rotationally invariant. One is guaranteed that the $so(3)$ algebra is preserved by the twisted angular momenta under the twisted brackets. On the other hand those operators which, at the undeformed level, are rotationally invariant (since they commute with the ordinary angular momentum generators under the ordinary brackets) can become anomalous. This means that their twisted commutators with respect to the twisted angular momenta can be non-vanishing. The non-zero result, called the deformation "anomaly", vanishes when the deformation parameter goes to zero. The anomalous operators are expanded in Taylor-series of the deformation parameter $\vec{\rho}$. A specific choice of the higher-order contributions can be made in order to minimize the overall anomaly (a similar feature is also encountered for standard quantum anomalies). These considerations apply for both the deformation of (undeformed) rotationally invariant primitive elements, as well as (undeformed) rotationally invariant composite operators (the operator $\vec{L}^2$, which is a Casimir of the $so(3)$ subalgebra, but not a Casimir of the whole Euclidean algebra $e(3)$, is perhaps the most obvious example).

In an Appendix we provide some motivations for the special role played by both the twist-deformed generators and the twist-deformed brackets. On the other hand the

*We recall that the Wigner's Quantization is based on super-Lie algebra valued operators acting on a vacuum state which corresponds to a lowest-weight representation; the ordinary quantization is recovered for a specific value of the lowest weight, which is nothing else than the Wigner's vacuum energy, see [1] and references therein for details.

connection between the abelian twist deformation and the non-commutative quantum mechanics results from the fact that the ordinary commutator between twist-deformed coordinates gives a constant matrix θ_{ij}. This is a constant element of the Universal Enveloping Algebra and depends on the deformation parameter $\vec{\rho}$. Our work is naturally motivated by the recent upsurge of interest in Noncommutative (NC) theories, both from the condensed matter physics and quantum gravity point of view. In the former case it has been known for a very long time that the guiding center coordinates of an electron moving in a plane, but subjected to a constant (i.e. uniform and static) magnetic field, give rise to noncommutativity [3]. This can have important consequences for example in QHE [4]. Besides, it can also arise due to Berry curvature effects appearing from the breaking of time-reversal symmetry in ferromagnetic systems or from the breaking of spatial-inversion symmetry in materials like GaAs crystals, as it has been been shown by Xiao et. al. [5]. In both these cases, the noncommutativity is of Moyal type, with time being the ordinary c-numbered variable. On the other hand, it has been argued by Doplicher et. al. [6], by bringing in considerations of both general relativity and quantum physics, that the nature of space-time is expected to be fuzzy at the Planck-length scale. Similar conclusions were also drawn by [7] from low energy considerations of string theory. Moyal type of Noncommutativity is one of the simplest types where these features can be realized.

On the other hand, we have still to face the perennial problem of rotational/Lorentz symmetry in NC theories defined in more than 2D. As already recalled in this paper we shall be basically considering the $3D$ problem, where the basic NC-ty among the spatial coordinates only is given by

$$[x_i, x_j] \;=\; i\theta_{ij}. \tag{1}$$

Clearly, the vector dual to θ_{ij}, i.e.

$$\theta_i \;=\; \frac{1}{2}\epsilon_{ijk}\theta_{jk}, \tag{2}$$

is pointing towards a particular direction, thereby violating the $SO(3)$ symmetry (note that the $2D$ case is safe from this problem). Nevertheless, it has been shown in the literature [8, 9] that this symmetry can be restored in a Hopf algebraic setting by using a Drinfeld's twist, such that θ_{ij} remains invariant under the twisted action of the rotation. This is in conformity with the usual philosophy of the twisted approach, where the matrix $\Theta = \{\theta_{ij}\}$ is regarded as a (matrix-valued) new constant of Nature like $\hbar, G, c$, etc. This is in contrast to other approaches followed in literature (see for example [10]). In the relativistic field theory this implies, in a similar manner, that the Poincaré symmetry itself is restored, so that the usual Wigner classification of particles remains unchanged. Since these results there has been a flurry of activities in this direction. Despite that, it was still however not clear how one could investigate the simple QM in this framework[†].

[†]It should be recalled, in this context, that the x_i's are operators in QM, while they are mere c-numbered labels for the continuous degrees of freedom in QFT and are not counted as members of the configuration space of variables. Consequently, we have to impose the Heisenberg algebra between the coordinates and the conjugate momenta in QM, i.e. $[x_i, p_j] = i\hbar\delta_{ij}$, but not in the case of QFT.

For instance it was not clear how to define a rotationally invariant potential even in the above mentioned framework of the twisted Hopf algebra. This is an important question, considering the fact that the exact solution of the energy-spectrum of a particle, confined in a noncommutative disc, has already been worked out using the method of piece-wise constant potential [11], which was subsequently used to study the thermodynamics of a system of particles confined in such a disc [12]. This analysis had the additional virtue of being carried on in a purely operatorial level, avoiding the pitfalls associated with the inequivalences between Moyal or Voros star product, which are currently debated in the literature [13]. This gives the main motivation for the present work, whose main results have been sketched before.

The scheme of the paper is as follows. In Section **2** we link (undeformed) Hopf algebras and Second Quantization, pointing out why some operators should be regarded as "primitive elements", while other operators should keep their "composite" property. In Section **3** we review the needed facts and formulas concerning the abelian Drinfeld twist. In Section **4** we discuss the twisted rotations presenting general formulas for the twisted brackets of the twisted angular momentum. In Section **5** we present the anomalous twist-deformed commutators for (twisted) primitive elements such as the quadratic (harmonic), quartic and Coulomb potentials and for the deformation of the $\vec{L}^2$ composite operator. In the Appendix we give heuristic considerations motivating the use of both twisted generators and twisted brackets. Finally, in the Conclusions we make some extra comments on the results here found.

2 Undeformed Hopf Algebras and Second Quantization

Before addressing the problem of twisting Hopf algebras in association to NC Quantum Mechanics, we need to learn how to apply undeformed Hopf algebras to Second Quantization. We work in the framework of the Hopf algebra structure of the Universal Enveloping Algebra of a Lie algebra (the Lie algebra itself is regarded as a dynamical symmetry of a quantum mechanical system). Our discussion has a general validity. For simplicity it will be illustrated with the basic examples of the Euclidean Lie algebras $e(2)$ and $e(3)$.

Additive operators (whose eigenvalues in a multi-particle state are the sum of the single-particle eigenvalues) have to be assumed as "primitive elements" of the dynamical symmetry algebra (i.e., as generators of the Lie algebra). This is because the additivity of the eigenvalues is encoded in the undeformed coproduct. Indeed,

$$\Delta(\Omega) \;=\; \Omega\otimes\mathbf{1}+\mathbf{1}\otimes\Omega \tag{3}$$

encodes the additivity of the eigenvalues ($\omega_{1+2}=\omega_1+\omega_2$) for the operator Ω.
This remark is still valid if the additive operator under consideration is a Casimir operator. The Hamiltonian H of a free-particle system is an additive operator. Therefore, $H=\frac{(\vec{p})^2}{2m}$

This distinction carries over to NC-QFT where, although the coordinates are promoted to the level of operators, they are certainly not valued in the same space as the field or other composite operators. Moreover, there is no conjugate momentum p_i to the coordinate x_i.

has to be regarded as a primitive element of the dynamical symmetry algebra despite its "composite" nature. It is also a Casimir operator of the Euclidean algebra. From physical considerations we are forced to reject the Hopf algebra equivalence $H = \vec{p}^2$ (for simplicity we set $m = \frac{1}{2}$) which would amount to consider $\vec{p}$ as an element of the Lie algebra, with H beloging to the Enveloping algebra. This Hopf algebra equality would imply the unphysical coproduct rule for H

$$\Delta(H) \;=\; \Delta(\vec{p}^2) = \Delta(\vec{p}) \cdot \Delta(\vec{p}) = \vec{p}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \vec{p}^2 + 2\vec{p} \otimes \vec{p} \neq H \otimes \mathbf{1} + \mathbf{1} \otimes H. \tag{4}$$

Note that a relation like (4) makes perfect sense (in physical, as well as in mathematical considerations) by replacing both $\vec{p}$ with the $3D$ angular momenta $\vec{L}$ and the free Hamiltonian H with the $so(3)$ Casimir operator $\vec{L}^2$. Assuming (as it has to be done) that the components of $\vec{L}$ are Lie-algebra primitive elements, the coproduct

$$\Delta(\vec{L}^2) \;=\; \Delta(\vec{L}) \cdot \Delta(\vec{L}) = \vec{L}^2 \otimes \mathbf{1} + \mathbf{1} \otimes \vec{L}^2 + 2\vec{L} \otimes \vec{L} \tag{5}$$

reflects the fact that $\vec{L}^2$ is not an additive operator since, for a composite system, we have that $(\vec{L}_{1+2})^2 = (\vec{L}_1 + \vec{L}_2)^2$.

Unlike H, which has to be assumed as a primitive element, $\vec{L}^2$ is a genuine composite operator. As this example shows, the distinction between a "primitive operator" versus a "composite operator" cannot be done in purely mathematical terms. Rather, the mathematical setting has to be accommodated to grasp the physical properties of the system under investigation.

Additive operators have a direct interpretation in terms of their primitive coproducts. Composite operators, such as $\vec{L}^2$, have no such direct interpretation. In this particular example, the eigenvalues of the composite system are obtained by decomposing into direct sums the tensor products of the subsystems with the help of the Clebsch-Gordan coefficients.

When dealing with the Second Quantization we have to specify at first the single-particle states. This can be done by giving a complete set of mutually commuting observables. One should note that these observables can be either "primitive", as well as "composite" operators in the sense specifed above. The discussion can be done in general. It is however useful to work out some specific examples that will be used in the following. Let us consider the Euclidean Lie algebras $e(2)$ and $e(3)$, respectively.

$e(2)$ admits the three generators p_1, p_2, L, satisfying the commutation relations

$$\begin{aligned} [p_1, L] &= -ip_2, \\ [p_2, L] &= ip_1, \\ [p_1, p_2] &= 0. \end{aligned} \tag{6}$$

$e(2)$ admits only one Casimir operator, $\mathcal{C} \equiv \vec{p}^2 = {p_1}^2 + {p_2}^2$. Indeed $[\vec{p}^2, L] = 0$.

The Casimir corresponds to the energy E of a non-relativistic, free, two-dimensional particle (whose mass has been normalized, as before, to $m = \frac{1}{2}$). Since the free energy is an additive operator, the Casimir operator $\mathcal{C}$ has to be added to the dynamical symmetry Lie algebra. For that we need to enlarge $e(2)$ by defining

$$\overline{e(2)} \;=\; e(2) \oplus u(1), \tag{7}$$

whose primitive generators are $\{p_1, p_2, L, \mathcal{C}\}$. $\mathcal{C}$ can be consistently identified with $\vec{p}^2$ as far as Lie-algebra and single-particle eigenvalues are concerned. We force this identification by imposing that the set of mutually commuting operators $p_1, p_2, \mathcal{C}$ admits the compatible set of respective eigenvalues $(\sqrt{E}\cos\alpha, \sqrt{E}\sin\alpha, E)$. This is not enough to completely specify the states in the Hilbert space because we still need to take into account the information carried on by the angular momentum L (whose eigenvalues are the integers m). This can be done as follows. At first, without loss of generality, we fix the "reference frame" specified by the eigenvalues $p_1 = 0$, $p_2 = \sqrt{E}$ (recovered by setting $\alpha = \frac{\pi}{2}$). Next, we consider the little (Lie) group of transformations respecting the reference frame and their associated Lie-algebra, Hermitian, operators. We can now find a complete set of observable operators which are mutually "weakly commuting" when the reference frame constraint is taken into account. In the example above, mutually "weakly commuting" observables are given by p_2, L, since $[p_2, L] = ip_1 \approx 0$, when $p_1 \equiv 0$ is taken into account.

We can extend these considerations to the less trivial case of the three-dimensional Euclidean algebra $e(3)$, whose generators $(p_1, p_2, p_3, L_1, L_2, L_3)$ satisfy the commutation relations

$$\begin{aligned} [p_i, p_j] &= 0, \\ [p_i, L_j] &= i\epsilon_{ijk}p_k, \\ [L_i, L_j] &= i\epsilon_{ijk}L_k \end{aligned} \tag{8}$$

(the L_i's are the generators of the $so(3)$ subalgebra).

$e(3)$ admits two Casimir operators, $\mathcal{C}_1$, $\mathcal{C}_2$, given respectively by

$$\begin{aligned} \mathcal{C}_1 &= \vec{p}^2, \\ \mathcal{C}_2 &= \vec{L}\vec{p}. \end{aligned} \tag{9}$$

One should note that $\vec{L}^2$ is a Casimir operator of the $so(3)$ subalgebra; on the other hand it is not a Casimir operator for $e(3)$.

We can repeat the same construction as in the $e(2)$ case, enlarging the algebra to $\overline{e(3)}$, by the addition of $\mathcal{C}_1$, $\mathcal{C}_2$ as primitive elements,

$$\overline{e(3)} = e(3) \oplus u(1) \oplus u(1). \tag{10}$$

The identification (9) is assumed to hold in the Lie algebra sense, but not in the Hopf algebra sense.

By setting E (the energy) to be the eigenvalue of the $\mathcal{C}_1$ Casimir operator, without loss of generality we can work within the $p_1 = 0, p_2 = 0, p_3 = \sqrt{E}$ reference frame. In this reference frame $\vec{L}\vec{p} \equiv L_3 p_3$, such that its eigenvalues are expressed by $m\sqrt{E}$. A set of mutually "weakly commuting" observables, respecting the given reference frame, is given by $p_3, L_3, \vec{L}^2$, with eigenvalues $\sqrt{E}$,m, $l(l+1)$, respectively. Indeed

$$\begin{aligned} [p_3, L_3] &= 0, \\ [L_3, \vec{L}^2] &= 0, \\ [p_3, \vec{L}^2] &\approx 0. \end{aligned} \tag{11}$$

A state of the system is uniquely specified in terms of its free energy E, the orbital angular momentum l and its component along the third axis m.

In the set of three, mutually weakly-commuting operators which specify the state of the (single-particle) system, two of them (p_3, L_3) are primitive operators, while the remaining one $(\vec{L}^2)$ is a composite operator.

3 The abelian Drinfeld twist

In this Section we will recall the basic formulas concerning the abelian Drinfeld twist deformation of the Universal Enveloping Algebra $\mathcal{U}(\mathbf{g})$ of a given Lie algebra $\mathbf{g}$. For our purposes the twist is expressed by $\mathcal{F} \in \mathcal{U}(\mathbf{g}) \otimes \mathcal{U}(\mathbf{g})$, such as

$$\begin{aligned} \mathcal{F} &= \exp(i\rho_{ij} p_i \otimes p_j), \\ \rho_{ij} &= \epsilon_{ijk}\rho_k, \end{aligned} \tag{12}$$

where $\vec{\rho}$ is a dimensional c-number and p_i $(i,j=1,2,3)$ are the three-dimensional momenta. It is obviously required that $p_i \in \mathbf{g}$.

The twist induces a deformation in the Hopf algebra $\mathcal{U}(\mathbf{g}) \rightarrow \mathcal{U}^{\mathcal{F}}(\mathbf{g})$ (see [14]). Particularly, the co-structures are deformed. The deformed co-structures (coproduct, counit and antipode), applied to an element $g \in \mathbf{g}$, are respectively given by

$$\begin{aligned} \Delta^{\mathcal{F}}(g) &= \mathcal{F}\Delta(g)\mathcal{F}^{-1}, \\ \varepsilon^{\mathcal{F}}(g) &= \varepsilon(g), \\ S^{\mathcal{F}}(g) &= \chi S(g)\chi^{-1}, \end{aligned} \tag{13}$$

where

$$\chi = f^{\alpha}S(f_{\alpha}) \in \mathcal{U}(\mathbf{g}) \tag{14}$$

(we are denoting, as usual, $\mathcal{F} = f^{\alpha} \otimes f_{\alpha}$, $\mathcal{F}^{-1} = \overline{f}^{\alpha} \otimes \overline{f}_{\alpha}$).

The generators of $\mathcal{U}^{\mathcal{F}}(\mathbf{g})$ are expressed as

$$g^{\mathcal{F}} = \overline{f}^{\alpha}(g)\overline{f}_{\alpha}. \tag{15}$$

The $\mathcal{F}$-deformed brackets in $\mathcal{U}^{\mathcal{F}}(\mathbf{g})$ are defined through

$$[g^{\mathcal{F}}, h^{\mathcal{F}}]_{\mathcal{F}} = {g^{\mathcal{F}}}_1 h^{\mathcal{F}} S(g^{\mathcal{F}})_2, \tag{16}$$

where the Sweedler's notation

$$\Delta^{\mathcal{F}}(g^{\mathcal{F}}) = (g^{\mathcal{F}})_1 \otimes (g^{\mathcal{F}})_2 \tag{17}$$

has been used.

The $\mathcal{F}$-deformed brackets satisfy the Jacobi identity.

A more complete list of the properties of the twist-deformed Hopf algebra $\mathcal{U}^{\mathcal{F}}(\mathbf{g})$ is encountered in [14].

The Universal Enveloping Algebra of the following Lie algebras can be deformed in terms of the (12) abelian twist. We have

i) the Heisenberg algebra $h_B(3)$, whose generators are $\hbar, x_i, p_i$ (for $i = 1, 2, 3$). $\hbar$ is a central element and the only non-vanishing commutation relations are given by

$$[x_i, p_j] \;=\; i\delta_{ij}\hbar; \tag{18}$$

ii) the Euclidean algebra $e(3)$ (considered in the previous Section), whose generators are p_i, L_i. Its non-vanishing commutators are given by

$$\begin{aligned}[p_i, L_j] &= i\epsilon_{ijk}p_k,\\ [L_i, L_j] &= i\epsilon_{ijk}L_k.\end{aligned} \tag{19}$$

This algebra can be induced by the $h_B(3)$ Heisenberg algebra after setting

$$L_i \;=\; \frac{1}{\hbar}\epsilon_{ijk}x_jp_k \tag{20}$$

and interpreting the L_i's as primitive elements[‡]. Similarly, the extended algebra $\overline{e(3)}$ introduced in (10) can be twist-deformed under (12);

iii) the algebra g, whose primitive elements are the Heisenberg algebra generators $\hbar, x_i, p_i$ and the angular momentum generators L_i whose commutation relations, as before, can be induced by the (20) position;

iv) the "oscillator" algebra *osc*, given by the set of primitive elements $\hbar, x_i, p_i, L_i, H, K, D$. The commutators involving the generators H, K, D can be read from the positions

$$\begin{aligned}H &= \frac{1}{\hbar}\vec{p}^2,\\ D &= \frac{1}{2\hbar}(\vec{x}\vec{p}+\vec{p}\vec{x}),\\ K &= \frac{1}{\hbar}\vec{x}^2\end{aligned} \tag{21}$$

(the $\hbar$ at the denominator in the r.h.s. expressions is required in order to compensate the corresponding term coming from the (18) commutators). H, K, D defines the $sl(2)$ subalgebra. The complete set of non-vanishing commutators among primitive elements of *osc* is given by

$$\begin{aligned}[x_i, p_j] &= i\delta_{ij}\hbar,\\ [x_i, L_j] &= i\epsilon_{ijk}x_k,\\ [p_i, L_j] &= i\epsilon_{ijk}p_k,\\ [x_i, H] &= 2ip_i,\\ [x_i, D] &= ix_i,\end{aligned}$$

[‡]As recalled in the previous Section, the notion of "primitive elements" is used to underline the fact that the generators of the Lie algebra should not be regarded as composite operators of the Hopf algebra structure.

$$\begin{aligned}
[p_i, D] &= -ip_i, \\
[p_i, K] &= -2ix_i, \\
[H, D] &= -2iH, \\
[H, K] &= -4iD, \\
[D, K] &= -2iK. \qquad (22)
\end{aligned}$$

The Hamiltonian of the harmonic oscillator is given by a linear combination of H and K;

v) a finite Lie algebra g_b of hermitian operators which can all be regarded as primitive elements and recovered from at most bilinear combinations in $\vec{x}$ and $\vec{p}$, is given by the set of generators $\hbar, x_i, p_i, P_{ij}, X_{ij}, {M^+}_{ij}, {M^-}_{ij}$. The commutation relations involving $P_{ij}, X_{ij}, {M^+}_{ij}, {M^-}_{ij}$ can be read by assuming

$$\begin{aligned}
P_{ij} &= \frac{1}{\hbar} p_i p_j, \\
X_{ij} &= \frac{1}{\hbar} x_i x_j, \\
{M^+}_{ij} &= \frac{1}{\hbar}(x_i p_j + p_j x_i), \\
{M^-}_{ij} &= \frac{i}{\hbar}(x_i p_j - p_j x_i); \qquad (23)
\end{aligned}$$

vi) the above construction can be further generalized. Any Lie algebra containing $h_B(3)$ as a subalgebra and at least one primitive element which is expressed as a trilinear (or k-linear, for $k \geq 3$) combination in $\vec{x}$ and $\vec{p}$ is necessarily infinite-dimensional. Indeed, the closure of the commutation relations of this generator with the previous ones requires that new higher-order multilinear terms have to be included as primitive elements. This procedure never stops, leading to an infinite-dimensional Lie algebra. This algebra can be regarded as the unfolded algebra of primitive elements (the multilinear combinations in terms of $\vec{x}$ and $\vec{p}$ is its folded version, in analogy of what happens, in a different context, with finite W-algebras [15] or the unfolded version of higher-spin algebras, see [16]). One should note that a primitive element which is k-linear in $\vec{x}, \vec{p}$, requires the $\frac{1}{\hbar^{k-1}}$ factor (for instance, a primitive element can be associated to $\frac{1}{\hbar^3}(\vec{x}^2)^2$).

4 Twisted rotations

The abelian Drinfeld twist (12) induces, through eq. (15), the following deformation of the space coordinates

$$ {x_i}^{\mathcal{F}} = x_i - \epsilon_{ijk}\rho_k \hbar p_j. \qquad (24)$$

This deformation corresponds to the Bopp shift and one should note that the second term in the r.h.s. is quadratic in the Heisenberg algebra generators. This result was also obtained in [1]. The shift maps $x_i \in h_B(3)$ into ${x_i}^{\mathcal{F}} \in \mathcal{U}(h_B(3))$. Concerning the p_i momenta, they undergo no deformation: $p_i^{\mathcal{F}} = p_i$.

The non-commutative quantum mechanics (for a constant operator θ_{ij}) is recovered from the abelian twist. Indeed

$$[{x_i}^{\mathcal{F}}, {x_j}^{\mathcal{F}}] \;=\; i\theta_{ij}, \tag{25}$$

where

$$\theta_{ij} \;=\; 2\hbar^2\epsilon_{ijk}\rho_k, \tag{26}$$

with θ_{ij} an operator belonging to $\mathcal{U}(h_B(3))$.

Similarly, but in the "opposite" direction, the $\mathcal{F}$-commutator of the ordinary coordinates produces

$$[x_i, x_j]_{\mathcal{F}} \;=\; -\frac{1}{2}i\theta_{ij}. \tag{27}$$

The $\mathcal{F}$-commutator among twisted space coordinates is vanishing

$$[{x_i}^{\mathcal{F}}, {x_j}^{\mathcal{F}}]_{\mathcal{F}} \;=\; 0. \tag{28}$$

The twisted coproduct of the space coordinates and of the twisted space coordinates is respectively given by

$$\begin{aligned} \Delta^{\mathcal{F}}(x_i) &= x_i \otimes \mathbf{1} + \mathbf{1} \otimes x_i + \epsilon_{ijk}\rho_k(\hbar \otimes p_j - p_j \otimes \hbar), \\ \Delta^{\mathcal{F}}({x_i}^{\mathcal{F}}) &= {x_i}^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes {x_i}^{\mathcal{F}} - 2\epsilon_{ijk}\rho_k p_j \otimes \hbar. \end{aligned} \tag{29}$$

If the algebra admits as primitive elements, besides the p_i's, the angular momentum operators L_i, their deformation ${L_i}^{\mathcal{F}}$, induced by the (12) twist is given by

$$\begin{aligned} {L_i}^{\mathcal{F}} &= L_i + K_i, \\ K_i &= \rho_k p_i p_k - \rho_i p_k p_k. \end{aligned} \tag{30}$$

The extra-term K_i can also be written as

$$K_i \;=\; -\rho_j \vec{p}^2 \Pi_{ij}, \tag{31}$$

in terms of the Π_{ij} projector

$$\Pi_{ij} \;=\; (\delta_{ij} - \frac{p_i p_j}{\vec{p}^2}). \tag{32}$$

The twisted coproduct of the (twisted) angular momentum reads as

$$\begin{aligned} \Delta^{\mathcal{F}}(L_i) &= L_i \otimes \mathbf{1} + \mathbf{1} \otimes L_i + \rho_k(p_i \otimes p_k - p_k \otimes p_i), \\ \Delta^{\mathcal{F}}({L_i}^{\mathcal{F}}) &= {L_i}^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes {L_i}^{\mathcal{F}} + 2\rho_k p_i \otimes p_k - 2\rho_i p_k \otimes p_k. \end{aligned} \tag{33}$$

The original $su(2)$ rotational algebra is recovered in terms of the $\mathcal{F}$-commutator of the twisted angular momentum. We have indeed

$$[{L_i}^{\mathcal{F}}, {L_j}^{\mathcal{F}}]_{\mathcal{F}} \;=\; i\epsilon_{ijk}{L_k}^{\mathcal{F}}. \tag{34}$$

As a consequence we get the first result, namely that the rotational symmetry is preserved by the (12) twist-deformation.

One can also check that

$$[x_i^{\mathcal{F}}, L_j^{\mathcal{F}}]_{\mathcal{F}} = i\epsilon_{ijk}x_k^{\mathcal{F}},$$
$$[p_i^{\mathcal{F}}, L_j^{\mathcal{F}}]_{\mathcal{F}} = [p_i, L_j^{\mathcal{F}}]_{\mathcal{F}} = i\epsilon_{ijk}p_k^{\mathcal{F}}, \qquad (35)$$

showing that both $x_i^{\mathcal{F}}$ and p_i have vectorial transformation properties under the deformed brackets. The situation is thus different from the case of the undeformed brackets where $x_i^{\mathcal{F}}$, unlike p_i, fails to transform as a vector [1].

The next important point is to check whether the operators which are rotationally invariant in the undeformed case, keep the rotational invariant property even in the deformed case or otherwise acquire an anomalous term which disappears in the limit $\vec{\rho} \to 0$. We investigate, specifically, the commutation relations

$$[L_i{}^{\mathcal{F}}, B^\sharp]_{\mathcal{F}} \qquad (36)$$

for an operator $B^\sharp$ belonging to the Universal Enveloping Algebra of a Lie algebra containing the Euclidean algebra $e(3)$ as a subalgebra and such that $B^\sharp$ is expanded in $\vec{\rho}$ Taylor series:

$$B^\sharp = B_0 + B_1 + B_2 + \ldots, \qquad (37)$$

with B_k k-linear in $\vec{\rho}$. Here $B_0 \equiv B$ denotes the undeformed limit for $\vec{\rho} \to 0$ of $B^\sharp$ (we can therefore say that the operator $B^\sharp$ is the deformation of B).

The rotational invariance in the undeformed limit requires that the following relation involving ordinary commutators and angular momentum operators has to be satisfied

$$[L_i, B_0] = 0. \qquad (38)$$

With a little algebra one can easily prove that the deformed commutator (36) can be expressed in terms of ordinary commutators:

$$[L_i{}^{\mathcal{F}}, B^\sharp]_{\mathcal{F}} = [L_i - K_i, B^\sharp] + M_{ik}[p_k, B^\sharp], \qquad (39)$$

where K_i enters (30) and M_{ik} is given by

$$M_{ik} = 2\rho_k p_i - 2\rho_i p_k. \qquad (40)$$

The r.h.s. in (39) is a consequence of the equality

$$2K_i - M_{ik}p_k = 0. \qquad (41)$$

The meaning of the $\mathcal{F}$-deformed brackets for non-commutative theories is discussed in the Appendix.

It is worth pointing out that the Taylor-expanded series (37) starting with B_0 does not necessarily coincide with the $\mathcal{F}$-deformed operator $B_0{}^{\mathcal{F}}$ (which can also be understood as Taylor-expanded). In the next Section we will discuss this point in more detail.

For completeness we write here the $\mathcal{F}$-deformed operators $H^{\mathcal{F}}, D^{\mathcal{F}}, K^{\mathcal{F}}$ obtained by applying the (12) twist to the H, D, K primitive elements of the oscillator algebra *osc* given in (22). We have

$$\begin{aligned} H^{\mathcal{F}} &= H, \\ D^{\mathcal{F}} &= D, \\ K^{\mathcal{F}} &= K - 2\epsilon_{ijk}\rho_k x_i p_j + \hbar[\vec{\rho}^2\vec{p}^2 - (\vec{\rho}\vec{p})^2]. \end{aligned} \tag{42}$$

5 Anomalous operators

For our purposes it is useful to set

$$\begin{aligned} L_i(B_n) &= [L_i, B_n], \\ T_i(B_n) &= -[K_i, B_n] + M_{ik}[p_k, B_n]. \end{aligned} \tag{43}$$

An undeformed rotationally invariant operator B such that

$$[L_i, B] = 0 \tag{44}$$

can develop, under deformation, an anomaly A_i which is expressed through

$$[L_i{}^{\mathcal{F}}, B^{\sharp}]_{\mathcal{F}} = A_i \tag{45}$$

(as discussed in the previous Section, $B^{\sharp}$ is the deformation of B).

The anomaly A_i can be expanded in powers of the deformation parameter $\vec{\rho}$. We have

$$\begin{aligned} (A_i)_0 &= L_i(B_0) = 0, \\ (A_i)_n &= L_i(B_n) + T_i(B_{n-1}), \end{aligned} \tag{46}$$

with $(A_i)_n$ the n-th order contribution in $\vec{\rho}$.

Let us consider now the deformation of the rotationally invariant primitive elements H, D, K of the *osc* oscillator algebra (22). We get that

$$H^{\sharp} = H^{\mathcal{F}} = H \tag{47}$$

is rotationally invariant under twist-deformed rotations since

$$[L_i{}^{\mathcal{F}}, H^{\mathcal{F}}]_{\mathcal{F}} = 0. \tag{48}$$

Similarly,

$$D^{\sharp} = D^{\mathcal{F}} = D \tag{49}$$

is rotationally invariant under twist-deformed rotations since

$$[L_i{}^{\mathcal{F}}, D^{\mathcal{F}}]_{\mathcal{F}} = 0. \tag{50}$$

On the other hand we get that K gets anomalous since

$$K^\sharp = K - 2\epsilon_{ijk}\rho_k x_i p_j - \hbar(\vec{\rho}\vec{p})^2 + \gamma\hbar\vec{\rho}^2\vec{p}^2, \tag{51}$$

which coincides with $K^{\mathcal{F}}$ for the special value $\gamma = 1$, namely

$$K^\sharp|_{\gamma=1} = K^{\mathcal{F}}, \tag{52}$$

is such that

$$[L_i^{\mathcal{F}}, K^\sharp]_{\mathcal{F}} = 4\hbar\rho_i. \tag{53}$$

The r.h.s. term $4\hbar\rho_i$, which is independent of γ, is the (constant) anomalous operator.[§]

The analysis of the anomaly can be performed also for composite operators. Let us consider the Euclidean algebra $e(3)$ defined in (8). We investigate the $(\vec{L}^2)^\sharp$ deformation of the composite operator $\vec{L}^2$, which is the Casimir of the $so(3)$ subalgebra. In accordance with the (37) expansion and the (39) equation, the $(\vec{L}^2)^\sharp$ Taylor-expansion in $\vec{\rho}$ stops at the second order for a minimal anomaly. We indeed get

$$(\vec{L}^2)^\sharp = (\vec{L}^2)_0 + (\vec{L}^2)_1 + (\vec{L}^2)_2, \tag{54}$$

with

$$\begin{aligned}
(\vec{L}^2)_0 &= \vec{L}^2,\\
(\vec{L}^2)_1 &= \alpha_1(\vec{\rho}\vec{p})(\vec{p}\vec{L}) + \alpha_2\vec{p}^2(\vec{\rho}\vec{L}),\\
(\vec{L}^2)_2 &= \beta_1\vec{\rho}^2(\vec{p}^2)^2 + \beta_2(\vec{\rho}\vec{p})^2\vec{p}^2.
\end{aligned} \tag{55}$$

The anomalous deformed commutator, given by

$$\begin{aligned}
[L_i^{\mathcal{F}}, (\vec{L}^2)^\sharp]_{\mathcal{F}} = {} & 4\rho_i\vec{p}^2 + 2i\epsilon_{kjl}\rho_k p_l L_j - 2i(\vec{\rho}\vec{p})\epsilon_{ijl}p_l L_j + \\
& \alpha_1 i\epsilon_{ijk}\rho_j p_k(\vec{p}\vec{L}) + \alpha_2 i\epsilon_{ijk}(\vec{p}^2)\rho_j L_k + (2\beta_2 - \alpha_2)i\epsilon_{ijk}\rho_j p_k(\vec{\rho}\vec{p})\vec{p}^2,
\end{aligned} \tag{56}$$

does not depend on the β_1 coefficient. The minimal anomaly is recovered by setting

$$\alpha_1 = \alpha_2 = \beta_2 = 0. \tag{57}$$

The minimal anomaly is therefore given by the first line in the r.h.s. of (56).

The deformed rotational anomaly can be discussed for more general potential terms. Let us consider the addition of an an anharmonic quartic term, given by

$$B = B_0 = \lambda\frac{(\vec{x}^2)^2}{\hbar^3}, \tag{58}$$

to the harmonic oscillator potential. In the above formula λ is a positive coupling constant which, for simplicity, will be set equal to 1. The $\frac{1}{\hbar^3}$ factor is introduced, as recalled at

[§]In this context it should be recalled that ρ_i, despite its appearance, transforms as a scalar under rotations. In the same spirit, despite its appearance, $(\vec{\rho}\vec{p})$ is not a scalar under rotations.

the end of Section **3**, in order to make (58) a primitive element of an infinite-dimensional Lie algebra g containing the oscillator algebra osc (22) as a finite subalgebra (since $\hbar$ is a central element of the Lie algebra g one can evaluate it by setting, as usual, $\hbar = 1$).

The $B^\sharp$ expansion starting from B_0 given in (58) produces a minimal anomaly for $B_k = 0$ for $k \geq 5$. By using the iterative procedure (46), we obtain that

$$B_1 \equiv \left(\frac{(\vec{x}^2)^2}{\hbar^3}\right)_1 \;=\; \frac{4i}{\hbar^2}(p_i(\vec{\rho}\vec{x}) - (\vec{\rho}\vec{x})x_i)\vec{x}^2 \tag{59}$$

gives the minimal anomaly of the first order, expressed by

$$(A_i)_1 \;=\; \frac{8}{\hbar}(2\rho_i\vec{x}^2 - x_i(\vec{\rho}\vec{x})). \tag{60}$$

At the next order we have that

$$B_2 \equiv \left(\frac{(\vec{x}^2)^2}{\hbar^3}\right)_2 \;=\; \frac{2}{\hbar}\left(2(\epsilon_{ijk}\rho_i p_j x_k)(\epsilon_{lmn}\rho_l p_m x_n) - (\vec{\rho}\vec{p})^2\vec{x}^2\right) + i\alpha(\vec{\rho}\vec{p})(\vec{\rho}\vec{x}) \tag{61}$$

produces an anomalous term $(A_i)_2$, given by

$$\begin{aligned}(A_i)_2 \;=\;& \frac{16i}{\hbar}\rho_i(\epsilon_{ljk}\rho_l p_j x_k)(\vec{p}\vec{x}) - 16\rho_i(\epsilon_{ljk}\rho_l p_j x_k) + \\ & -(4+\alpha)(\vec{\rho}\vec{p})(\epsilon_{ijk}\rho_j x_k) - (12+\alpha)(\epsilon_{ijk}\rho_j p_k)(\vec{\rho}\vec{x}).\end{aligned} \tag{62}$$

The choice $\alpha = -4$ (respectively, $\alpha = -12$) makes disappear the third (fourth) term in the right hand side.

In the final example we consider the deformation of the Coulomb potential

$$B = B_0 \;=\; \frac{1}{r}, \tag{63}$$

with $r = \sqrt{\vec{x}^2}$.¶

Unlike the expansion for the (58) anharmonic oscillator potential, the iterative procedure in this case never stops ($B_k \neq 0$ at all orders).

Due to

$$T_i(B_0) \;=\; i\hbar\rho_k(p_i x_k - p_k x_i)\frac{1}{r^3} + \hbar^2\left(\frac{\rho_i}{r^3} - 3x_i\frac{\vec{\rho}\vec{x}}{r^5}\right), \tag{64}$$

we can set

$$B_1 = \left(\frac{1}{r}\right)_1 \;=\; \alpha\hbar\epsilon_{ljk}\rho_l p_j x_k\frac{1}{r^3}, \tag{65}$$

so that

$$L_i(B_1) \;=\; i\alpha\hbar(-\vec{\rho}\vec{p}x_i + p_i\vec{\rho}\vec{x})\frac{1}{r^3}. \tag{66}$$

¶It is worth to mention that it is consistent to produce the infinite-dimensional Lie algebra g of primitive elements obtained by repeatedly applying the commutation relations to the generating elements $\hbar, x_i, p_i, \frac{1}{r}$.

By choosing

$$\alpha = -1, \tag{67}$$

the term proportional to $\hbar$ in (64) can be reabsorbed. On the other hand, one can easily see that the anomalous term $\hbar^2(\frac{\rho_i}{r^3} - 3x_i\frac{\vec{\rho}\vec{x}}{r^5})$ cannot be reabsorbed by the contributions coming from the higher order terms $\left(\frac{1}{r}\right)_k$ for $k > 1$.

Summarizing, we get

$$\left(\frac{1}{r}\right)^\sharp = \frac{1}{r} - \hbar\epsilon_{ljk}\rho_l p_j x_k \frac{1}{r^3} + O(\hbar^2), \tag{68}$$

satisfying the anomalous twist-deformed commutator (with minimal anomaly)

$$\left[{L_i}^{\mathcal{F}}, \left(\frac{1}{r}\right)^\sharp \right]_{\mathcal{F}} = \hbar^2(\frac{\rho_i}{r^3} - 3x_i\frac{\vec{\rho}\vec{x}}{r^5}) + O(\hbar^3). \tag{69}$$

6 Conclusions

In this work we investigated the Non-commutative Quantum Mechanics as a result of an abelian Drinfeld twist provided by formula (12). The twist deforms the Hopf algebra defined on the Universal Enveloping Algebra of a suitable Lie algebra. The Lie algebra under consideration, named *dynamical Lie algebra*, contains composite operators of the Heisenberg algebra operators $x_i, p_i, \hbar$. Nevertheless, these composite operators should be treated as primitive elements, i.e. as generators, of the dynamical Lie algebra. Besides the Hamiltonian, the dynamical Lie algebra also includes, among its generators, the momenta p_i in order to have a well-defined action of the (12) twist on its Enveloping Algebra endowed with the Hopf algebra structure. The dynamical Lie algebra is either finite if its primitive elements result from composite operators at most quadratic in x_i, p_i; it is an infinite Lie algebra otherwise.

We gave motivations for the use of $\mathcal{F}$-deformed generators and $\mathcal{F}$-deformed commutators in dealing with the twist-deformed Enveloping Algebra and pointed out the connection between twist and non-commutativity, which is provided by the equations (25) and (26), with θ_{ij} a constant operator belonging to $\mathcal{U}(h_B(3))$.

The $\mathcal{F}$-deformed angular momenta ${L_i}^{\mathcal{F}}$ close the $so(3)$ algebra under $\mathcal{F}$-deformed brackets. On the the other hand, several operators which at the undeformed level are rotationally invariant are anomalous in the deformed case, with the anomaly expressed by equation (45). We discussed various examples of anomalous operators. For the (deformed) $3D$ harmonic oscillator potential the anomaly is a linear constant operator, see (53). In more general cases the anomaly is an operator which belongs to the Enveloping Algebra and is not necessarily constant. The concept of "minimal anomaly" can be introduced. It corresponds to the specific choice, made so that to minimize the r.h.s. of equation (45), of the higher-order terms in the $\vec{\rho}$ Taylor-expansion of the deformed operator $B^\sharp$. In some cases the notion of minimal anomaly becomes ambiguous. This is the case for instance of formula (62). A (different) term contributing to the anomaly is eliminated by choosing the arbitrary parameter α to be either $\alpha = -4$ or $\alpha = -12$.

We also discussed the anomalous property of the twist-deformed Coulomb potential and of the twist-deformed $so(3)$ composite Casimir operator $\vec{L}^2$, regraded as belonging to $\mathcal{U}(e(3))$, the Enveloping Algebra of the three-dimensional Euclidean algebra.

In a different (not involving the Drinfeld twist) context from ours, non-commutative Quantum Mechanics has been studied in several works, see e.g. [17], where the non-commutative hydrogen atom was discussed, and [18]. In [19] (see also [20] and [21]) the classical counterpart of the non-commutative quantum mechanics is shown to be a constrained system. In [22] investigations of a dynamical (i.e. non-constant) non-commutative matrix θ_{ij} were made.

Acknowledgments

We have profited of clarifying discussions with P. Aschieri on the properties of the $\mathcal{F}$-deformed commutators. We acknowledge several useful discussions with P. G. Castro. Z. K. and F. T. are grateful to the S. N. Bose National Center for Basic Sciences of Kolkata for hospitality. B. C. acknowledges a TWAS-UNESCO associateship appointment at CBPF and CNPq for financial support. The work was supported by Edital Universal CNPq, Proc. 472903/2008-0 (Z.K., F.T).

Appendix

The use of the $\mathcal{F}$-deformed brackets (16) in association with Non-commutative theories can be argued to appear naturally according to the following heuristic considerations. Let us consider a general Lie algebra $\mathbf{g}$ corresponding to a group G and its Universal Enveloping Lie algebra $\mathcal{U}(\mathbf{g})$. Let the generators satisfy the commutation relations

$$[T_a, T_b] \;=\; i f_{ab}^c T_c. \tag{70}$$

The adjoint representation of the group G is obtained by the adjoint action of the group element, according to

$$T_a \rightarrow g T_a g^{-1} \;=\; D_{ab} T_b. \tag{71}$$

The matrices D provide the adjoint representation for $g \in G$ and satisfy $D(g_1)D(g_2) = D(g_1 g_2)$. In the infinitesimal version, near the identity, we can express g as

$$g \;\approx\; \mathbf{1} + i\omega_a T_a, \tag{72}$$

so that the above transformation (71) reads as

$$T_a \rightarrow T'_a \;=\; T_a + \delta T_a, \tag{73}$$

with $\delta T_a = i\omega_b [T_b, T_a]$.

Before we generalize this construction to the NC case, we need to recast the commutative case itself in a bit different setting. For that we can work with the group G, endowed with a Hopf group-algebra structure [14]. In the undeformed case, the undeformed coproduct of g is $\Delta_0(g) = g \otimes g$ and the antipode is $S(g) = g^{-1}$. Near the identity, it is equivalent to the following assignments of antipodes to the Lie algebra generators T_a and the identity $\mathbf{1}$:

$$S(T_a) = -T_a, \qquad S(\mathbf{1}) = \mathbf{1}. \tag{74}$$

Now one can easily see that the finite form of the adjoint action on the group element is

$$T_a \to gT_a g^{-1} \;\; = \;\; gT_a S(g). \tag{75}$$

It is therefore associated with the coproduct $\Delta_0(g) = g \otimes g$.

When considering again g to be close to the identity, the corresponding infinitesimal version of the coproduct is given by

$$\Delta_0(g) \;\; = \;\; \mathbf{1} \otimes \mathbf{1} + i\omega_b \Delta_0(T_b), \tag{76}$$

where

$$\Delta_0(T_b) \;\; = \;\; T_b \otimes \mathbf{1} + \mathbf{1} \otimes T_b \tag{77}$$

is the undeformed coproduct of the Lie-algebra generator T_b. Just like (75) the infinitesimal transformation of T_a can be written with the help of the antipode

$$T_a \to T'_a \;\; = \;\; \mathbf{1}T_a S(\mathbf{1}) + i\omega_b(T_b T_a S(\mathbf{1}) + \mathbf{1}T_a S(T_b)) = T_a + i\omega_b [T_b, T_a]. \tag{78}$$

From this perspective the commutator brackets are associated with the undeformed coproduct. Once this standpoint is adopted, it is natural to expect that the deformed coproduct arising in NC theories should be associated with the deformed brackets.

To that end, let us consider the deformed coproduct obtained by the (12) Drinfeld abelian twist expressing the usual Moyal type of noncommutativity. We have

$$\Delta_0(g) \to \Delta_{\mathcal{F}}(g) \;\; = \;\; \mathcal{F}\Delta_0(g)\mathcal{F}^{-1} \tag{79}$$

with $\mathcal{F}$ given by (12). Considering again a group element close to the identity, we can write

$$\Delta_{\mathcal{F}}(g) \;\; = \;\; \mathbf{1} \otimes \mathbf{1} + i\omega_b \Delta_{\mathcal{F}}(T_b). \tag{80}$$

Let the corresponding deformed coproduct for the Lie algebra generator T_b be denoted in the Sweedler's notation (17), as

$$\Delta_{\mathcal{F}}(T_b) \;\; = \;\; \xi_1 \otimes \xi_2. \tag{81}$$

Repeating the same steps as before it can be easily seen that, in its infinitesimal form, the transformation rule corresponding to the deformed coproduct of a generic element $A \in \mathcal{U}(\mathbf{G})$ is

$$A \to A' \;\; = \;\; A + \delta A, \tag{82}$$

with

$$\delta A = i\omega_b [T_b, A]_{\mathcal{F}}. \tag{83}$$

The deformed bracket reads as

$$[T_b, A]_{\mathcal{F}} = \xi_1 A S(\xi_2). \tag{84}$$

However, it can be easily seen that the algebra will not close under the deformed brackets, unless the original generators T_a's are deformed further as [14] (see also formula (15))

$$T_a^{\mathcal{F}} = \overline{f}^{\alpha}(T_a)\overline{f}_{\alpha}. \tag{85}$$

These deformed generators span a linear subspace of the deformed Hopf algebra $\mathcal{U}^{\mathcal{F}}(\mathbf{g})$ and their deformed brackets induce on them a Lie-algebraic structure. One has to note at least two important differences of these deformed generators and brackets in contrast to the undeformed ones. Firstly, the exponentiation of the deformed generators does not yield elements of the Lie group, so that only the infinitesimal version of the symmetry transformations are considered. Secondly, the deformed brackets are not manifestly antisymmetric, as it can be easily verified.

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