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A New Evolution Equation

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abstract

We propose a new evolution equation for the gluon density relevant for the region of small x_{B}. It generalizes the GLR equation and allows deeper penetration in dense parton systems than the GLR equation does. This generalization consists of taking shadowing effects more comprehensively into account by including multigluon correlations, and allowing for an arbitrary initial gluon distribution in a hadron. We solve the new equation for fixed α_{s}. We find that the effects of multigluon correlations on the deep-inelastic structure function are small.

Key-words: Evolution equation; Shadowing corrections; Multigluon correlations; High twists contributions; Perturbative QCD; Low x.

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1 Introduction

Our objective in this paper is to derive a new evolution equation describing the behavior of large partonic densities. The fact that parton densities increase as the Bjorken scaling variable x_B decreases follows directly from linear evolutions equations such as the GLAP [1] or BFKL [2] equation, and has been experimentally observed by both HERA collaborations [3]. They observe a powerlike behavior of the deep inelastic structure function

$$F_2(x_B, Q^2) \sim x_B^{-\omega_0}, \tag{1}$$

with $\omega_0 \simeq 0.3 - 0.5$. This behavior is predicted by the BFKL equation and is not inconsistent with the GLAP equation. (Such powerlike growth at small x can be imitated by a solution of the GLAP equation with a distribution that is constant in x at a low initial scale Q_0.)

¿From a physical point of view such behavior is inconsistent with unitarity. This fact necessitates a change in the evolution equation in the region of small x_B. The first attempt, more than ten years old, to write down a new equation led to the nonlinear GLR equation [4]. See [5] for a more extensive review.

We will explain in this paper that the GLR equation only includes two-gluon correlations in the parton cascade. Multigluon correlations should be essential to solving the small x_B problem, at least theoretically [6, 7]. It is the aim of this paper to take such correlations into account.

In section 2 we derive and explain the limitations of the GLR equation. We consider multigluon correlations in section 3 by employing the relation between these correlations and high twist contributions to the deep-inelastic structure function. In section 4 we suggest the evolution equation which takes these correlations into account. Some particular solutions and a comparison with the GLR equation are discussed in section 5, while we discuss the general solution of the new evolution equation in section 6. We summarize and conclude in section 7.

2 The GLR Equation

In the region of small x_B and large Q^2 we face a system of partons at mutually small distances (thus the QCD coupling α_s is still small), but dense enough that the usual perturbative QCD (pQCD) methods cannot be applied. The physics that governs this region is non-perturbative, but of a different nature than the one associated with large distances. The latter corresponds to the confinement region, and is usually analyzed using lattice field theory or QCD sum rules. In contrast, we encounter here a situation where new methods must be devised to analyze such a dense relativistic system of partons in a non-equilibrium state. We need, in fact, new quantum statistical methods to describe the behavior of such a system and to chart this unknown region. We are unfortunately only at the beginning of this exploration.

On the upside, we can approach this kinematical region in theory from the pQCD region, and assume that in a transition region between pQCD and high density QCD (hdQCD) we can study such a dense system in some detail. To illustrate what new physics one might anticipate in this transition region let us look at deep-inelastic scattering and compare with the pQCD results for this process. We can expect the following phenomena in the transition region:

(i) As x_B decreases the total cross section $\sigma(\gamma^* N)$ grows and, near the border with the hdQCD domain, becomes even comparable with the geometrical size of the nucleon $\sigma(\gamma^* N) \rightarrow \alpha_{em} \pi R_N^2$. Here the cross section should be a smooth function of $\ln Q^2$.

(ii) Although the parton language can be used to discuss the main properties of the process, interactions between partons become important, especially the annihilation process. This interaction induces screening (a.k.a. shadowing) corrections.

(iii) In this particular kinematical region such screening corrections are fortunately under theoretical control. They modify however the pQCD linear evolution equation. The correct evolution equation now becomes nonlinear.

We will now give a simple derivation of this nonlinear equation, based on physical concepts.

First some nomenclature. The quantity that is measured in deep-inelastic experiments is the structure function $F_2(x_B, Q^2)$. As is well-known, it can be expanded according to the Wilson Operator Product Expansion (OPE) as a series of terms consisting of coefficient functions multiplying matrix elements of local operators, in order of increasing twist (the twist of an operator is its dimension minus its spin). Terms with operators of twist τ are supressed by a factor $(Q^2)^{(1-\tau/2)}$. In this paper we are dealing with the gluon structure function (a.k.a gluon distribution function), which can be measured fairly directly in many experiments on diffraction dissociation and heavy quark production in deeply inelastic processes. As an explicit example, we mention how the gluon structure function can be extracted from the measurement of F_2 in deep-inelastic scattering. Based on the leading twist factorization one may show [8] that at small x_B the gluon density $G(x, Q^2)$ is approximately related to $F_2(x_B, Q^2)$ by

$$x_B G(x_B, Q^2) = \frac{1}{< e^2 >} \left[\frac{dF_2(x_B, Q^2)}{d \ln Q^2} - P^{FF}(\omega_0) F_2(x_B, Q^2) \right] \frac{1}{P^{FG}(\omega_0)} \tag{2}$$

with ω_0 defined in (1), $< e^2 >= 5/18$ for four active flavors and P^{FF}, P^{FG} certain combinations of Altarelli-Parisi splitting functions, see [8]. For simplicity, and because our investigation is mainly theoretical in nature, we will neglect operators containing quark fields beyond leading twist. In our kinematic region of interest gluons presumably dominate the physics. Thus we will consider a definition such as (2) valid for all twist, and will treat the gluon density in a nucleon as a structure function.

As we stated, the main new processes that we must consider at high density are parton-parton interactions. To incorporate these in our physical descriptions we must identify a new small parameter that controls the accuracy of calculations involving these interactions. Such a parameter is

$$W \; = \; \frac{\alpha_s}{Q^2} \cdot \rho \; , \tag{3}$$

where ρ is the gluon density in the transverse plane

$$\rho \; = \; \frac{x_B G(x_B, Q^2)}{\pi R_N^2} \; . \tag{4}$$

Here R_N characterizes the area of a hadron which the gluons populate. It is the correlation length of gluons inside a hadron. Naturally, this radius must be smaller than the radius of a hadron (proton). Since this paper is mostly devoted to the discussion of purely theoretical questions we will not specify further the value of R_N. However it should be stressed that R_N is nonperturbative in nature: all physics that occurs at distance scales larger than R_N is non-perturbative.

The first factor in (3) is the cross section for gluon absorption by a parton in the hadron. Hence W has the simple physical meaning of being the probability of parton (mainly gluon) recombination in the parton cascade. The unitarity constraint mentioned in the introduction can be represented as [4]

$$W \ \leq \ 1 \ . \tag{5}$$

Thus W is indeed the small parameter sought. The parton cascade can be expressed as a perturbation expansion in this parameter. This perturbative series can in fact be resummed [4] and the result understood by considering the structure of the QCD cascade in a fast hadron.

There are two elementary processes in the cascade that impact on the number of partons.

$$\text{splitting} \ \ (1 \to 2); \quad \text{probability} \ \propto \ \alpha_s \, \rho \ ; \tag{6}$$

$$\text{annihilation} \ \ (2 \to 1); \quad \text{probability} \ \propto \ \alpha_s^2 \, d^2 \, \rho^2 \ \propto \ \alpha_s^2 \, \frac{1}{Q^2} \, \rho^2 \ ,$$

where d is the size of the parton produced in the annihilation process. In the case of deep-inelastic scatterinq $d^2 \sim 1/Q^2$.

When x_B is not too small only the splitting of one parton into two counts because ρ is small. However as $x_B \to 0$ annihilation comes into play as ρ increases.

This simple picture allows us to write an equation for the change in the parton density in a 'phase space' cell of volume $\Delta \ln \frac{1}{x_B} \Delta \ln Q^2$:

$$\frac{\partial^2 \rho}{\partial \ln \frac{1}{x_B} \, \partial \ln Q^2} \ = \frac{\alpha_s N_c}{\pi} \, \rho \ - \ \frac{\alpha_s^2 \, \gamma \pi}{Q^2} \, \rho^2 \ , \tag{7}$$

where N_c is the number of colors. In terms of the gluon structure function

$$\frac{\partial^2 x_B G(x_B, Q^2)}{\partial \ln \frac{1}{x_B} \, \partial \ln Q^2} \ = \ \frac{\alpha_s N_c}{\pi} x_B G(x_B, Q^2) \ - \ \frac{\alpha_s^2 \, \gamma}{Q^2 \, R_N^2} \, (x_B G(x_B, Q^2))^2 \ . \tag{8}$$

This is the so-called GLR equation [4]. To determine the value of γ and the understand the kinematical range of validity of (8) this simple physical description does not suffice; rather one must analyze the process carefully in W-perturbation theory [4] [9]. The result for γ was found to be [9]

$$\gamma \ = \ \frac{81}{16} \ \ \text{for } N_c \ = \ 3.$$

We would like to emphasize that the main assumption in the above derivation was that

$$P^{(2)} \sim \rho^2, \tag{9}$$

where $P^{(2)}$ denotes the probability for two gluons in the parton cascade to have the same fraction of energy x and tranverse momentum (characterized by $r \simeq \ln Q^2$).

By assuming (9) we neglect all correlations between the two gluons other than the fact that they are distributed in the hadron disc of radius R_N. In the large x_B region this assumption is plausible because the correlations are power suppressed, and the densities involved are small. In the small x_B region we cannot justify (9), even when it holds for large x_B.

3 Induced multigluon correlations

In [6] it was shown that the problem is oversimplified when one assumes that the probability of annihilation is simply proportional to ρ^2 in deriving the GLR equation. It was found that

$$\frac{P^{(2)}}{\rho^2} \propto e^{\frac{1}{(N_c^2-1)^2}\sqrt{\frac{16 N_c \alpha_s}{\pi} \ln \frac{Q^2}{Q_0^2} \ln \frac{1}{x_B}}}, \tag{10}$$

where Q_0 is the initial virtuality in the parton cascade. This ratio increases with decreasing x_B. Consequently we must take dynamical correlations into account, which could change the GLR equation crucially.

The key to the calculation of parton correlations was suggested by Ellis, Furmanski and Petronzio in [10], and was developed further in [11]. It was shown that gluonic correlations are directly related to high twist contributions in the Wilson Operator Product Expansion (OPE) arising from so called quasi partonic operators. According to the OPE, the gluon structure function can be written as (see the remarks above (2))

$$\begin{aligned} x_B G(x_B, Q^2) \;&=\; x_B G^{(1)}(x_B, Q^2) + \tfrac{1}{Q^2 R_N^2} x_B^2 G^{(2)}(x_B.Q^2) \\ &\quad ... + ... \tfrac{1}{(Q\,R_N)^{2(n-1)}} x_B^n G^{(n)}(x_B, Q^2) ... \end{aligned} \tag{11}$$

where the n'th term results from the twist $2n$ quasi-partonic operator. The probability density $P^{(n)}$ to find n gluons in the cascade with the same x and Q^2 can be directly expressed through the n'th term in the above expansion ($P^{(n)} = x_B^n G^{(n)}(x_B, Q^2)/(\pi R_N^2)^n$, $P^{(1)} = \rho$).

We recently determined the anomalous dimensions γ_{2n} of these high twist operators [7] to next-to-leading order in the number of colors N_c. This was done by reducing the complicated problem of gluon-gluon interactions to rescattering of colorless gluon 'ladders' (Pomerons) in the t-channel. In [6] it was shown that this approach works for the case of the anomalous dimension of the twist 4 operator. The fact that there is no Pomeron 'creation' or 'absorption' in the t-channel means that we are dealing with a quantum mechanical problem (not a field theoretical one) in the calculation of the γ_{2n} anomalous dimension. The problem then amounts to calculating the ground state energy of an n-particle system with an attractive interaction given by a four-particle contact term (see Fig.1) of strength λ. Its value can be calculated to be

$$\lambda \ = \ 4\,\frac{\alpha_s N_c}{\pi}\,\delta, \tag{12}$$

where $\delta = 1/(N_c^2 - 1)$ if one only takes color singlet ladders into account. Including the other color states renormalizes δ to 0.098 [6]. This effective theory is two-dimensional (the two dimensions corresponding to $\ln(1/x)$ and $\ln(Q^2)$) and is known as the Nonlinear Schrodinger Equation. It is well known that this model is exactly solvable. Translating the ground state energy of this model into the value of the anomalous dimension led to [7]

$$\gamma_{2n} \ = \ \frac{\bar{\alpha}_s n^2}{\omega}\{\,1 \ + \ \frac{\delta^2}{3}(\,n^2 \ - \ 1)\}, \tag{13}$$

where $\bar{\alpha}_s = \alpha_s N_c/\pi$, $\omega = N - 1$, N being the Mellin-conjugate variable to x_B

$$f(N) = \int_0^1 dx_B x_B^{N-1} f(x_B) \ \text{ or } \ f(\omega) = \int_0^\infty dy e^{-\omega y}[x_B f(x_B)], \tag{14}$$

where f is an arbitrary function and $y = \ln(1/x_B)$.

The answer (13) is only reliable when $\delta^2 n^2/3 \ll 1$ [7]. Thus to check selfconsistency we must first generalize the GLR equation based on (13) and understand what value of n is important for the deep-inelastic structure function. If the answer is inconsistent with the condition $\delta^2 n^2/3 \ll 1$ we must try and find the expression for the anomalous dimension valid for any n.

Let us note that if we neglect the term proportional δ^2 in (13) we can estimate $x_B^n G^{(n)}(x_B, Q^2)$ via the inverse Mellin transform

$$x_B^n G^{(n)}(x_B, Q^2) \ = \ \frac{1}{2\pi i}\int_C d\omega e^{(\omega y + \gamma_{2n}(\omega) r)} M^{(n)}(\omega, Q^2 = Q_0^2)\ , \tag{15}$$

where $y = \ln(1/x_B)$, $r = \ln(Q^2/Q_0^2)$, Q_0 being the initial virtuality in the parton cascade. The contour is to the right of all singularities in M as well as to the right of the saddle point (ω_S) which is given by

$$\frac{d}{d\omega}\{\omega y + \gamma_{2n} r\}|_{\omega = \omega_S} = 0. \tag{16}$$

Thus in the saddle point approximation for $\delta = 0$

$$x_B^n G^{(n)}(x_B, Q^2) \ \sim \ [\,x_B G(x_B, Q^2)\,]^n, \tag{17}$$

in particular $P^{(2)} = \rho^2$. We also assume here the factorization of the matrix element $M^{(n)} \ = \ (\,M^{(1)}\,)^n$; this expresses the physical assumption that there are no correlations between gluons other than the fact that they are distributed in a disc of radius R_N. We will comment more about this assumption further on. In this sense the GLR equation is only the lowest order approximation in δ^2 even if we assume the factorization of the matrix element and thus, strictly speaking, is valid only in the limit of a large number of colors. The term proportional to δ^2 in (13) is clearly responsible for induced gluon correlations and we take it seriously in this paper. In next section we will therefore generalize the GLR equation.

4 A New Evolution Equation

The first step in such a generalization is to make an *ansatz* for $P^{(n)}$ using the same approach as for the GLR equation, viz. the competition of two processes in the parton cascade. Thus, in analogy to the derivation of (7) we write

$$\frac{\partial^2 \; P^{(n)}(x_B, Q^2)}{\partial \ln \frac{1}{x_B} \, \partial \ln Q^2} \;=\; C_{2n} \cdot P^{(n)}(x_B, Q^2) \;-\; n \cdot \frac{\alpha_s^2 \, \gamma \pi}{Q^2} \, P^{(n+1)}(x_B, Q^2) \;, \tag{18}$$

where $C_{2n} = \gamma_{2n} \, \omega$. The factor n in front of the second term on the right hand side of (18) reflects the fact that in the Born approximation $n+1$ gluons annihilate in n gluons through the subprocess $gluon + gluon \to gluon$, which corresponds to the $two - ladder \to one - ladder$ transition with the strength of the triple ladder vertex γ. There are n such possibilities due to the time ordering of gluon emission. Note that the infinitely recursive set of equations can be cut off at any level by imposing e.g. $P^{(n)} = P^{(n-m)} P^{(m)}$. The GLR equation is the case $P^{(2)} = (P^{(1)})^2$. Since we operate under the assumption that high twists are essential for small enough x_B we must however consider the whole series in (11), which we now do using eqs. (18).

Let us introduce the generating function

$$g(x_B, Q^2, \eta) \;=\; \sum_{n=1}^{\infty} e^{n\eta} g^{(n)} \;, \tag{19}$$

where $g^{(n)} = x_B^n G^{(n)}(x_B, Q^2)$. Comparing with (11) we see that for the full structure function

$$x_B G(x_b, Q^2) = Q^2 R_N^2 \, g(x_B, Q^2, \eta = -\ln(Q^2 R_N^2)). \tag{20}$$

The recursive set of equations (18) can be summarized in one equation for g:

$$\frac{\partial^2 g(x_B, Q^2, \eta)}{\partial \ln \frac{1}{x_B} \partial \ln Q^2} = \bar{\alpha}_s \frac{\partial^2 g}{\partial \eta^2} + \frac{\bar{\alpha}_s \delta^2}{3} (\frac{\partial^4 g}{\partial \eta^4} - \frac{\partial^2 g}{\partial \eta^2}) - \alpha_s^2 \gamma e^{-\ln(Q^2 R_N^2)} e^{-\eta} (\frac{\partial g}{\partial \eta} - g). \tag{21}$$

To solve this linear, 4th order partial differential equation in three variables, we must impose some boundary and initial conditions, on the Q^2 and x_B behavior respectively.

The boundary condition is straightforward

$$\text{For } \; \eta, \; \ln(\frac{1}{x_B}) \; \text{ fixed;} \quad g(x_B, Q^2, \eta) \; \stackrel{\ln Q^2 \to \infty}{\to} \; e^{\eta} \, g_{LLA}(x_B, Q^2), \tag{22}$$

where g_{LLA} is the solution of the standard GLAP evolution equation for the leading twist gluon density in leading $\ln Q^2$ approximation.

The initial condition is much more difficult, because we need $g(x_B = x_B^0, Q^2, \eta)$ for solving (21), whereas experimentally we can only measure the structure function, which is at fixed η. In other words, we need detailed information on the gluon distribution in a hadron at large x_B. We can make the following suggestion (although others are possible)

$$g(x_B^0, Q^2, \eta) = \tag{23}$$

$$\sum_{n=1}^{\infty} e^{n\eta}\frac{(-1)^n}{n!}\cdot[\,g_{LLA}(x_B^0,Q^2)\,]^n \;=\; 1-\exp(-e^{\eta}g_{LLA}(x_B^0,Q^2)\,).$$

This can be recognized as the usual eikonal approximation for the virtual gluon-hadron interaction [12]. The virtues of this expression lie in the fact that it is simple, and that it has the transparent physical meaning of reflecting the assumption that there are no correlations between gluons with $x\sim 1$ other than that they are distributed in a hadron disc of radius R_N.

If one replaces the hadron with a nucleus, one can prove such an approach, which corresponds to the so-called Glauber Theory of shadowing corrections. In the deep-inelastic scattering case an expression of this type was discussed by A. Mueller in [12].

To summarize this section, we have proposed a new evolution equation which has two new features over and above the GLR equation:

(i) It includes induced multigluon correlations.

(ii) It allows an arbitrary initial condition not necessarily an eikonal one, unlike the case of the GLR equation. We recall that the GLR equation has been proven only under the assumption of the factorization property of the matrix elements, which corresponds to an eikonal initial condition [4]. By including all twists in our evolution equation we overcome the need for a reductive initial condition, such as (23). One could e.g. try to solve (21) using an initial condition with correlated gluons at large x and study the consequences of its evolution, with or without δ. Because such initial correlations must be very small, and because our main interest lies in comparing with the GLR equation, we do not pursue this line of inquiry here.

5 Approximate Solutions

In the next section we will discuss the general solution to eq. (18). Here we will give approximate solutions for various special cases. In particular we try to answer the following questions: how does the nonlinear GLR equation follow from our present linear equation; what value of n is typically relevant in the sum (19) and how do the corrections to the ordinary GLAP evolution due to (18) differ from those due to the GLR equation?

5.1 The GLR Equation from the Generalized Equation

The first question that arises is how the nonlinear GLR equation is contained in the linear equation (18) if we neglect the term proportional to δ^2. Specifically, we would like to establish the equivalence of

$$\frac{\partial^2 g(x_B,Q^2,\eta)}{\partial\ln\frac{1}{x_B}\partial\ln Q^2} \;=\; \bar{\alpha}_s\frac{\partial^2 g}{\partial\eta^2}-\alpha_s^2\gamma\, e^{-\ln(Q^2R_N^2)}e^{-\eta}(\frac{\partial g}{\partial\eta}-g), \tag{24}$$

to the nonlinear GLR equation. Let us parametrize the solution in the form

$$g(x_B,Q^2,\eta) \;=\; \Phi(\;e^{\eta}F(y=-\ln x_B, r=\ln(Q^2R_N^2)\;))\;. \tag{25}$$

Because we dropped the gluon-correlation term from (18) we can impose the 'no-correlation' initial condition at $x_B \sim 1$. This is the only initial condition the GLR equation allows [4]. In the set of "fan" diagrams (Fig.3) that the GLR equation sums the initial distribution has the form

$$g(x_B^0, Q^2, \eta) \;=\; \sum_{n=1}^{\infty} e^{n\eta}(-1)^n \cdot [\, g_{LLA}(x_B^0, Q^2)\,]^n \;, \tag{26}$$

which gives

$$\Phi(t) \;=\; \frac{t}{1\,+\,t} \;. \tag{27}$$

The absence of the $1/n!$ in the above equation compared with (23) is explained as follows. The $1/n!$ in (23) enforces the correct time ordering of the produced partons in the parton cascade, related to diagrams of production of n parton shadows (see Fig.4 which shows a case in which three parton shadows are produced). In the fan diagram of Fig.3 we do not have to enforce the correct time ordering because it is already included via the vertex γ [1]. Thus the initial condition of eq. (27) just corresponds to the sum of "fan" diagrams, of which Fig.3 is the lowest order example, with the assumption that there are no correlations between gluons inside the proton. We will see in section 6 that the main properties of the full solution of eq. (18) do not depend on the form of the initial condition. The reduction to the GLR equation does however.

With (27) it is now straightforward to check that eq. (24) reduces to the GLR equation to first order in e^{η} (recall that finally we must put $\eta = -\ln(Q^2R_N^2) \ll 1$), with $F(y,r) = xG(x,Q^2)$.

5.2 Relevant Twists in the Solution

Here we try to follow the recipe mentioned below eq. (14): to see if our approach is consistent we must determine what values of n are relevant in the solution to (21). Should those values not be consistent with $\delta^2 n^2/3 \ll 1$ then we must generalize the expression (13) for γ_{2n} such that it is valid for all n. If they are consistent we can maintain (13) and thus (21).

However, to determine the relevant n's we need the general solution to eq. (21), which is presented in section 6. Here we will perform some rough estimates.

Let us first take a 'worst case' scenario by letting the term proportional to δ^2 dominate, and by dropping the first term and last term on the RHS of (21):

$$\frac{\partial^2 g(x_B, Q^2, \eta)}{\partial \ln \frac{1}{x_B} \partial \ln Q^2} \;=\; \frac{\bar{\alpha}_s \delta^2}{3}\left(\frac{\partial^4 g}{\partial \eta^4} - \frac{\partial g}{\partial \eta^2}\right) \tag{28}$$

To solve this equation we perform a Laplace transform in the variable η

$$\tilde{g}(x_B, Q^2, p) \equiv \int_{-\infty}^{0} d\eta \; e^{p\eta} g(x_B, Q^2, \eta). \tag{29}$$

[1]The strength of the three Pomeron vertex γ was calculated in ref. [9] using the AGK cutting rules [13], which are equivalent to time-ordering.

Then

$$\frac{\partial^2 \tilde{g}(y,r,p)}{\partial y \partial r} \;=\; \frac{\bar{\alpha}_s \delta^2}{3} \cdot p^2(p^2-1)\; \tilde{g}(y,r,p), \tag{30}$$

where $y = \ln(1/x_B)$, $r = \ln(Q^2/Q_0^2)$. The solution to (30) is the Bessel function $I_0(2\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}p^2(p^2-1)yr})$. The most general solution is then

$$g(y,r,\eta) \;=\; \int_C \frac{dp}{2\pi i}\, e^{-p\eta}\phi(p)\, I_0(\, 2\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}p^2(p^2-1)yr}\ )\ , \tag{31}$$

where the contour C runs to the right of all singularities in p. The function $\phi(p)$ is fixed by imposing an initial condition (e.g. (23))

$$g(0,r,\eta) = \frac{1}{2\pi i}\int_C dp\; e^{-p\eta}\phi(p). \tag{32}$$

One may write the solution in fact directly in terms of the initial condition:

$$g(y,r,\eta) \;=\; \int_{-\infty}^{0} d\eta' G(y,r,\eta-\eta')g(0,r,\eta'), \tag{33}$$

with the Green's function

$$G(y,r,\eta-\eta') \;=\; \frac{1}{2\pi i}\int dp\; e^{-p(\,\eta\,-\,\eta'\,)} I_0(\, 2\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}p^2(p^2-1)yr}\ )\ . \tag{34}$$

¿From eq. (19) we see that large typical n corresponds to small typical η, which in turn by (31) corresponds to large typical p ("p_0"). Thus we must find $\delta^2 p_0^2 \ll 1$ to trust eq. (13).

Let us investigate (33) to find the most relevant values of η'. The function $g(0,r,\eta')$ falls down monotonously for $\eta' \to -\infty$ (the first term in (19) dominates) where it behaves as $\exp(\eta')$. Next, we can show that the function $G(y,r,\eta-\eta')$ has a maximum at $\eta = \eta'$ whose width is of the order of $(8\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}yr})^{(1/2)}$. To see this, evaluate (34) using the asymptotic form for $I_0(z) \sim e^z/\sqrt{2\pi z}(1+\ldots)$. Eq. (34) then becomes (neglecting the unimportant non-exponential prefactor) in the asymptotic form

$$G(y,r,\eta-\eta') \;=\; \frac{1}{2\pi i}\int dp\; e^{-p(\,\eta\,-\,\eta'\,)\,+\,2\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}p^2(p^2-1)yr}}\ . \tag{35}$$

The equation that determines the saddle point p_S here is

$$-\,(\eta\,-\,\eta')\;+\;\frac{2(2p_S^2-1)}{\sqrt{p_S^2-1}}\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}\;=\;0 \tag{36}$$

or

$$2\sqrt{p_S^2-1}\;+\;\frac{1}{\sqrt{p_S^2-1}}\;=\;\frac{\eta\,-\,\eta'}{2\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}}\ . \tag{37}$$

In the dangerous case that p_S is large this leads to

$$p_S \;=\; \frac{\eta\,-\,\eta'}{4\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}}\ . \tag{38}$$

With (35) this becomes

$$G(y,r,\eta-\eta') \;=\; \left(8\pi\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}\right)^{-1/2} \cdot e^{\;-\,\frac{(\eta\,-\,\eta')^2}{8\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}}}\ . \tag{39}$$

Thus, in summary, either or both the regions $\eta \sim 0$ and $\eta' \sim \eta$ give dominant contributions in (34). For $\eta' \sim 0$ we have

$$g(y,r,\eta) \sim e^{-\frac{\eta^2}{8\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}}} .$$

while for $\eta' \sim \eta$

$$g(y,r,\eta) \propto e^{-|\eta|} .$$

The first of these can dominate the second only for a restricted range in η, namely

$$|\eta| < 8\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr}, \tag{40}$$

which implies, with (38)

$$p_S \ll 2 . \tag{41}$$

Outside of the range (40) the typical value of $(\eta - \eta')$ is at large η of order $(8\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr})^{(1/2)}$, yielding

$$p_S \sim \frac{1}{(2\sqrt{\frac{\bar{\alpha}_s\delta^2}{3}\cdot yr})^{(1/2)}} \ll 1 \ \text{for} \quad yr \gg 1. \tag{42}$$

We conclude, based on the simplified model equation (24) that typical values of p, and thus n, are small, and therefore we should be able to use eq. (13). We will see that this conclusion holds when we consider the full solution in section 6.

5.3 Estimates for Corrections to the GLR Equation

In this subsection we want to estimate the possible size of corrections to the GLR equation. Recall that the GLR equation sums the contributions of fan diagrams (Fig.3) while the generalized equation includes more general graphs, which are all of the type shown in Fig.5 at large Q^2. To estimate the correction let us parametrize the full solution as

$$g(x_B,Q^2,\eta) = \Phi(\ e^{\eta}F(y,r)\ ) + \Delta g(y,r,\eta)\ , \tag{43}$$

where the first term is the solution to the GLR equation found in subsection 5.1, and the second term is considered to be a small perturbation. Equation (21) becomes then for Δg

$$\frac{\partial^2\Delta g(y,r,\eta)}{\partial y\partial r} = \bar{\alpha}_s\frac{\partial^2\Delta g}{\partial\eta^2} + \frac{\bar{\alpha}_s\delta^2}{3}(\frac{\partial^4\Phi}{\partial\eta^4} - \frac{\partial^2\Phi}{\partial\eta^2}) - \Phi''\,[\,F'_y\,F'_r - \bar{\alpha}_sF^2\,]e^{2\eta}\ , \tag{44}$$

where Φ' denotes $\partial\Phi/\partial t$. We neglected the contribution of the $\partial\Delta g/\partial\eta - \Delta g$ term in eq. (44) since the coefficient in front of this term contains an extra power of $\alpha_{\rm S}$ which we treat as a small parameter. The initial conditions for $\Delta g(y,r,\eta)$ are $\Delta g(0,r,\eta)=\ \Delta g(y,0,\eta) = 0$, because we assume that all boundary conditions have been fulfilled by Φ. The last term in eq. (44) can be neglected for two reasons: (i) at $\eta = -\ln(Q^2R_N^2)$ this term is suppressed and (ii) the difference in brackets is small in both the semicassical and the EKL [8] approach. Substitution of the explicit form of Φ in (27) yields

$$\frac{\partial^2\Delta g(y,r,\eta)}{\partial y\partial r} = \bar{\alpha}_s\frac{\partial^2\Delta g}{\partial\eta^2} + \frac{\bar{\alpha}_s\delta^2}{3}\cdot\frac{12t^2(t-1)}{(1+t)^5}\ , \tag{45}$$

where $t = e^{\eta} F(y,r)$. We can simplify the equation if we keep in mind that the value of η is large and negative in the deep-inelastic structure function. Thus $t \ll 1$ and

$$\frac{\partial^2 \Delta g(y,r,\eta)}{\partial y \partial r} = \bar{\alpha}_s \frac{\partial^2 \Delta g}{\partial \eta^2} - 12\, \frac{\bar{\alpha}_s \delta^2}{3} F^2 e^{2\eta} \; . \tag{46}$$

Now the η depedence of Δg is trivial: $\Delta g = e^{2\eta} \Delta F(y,r)$. Thus

$$\frac{\partial^2 \Delta F(y,r)}{\partial y \partial r} = 4\, \bar{\alpha}_s \Delta F(y,r) - 12\, \frac{\bar{\alpha}_s \delta^2}{3} F^2 \; . \tag{47}$$

Using similar techniques as in the previous subsection, but now for the y and r variables, it is straightforward to show that the general solution to this equation with the initial condition $\Delta F(0,r) = 0$ has the form

$$\Delta F = -12 \frac{\bar{\alpha}_s \delta^2}{3} \int_0^y dy' \int \frac{df}{2\pi i} \int \frac{d\omega}{2\pi i} e^{\omega y' + f r} \, \frac{\omega\, \tilde{F^2}(\omega, f)}{\omega f - 4\, \bar{\alpha}_s} \; , \tag{48}$$

where $\tilde{F^2}$ is the Laplace transform in y and r of the function $F^2(y,r)$. The contours for the f and ω integrals lie to the right of all singularities in these variables. We now need a reasonable estimate for $\tilde{F^2}$. Note that $\tilde{F^2}$ corresponds to the (Laplace tranform of) the properly normalized initial gluon distribution at low virtuality and small y. A rough estimate can be made usine the methods of [8].

$$\tilde{F^2} = \frac{A_G^2}{[\omega - 2\,\omega_0][\, f - 2\,\gamma(\omega = \omega_0)]} \; , \tag{49}$$

where A_G is the normalization factor for the gluon structure function, ω_0 is defined in eq. (1) and $\gamma(\omega) = \frac{\bar{\alpha}_s}{\omega}$ is the anomalous dimension for the leading twist operator. Here we consider the EKL solution to parametrize the data over a wide kinematic region, including GLR nonlinear corrections. Shortly we will discuss the case where both the GLR and multigluon corrections are considered small.

All contours in (48) are to the right of all singularities in ω and f. The integrand has two poles in ω, one corresponding to the initial condition ($2\omega_0$) and one from the equation ($4\bar{\alpha}_s/f$). We will demonstrate in the next subsection that the former is dominant for the choice $\omega_0 = 0.5$, and that restricting ourselves to its contribution is very good approximation. Under this assumption we perform the ω and f integrals, and obtain

$$\Delta g = -4 r \bar{\alpha}_s \delta^2 A_G^2 e^{2\eta} \int_0^y dy' e^{2\omega_0\, y' + 2\,\gamma(\omega = \omega_0)\, r} \; . \tag{50}$$

The factor r in front arises from the double pole in the f variable. The answer clearly satisfies the boundary conditions. We derive further

$$\frac{\Delta g(y,r,\eta)}{g(y,r,\eta)} \simeq -\frac{2\bar{\alpha}_s r}{\omega_0} \delta^2 e^{\eta} \cdot \frac{F^2(y,r) - F^2(y=0,r)}{F(y,r)} \; . \tag{51}$$

This implies

$$\frac{\Delta x_B G(x_B, Q^2)}{x_B G(x_B, Q^2)} = -\frac{2\bar{\alpha}_s r}{\omega_0} \delta^2 \cdot \frac{1}{Q^2 R_N^2} x_B G(x_B, Q^2) \; , \tag{52}$$

if the value of the structure function is large enough in the region of low x_B. Recall that the correct definition of η is $e^{\eta} = 1/(Q^2 R_N^2)$. Substituting in eq. (52) the value of gluon structure function from

HERA data [3] at $Q^2 = 15$ GeV^2 and $x_B = 10^{-4}$ ($x_B G \sim 30$) and a typical value of $R_N^2 = 5GeV^{-2}$ we obtain (with $\alpha_s = 0.25$)

$$\frac{\Delta x_B G(x_B, Q^2)}{x_B G(x_B, Q^2)} \sim 0.4\delta^2 \sim 6 \cdot 10^{-3} .$$

It is more instructive to compare the above correction to the gluon structure function with the one due to the GLR equation. In this case we consider both the correction to the GLAP equation due to the GLR shadowing and due to multigluon correlations as small. Thus we try to find the solution to the GLR equation in the form:

$$F(y, r) = F_{GLAP} + \Delta F_{GLR}.$$

For ΔF_{GLR} we can write the equation:

$$\frac{\partial^2 \Delta F_{GLR}}{\partial y\, \partial r} = \bar{\alpha}_{\rm S}\, \Delta F_{GLR} - \frac{\alpha_{\rm S}^2 \gamma}{Q^2 R_N^2}\, F_{GLAP}^2 \tag{53}$$

The solution is

$$\Delta F_{GLR} = -\frac{\alpha_{\rm S}^2 \gamma}{Q^2 R_N^2} \int_0^y dy' \int \frac{df}{2\pi i} \int \frac{d\omega}{2\pi i}\, e^{\omega y' + f r}\, \frac{\omega \tilde{F}_{GLAP}^2(\omega, f)}{\omega f - \bar{\alpha}_{\rm S}}, \tag{54}$$

where $\tilde{F}_{GLAP}^2$ is the Laplace image of function $F_{GLAP}^2(y, r)$. Again using (49) we get

$$\Delta F_{GLR} = -\frac{2\omega_0 \alpha_{\rm S}^2 \gamma}{3\bar{\alpha}_{\rm S} Q^2 R_N^2}\, A_G^2 \int_0^y dy' e^{2\omega_0 y'}\, [e^{2\gamma(\omega = \omega_0)} - e^{\gamma(\omega = \omega_0)}] \tag{55}$$

Assuming that the second term in the above equation is much smaller than the first we have

$$\Delta F_{GLR} = -\frac{\alpha_{\rm S} \gamma \pi}{3 Q^2 R_N^2 N_c}\, F_{GLAP}^2 \tag{56}$$

Finally, we can get for the ratio

$$\frac{\Delta x_B G(x_B, Q^2)}{\Delta(x_B G(x_B, Q^2))_{GLR}} = \frac{6 N_c^2 \ln(Q^2/Q_0^2)}{\pi^2 \omega_0 \gamma}\, \delta^2 \tag{57}$$

which gives a value of the order of 0.04 if $\omega_0 \sim 0.5$. We will return to a discussion of the corrections to the GLR equation in the next section where we will consider the general solution to the new equation at fixed $\alpha_{\rm S}$.

The above estimates seem in contradiction with the estimates in [14], where a large contribution from multigluon correlations was found. The method we use here is however quite diffent from the one in [14]. There the effect of including the new (pole) singularity in f (the Laplace conjugate variable to r) that results from the resummation of bubbles associated with the 4-Pomeron coupling was contrasted with the contribution from the two Pomeron cut at the level of the Green functions. Although the location of the singularities is quite close, their nature and residues are very different. This led to a large ratio of the contributions of these singularities to the Green functions. In the present case we include in our estimates the two-gluon source, i.e. the initial gluon distribution, and renormalize it in both cases to the same physical initial condition (here the EKL ansatz), thus absorbing the residues. As was remarked in [14], one can absorb the residues alternatively in R_N. Further, in closing the contours involved in performing the inverse Laplace transforms we closed on the singularities of the initial condition (being the rightmost singularities), and not on the propagator poles. Therefore our renormalized R_N^2 has no extra dependence on $\ln(1/x)$. We believe that our method is in the above sense more physical.

5.4 Numerical Estimates

In this subsection we will solve eq. (47) numerically. The method we use goes as follows. We first perform a Laplace transform with respect to y on (47), which leads to

$$\frac{\partial \Delta F(\omega,r)}{\partial r} = \frac{4\bar{\alpha}_{\rm S}}{\omega}\Delta F(\omega,r) - \frac{4\bar{\alpha}_{\rm S}\delta^2}{\omega}F^2(\omega,r) \ . \tag{58}$$

Using various *ansätze* for $F(y,r)$ we solve this equation using Numerov's method, and perform finally the inverse Laplace transform with respect to ω [2]. For the various *ansätze* we fit a parametrized form to $F^2(y,r)$ of which the Laplace transform can be taken analytically.

We will investigate three cases. The first is the EKL ansatz from the previous subsection

$$F(y,r) = A_G\, e^{\omega_0 y + f_0 r} \ , \tag{59}$$

where we take $\omega_0 = 0.5$, $f_0 = \bar{\alpha}_{\rm S}/\omega_0$, and $\bar{\alpha}_{\rm S} \simeq 0.25$. To correspond with the numbers given in the previous subsection we put $A_g \simeq 0.07$. We will use this case to check the accuracy of the estimate given earlier, and drop for this check the requirement $\Delta F(y=0,r) = 0$. Let us denote the numerical answer by $\Delta F^{\rm EKL}(num)$. Then for the pole answer we find

$$\Delta F^{\rm EKL}(pole) = \frac{-2\bar{\alpha}_{\rm S}\delta^2}{\omega_0}\, r\, e^{2\omega_0 y + 2\gamma_0 r} \ , \tag{60}$$

and for the leading term of the saddle point contribution

$$\Delta F^{\rm EKL}(saddle) \simeq \frac{2\bar{\alpha}_{\rm S}\delta^2}{\omega_0}\sqrt{\frac{f_S^3}{16\pi\bar{\alpha}_{\rm S} y}}\,\frac{1}{(f_S - 2\gamma_0)^2}e^{4\sqrt{\bar{\alpha}_{\rm S} y r}} \tag{61}$$

where $f_S = \sqrt{4\bar{\alpha}_{\rm S} y/r}$. Note that both contributions vanish as $r \to 0$. In Table 1 we list these three contributions for various x at a fixed $Q^2 = 15$ GeV$/c^2$. It is clear from this table that $\Delta F^{\rm EKL}(pole)$ is a very good approximation to $\Delta F^{\rm EKL}(num)$, in fact better than one might expect from $\Delta F^{\rm EKL}(saddle)$.

Next we treat a more realistic case. We now use for $F(y,r)$ alternatively the MRSD0' and MRSD-' [15] gluon distribution functions, which are, at the starting scale Q_0, constant as function of x, and behave as $x^{-0.5}$ respectively. They are both parametrized by

$$xG(x,Q^2) = A_G x^{\lambda_g}(1-x)^{\eta_g}(1+\gamma_g x) \ , \tag{62}$$

with the coefficients A_G, λ_g, η_g and γ_g given at $Q = Q_0$ in [15]. We kept this parametrization up until $Q^2 = 15$ GeV$/c^2$, but refitted the coefficients for every step in the Numerov procedure. The Laplace

[2] We thank Keith Ellis for providing subroutines that perform this last step.

transform of this parametrization is easily determined. Following the methods described in the above we determine $\Delta F(y,r)$. We checked that ΔF vanishes for small y. Furthermore eq. (58) was solved under the condition $\Delta F(y,r=0)=0$.

Similarly to the previous subsection we determined at $Q^2 = 15$ GeV/c^2 for values of x_B from 0.01 down to the LEP $\otimes$ LHC value of 10^{-5} the ratios

$$\frac{\Delta(x_B G(x_B,Q^2))}{x_B G(x_B,Q^2)} = \frac{1}{Q^2 R_N^2}\frac{\Delta F(y,r)}{x_B G(x_B,Q^2)} \tag{63}$$

and

$$\frac{\Delta(x_B G(x_B,Q^2))}{\Delta(x_B G(x_B,Q^2))_{GLR}} = \frac{\Delta F(y,r)}{\alpha_{\rm S}^2\gamma(x_B G(x_B,Q^2))^2}. \tag{64}$$

These ratios are given in Table 2. We infer from this table that corrections to the gluon structure function from multigluon correlations beyond the next-to-leading twist are small, at most 5% at small x. We see that the MRSD-' distribution leads to larger corrections than the MRSD0' one. As a fraction of the GLR correction both *ansätze* are small, about 10% for the MRSD-' case and 10-30% for the MRSD0' case.

Thus we confirm here the rough estimates from the previous section. However we note that estimates of the gluon correlation radius R_N range from 1 fm to 0.3 fm. The numbers in table 2 are accordingly easily adjusted.

6 The General Solution (for Fixed $\alpha_{\rm S}$)

In this section we will discuss the general solution to eq. (18) and its consequences for the gluon structure function.

We start by noting that the equation (18) can be written, at fixed $\alpha_{\rm S}$, as

$$\frac{\partial^2 x_B^n G^{(n)}(x_B,r+\eta_0)}{\partial y \partial(r+\eta_0)} = C_{2n}\cdot x_B^n G^{(n)}(x_B,r+\eta_0) - n\cdot\alpha_{\rm S}^2\gamma e^{-(r+\eta_0)}\quad x_B^{(n+1)}G^{(n+1)}(y,r+\eta_0), \tag{65}$$

where we used the fact that that $x_B^n G^{(n)}$ only depends on $\ln(Q^2/Q_0^2)$ with an arbitrary Q_0. The above form of the equation reflects the choice $\eta_0 = -\ln Q_0^2 R_N^2$. We focus now on the hypersurface $\eta_0=\eta$. The evolution equation (see eq. (21)) can then be written in the form:

$$\frac{\partial^2 g(y,\xi,\eta)}{\partial y\partial\xi} = \bar\alpha_s\frac{\partial^2 g}{\partial\eta^2} + \frac{\bar\alpha_s\delta^2}{3}(\frac{\partial^4 g}{\partial\eta^4} - \frac{\partial^2 g}{\partial\eta^2}) - \alpha_s^2\ \gamma\ e^{-\xi}(\frac{\partial g}{\partial\eta} - g)\ , \tag{66}$$

where $\xi = r+\eta$. The advantage of this form is that all explicit η dependence such as $\exp(-\eta)$ has been removed.

We can find $g(y,\xi=r+\eta,\eta)$ using a double Laplace transform with respect to y and η, namely

$$g(y,\xi,\eta) \;=\; \int\frac{d\omega\ dp}{(2\pi i)^2}\ e^{\omega\,y\,+\,p\,\eta}\ g(\omega,p,\xi)\ . \tag{67}$$

The function $g(\omega, \xi, p)$ obeys the equation

$$\omega \frac{\partial g(\omega, \xi, p)}{\partial \xi} = \{ \bar{\alpha}_S \, p^2 + \frac{\bar{\alpha}_S \delta^2}{3} (p^4 - p^2) - \alpha_S^2 \gamma (p - 1) \, e^{-\xi} \} \, g(\omega, \xi, p) \tag{68}$$

The solution to eq. (68) is:

$$g(\omega, \xi, p) = \int \frac{d\omega \, dp}{(2\pi i)^2} \, g(\omega, p) \cdot e^{(\frac{\bar{\alpha}_S}{\omega} p^2 + \frac{\bar{\alpha}_S \delta^2}{3\omega} p^2 (p+1)(p-1)) \xi + \frac{\bar{\alpha}_S^2 \gamma}{\omega} (p-1) (e^{-\xi} - 1)} \tag{69}$$

The function $g(\omega, p)$ must be determined from the initial condition, eq. (23), for $|\eta| \ll r$, $r \gg 1$ and $y = 0$ where the solution looks as follows:

$$g(y = 0, r, \eta) = \int \frac{d\omega dp}{(2\pi i)^2} \, g(\omega, p) \cdot e^{p \, \eta} e^{(r+\eta) \, \phi(p) + \frac{\alpha_S^2 \gamma}{3\omega} (p - 1) (e^{-r - \eta} - 1)} , \tag{70}$$

where

$$\phi(p) = \frac{\bar{\alpha}_S}{\omega} p^2 + \frac{\bar{\alpha}_S \, \delta^2}{3\omega} p^2 \, (p+1)(p-1) \ .$$

We now assert that

$$g(\omega, p) = \Gamma(- f(p)) \cdot \frac{d \, f(p)}{d \, p} \; e^{\frac{\alpha_S^2 \gamma}{\omega}} \tag{71}$$

satisfies the initial condition of eq. (23) with $g_{LLA}(x_B, Q_0^2) = \delta(y)$ at $r = \ln(Q^2/Q_0^2) = 0$. Here $f(p) = p + \phi(p)$ and $\Gamma(-f)$ is the Euler gamma function. We will prove this shortly.

Thus, finally, the solution to eq. (66) looks as follows:

$$g(y, \xi, \eta) = \int \frac{d\omega \, dp}{(2\pi i)^2} \, \Gamma(-f(p)) \cdot \frac{d \, f(p)}{d \, p} \cdot e^{f(p) \, \xi + \frac{\alpha_S^2 \gamma}{\omega} (p-1) \, e^{-\xi} - p \, r + \omega \, y} \tag{72}$$

or changing the integration from p to f,

$$g(\omega, p, \xi) = \int \frac{d\omega \, d \, f}{(2\pi i)^2} \, \Gamma(- f) \cdot e^{f \, \xi + \frac{\alpha_S^2 \gamma}{\omega} (p(f) - 1) \, e^{-\xi} - p(f) \, r + \omega \, y} , \tag{73}$$

where $p(f)$ is determined by

$$f = p(f) + \phi(p(f)) \ . \tag{74}$$

The contour of integration over f is defined in a such a way that all singularities in $\Gamma(-f)$ except the one at $f = 0$ are located to the right of the contour.

We can now verify the claim made in eq. (71). At $y = y_0$, $|\eta| \ll r$, $\xi \simeq r \gg 1$ we have

$$g(y_0, r, \eta) \simeq \int \frac{d\omega \, df}{(2\pi i)^2} \, e^{\omega y_0} \Gamma(-f) \, e^{-r p(f) + \xi f}. \tag{75}$$

We now close the f contour in the right half plane. Neglecting the δ^2 term we have $\phi(p(n)) \simeq \bar{\alpha}_S n^2/\omega$, and we get indeed

$$g(y_0, r, \eta) \simeq \sum_{n=1}^{\infty} \frac{(-)^n}{n!} e^{n\eta} \int \frac{d\omega}{2\pi i} e^{\omega y_0 + \frac{\bar{\alpha}_S n^2}{\omega}} \tag{76}$$

The structure function can be found from eq. (73) putting $\xi = 0$ (see eq. (20)). We obtain

$$g(y, \xi = 0, r) = \int \frac{d\omega \, d \, f}{(2\pi \, i)^2} \, \Gamma(- f) \cdot e^{\frac{\alpha_S^2 \gamma}{\omega} (p(f) - 1) - p(f) \, r + \omega \, y} . \tag{77}$$

Eq. (74) implies that $p(f) \rightarrow (\frac{3\,\omega\, f}{\bar{\alpha}_{\rm S}\delta^2})^{\frac{1}{4}}$. Therefore we can close the contour in f over the singularities of $\Gamma(-f)$. Next we must integrate over ω. We evaluate this integral using the method of steepest descent and the large f approximation for $p(f)$. Neglecting terms proportional to $\alpha_{\rm S}$ in the exponent we find the saddlepoint $\omega = (p_0 r/4y)^{4/3}$ where $p_0 = (3n/\bar{\alpha}_{\rm S}\delta^2)^{1/4}$. Then

$$g(y,\xi=0,r) \;=\; \sum_{n=1}^{\infty} \frac{(\,-\,1\,)^n}{n!}\, C(n,y) e^{-\,4^{-\frac{1}{3}}\,\frac{3}{4}\,(\,\frac{3\,n}{\bar{\alpha}_{\rm S}\delta^2}\,)^{\frac{1}{3}}\cdot(\,\frac{r^4}{y}\,)^{\frac{1}{3}}} \quad ,$$

where $C(n,y)$ is a pre-exponential, smooth factor. The above series clearly converges and the typical value of n in this series is of the order of unity. We thus confirm the estimates from the previous section, and see that we could trust our calculations of the anomalous dimension γ_{2n}. We note that this series has an infinite radius of convergence. Thus for the simplified model we consider and for the case of an eikonal initial condition we conclude that there is no analogy to a renormalon in the Wilson Operator Product Expansion.

Now we wish to consider the solution near $f \;\rightarrow\; 1$, to understand under which circumstances it suffices to take only this singularity into account. Note that this corresponds to the leading twist case, with GLAP evolution. In this region we can rewrite the solution in the following form. Substituting $f \;=\; 1 + t$ we obtain:

$$g(y,\xi=0,r) \;=\; \int \frac{d\omega}{2\pi i}\,\frac{dt}{2\pi i}\,\frac{1}{t}\cdot e^{\Psi} \tag{78}$$

where

$$\Psi \;=\; \omega\, y \;-\; r \;+\; \frac{\bar{\alpha}_{\rm S}}{\omega}\, r \;+\; t\; r\,\{\,\frac{2\bar{\alpha}_{\rm S}(\,1\;+\;\frac{\delta^2}{3}\,)}{\omega} \;+\; \frac{\alpha_{\rm S}^2\,\gamma}{\omega\, r} \;-\; 1\,\} \tag{79}$$

¿From this form of the exponent we see that for ω smaller than ω_{cr} where

$$\omega_{cr} \;=\; 2\,\bar{\alpha}_{\rm S}(\,1\;+\;\frac{\delta^2}{3}\,)\cdot\{\,1\;+\;\frac{\alpha_{\rm S}^2\,\gamma}{2\bar{\alpha}_{\rm S}(1+\delta^2/3)\,r}\,\} \equiv \omega_{cr}^0\,\{\,1\;+\;\frac{\alpha_{\rm S}^2\,\gamma}{\omega_{cr}^0\, r}\,\} \tag{80}$$

we cannot close the contour in t over singularities at positive t. (Here we introduce the parameter $\omega_{cr}^{\,0}$ in order to separate the effects of γ and δ.) For such ω one would need all singularities in f. To understand what happens in this region let us expand Ψ by writing $\omega \;=\; \omega_{cr} \;+\; \Delta$:

$$\Psi \;=\; \omega_{cr}\, y \;-\; r \;+\; \frac{\bar{\alpha}_{\rm S}}{\omega_{cr}}\, r \;+\; \Delta\,(\,y \;-\; \frac{\bar{\alpha}_{\rm S}}{\omega_{cr}^2}\, r \;-\; \frac{t\; r}{\omega_{cr}}\,) \tag{81}$$

Integration over Δ [3] gives rise to the delta function $\delta[\,\frac{r}{\omega_{cr}}\,(t \;-\; t_0\,)\,]$ where

$$t_0 \;=\; \frac{1}{\omega_{cr}\; r}\,[\,\omega_{cr}^2\, y - \bar{\alpha}_{\rm S}\, r\,] \tag{82}$$

Carrying out the integration gives

$$g(y,\xi=0,r) \;=\; \frac{\omega_{cr}}{2\pi r}\,\cdot\,\frac{1}{t_0}\,\cdot\, e^{\omega_{cr}\, y \;-\; r \;+\; \frac{\bar{\alpha}_{\rm S}}{\omega_{cr}}\, r} \tag{83}$$

[3] Alternatively one may expand to $O(\Delta^2)$ and use steepest descent.

Let us define the "critical line" [4] by

$$y_{cr} \;=\; \frac{1}{\omega^0_{cr}}\, r \;-\; \frac{\bar{\alpha}_{\rm S}}{(\,\omega^0_{cr}\,)^2}\, r \tag{84}$$

The gluon structure function is on this critical line

$$x_B G(x_B, Q^2) = R^2_N\, Q^2\; \frac{(\,\omega^0_{cr}\,)^2}{2\,\bar{\alpha}_{\rm S} r}\,[\frac{\delta^2}{3} \;+\; \frac{\alpha^2_{\rm S}\gamma}{4\pi\bar{\alpha}_{\rm S}\, r}\,]^{-1} \tag{85}$$

For $\delta \;=\; 0$ this becomes

$$x_B G(x_B, Q^2) = R^2_N\, Q^2\; \frac{2}{\gamma}\frac{N^2_c}{\pi^3}. \tag{86}$$

Note that this is similar to the solution of the GLR equation with running coupling [4], but not quite the same.

Thus, the structure of the solution to the new evolution equation with the initial condition eq. (23) looks as follows. In the kinematic region to the right of the critical line we can in fact solve the linear GLAP equation, but with the new initial condition eq. (86) on the critical line. To the left of the critical line we need the solution to the full equation. Note that if we would change the initial condition of the full evolution equation the value of the structure function on the critical line would also change.

We have thus achieved a new understanding of the role of the initial condition in the problem. In particular, we conclude that the solution on the critical line depends only on the initial condition in the region of $f \;\rightarrow\; 1$, i.e. it depends only on the initial condition for the GLAP equation. Implicitly we have used here the assumption, expressed in our initial condition (23), that the multigluon correlations are sufficiently small at large x. We recall that the original derivation of the GLR equation was based on this same assumption. We are not restricted to such an assumption for our equation. Clearly, if there were strong correlations between gluons at large x it would change the explicit form of the solution of our evolution equation. Nevetheless, the line of reasoning followed in this section would continue to hold.

For $\delta \;\neq\; 0$ we have two different situations. In the first, for

$$\frac{\delta^2}{3} \;\ll\; \frac{\alpha^2_{\rm S}\,\gamma}{2\bar{\alpha}_{\rm S}\, r}$$

the solution on the critical line is the same as in eq. (86). I.e. the only change that occurs in the solution of the case $\delta \;=\; 0$ is that there is a new equation for the critical line, eq. (84). Note that the HERA experiments correspond to this situation.

At very large values of r $(Q^2 \;\gg\; Q^2_0)$, when

$$\frac{\delta^2}{3} \;\gg\; \frac{\alpha^2_{\rm S}\,\gamma}{2\bar{\alpha}_{\rm S}\, r},$$

the solution on the critical line looks as follows:

$$x_B G(x_B, Q^2) = R^2_N\, Q^2\; \frac{3\,\omega^0_{cr}}{2\pi\delta^2\, r} \tag{87}$$

In this case we must solve the GLAP equation using eq. (87) as the boundary condition.

7 Conclusions.

The main result of the paper is the new evolution equation (21). It allows us to penetrate deeper into the region of high density QCD because we incorporate multigluon correlations into the evolution, and to answer questions which could not be answered before. For example one could now investigate the question of how well the Glauber theory for shadowing corrections in deep-inelastic scattering with a heavy nucleus works.

This present equation solves two theoretical problems which arise in the region of high parton density: (i) It takes induced multigluon correlations into account, which originate from parton-parton (mainly gluon-gluon) interactions at high enegy and can be calculated in the framework of perturbative QCD, and (ii) it allows for an arbitrary initial gluon distribution, which is nonperturbative in nature.

We have found the general solution to the new equation for the case of an eikonal initial condition and fixed α_s. We found no evidence for a "renormalon" in the twist expansion.

Our numerical estimates show that the effect of multigluon correlations is rather small in the accessible region of energy. We have seen evidence for this by using approximate methods and the general solution to the new equation.

We have shown that the general solution confirms the strategy developed for the GLR equation: we have calculated the new critical line for the generalized equation and shown that to the right of this critical line we can solve the linear GLAP equation with a new boundary condition on this line. We found this boundary condition taking into account the multigluon correlations. This approach, developed in this paper simplifies also the solution to the GLR equation and allows us to understand how solutions to the GLR equation depend on the initial conditions. This is essentially a consequence of the linearization of the GLR equation in 21.

We have not discussed here the behaviour of the solution in the region to the left of the critical line, where multigluon correlations should come more forcefully into play. We plan to do this in later publication. We hope that the solution in the latter kinematic region will have a significant impact on understanding the scale of the shadowing correction and the importance of multigluon correlations in the so-called Regge domain. This must be understood in order to provide a matching between soft and hard processes.

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FIGURE CAPTIONS

Figure 1.

Rescattering of Pomerons in the t-channel.

Figure 2.

Pictorial representation of the generalized evolution equation.

Figure 3.

"Fan" diagram.

Figure 4.

Production of three gluon shadows in a parton cascade.

Figure 5.

Example of type of multigluon interactions that the generalized evolution equation takes into account.

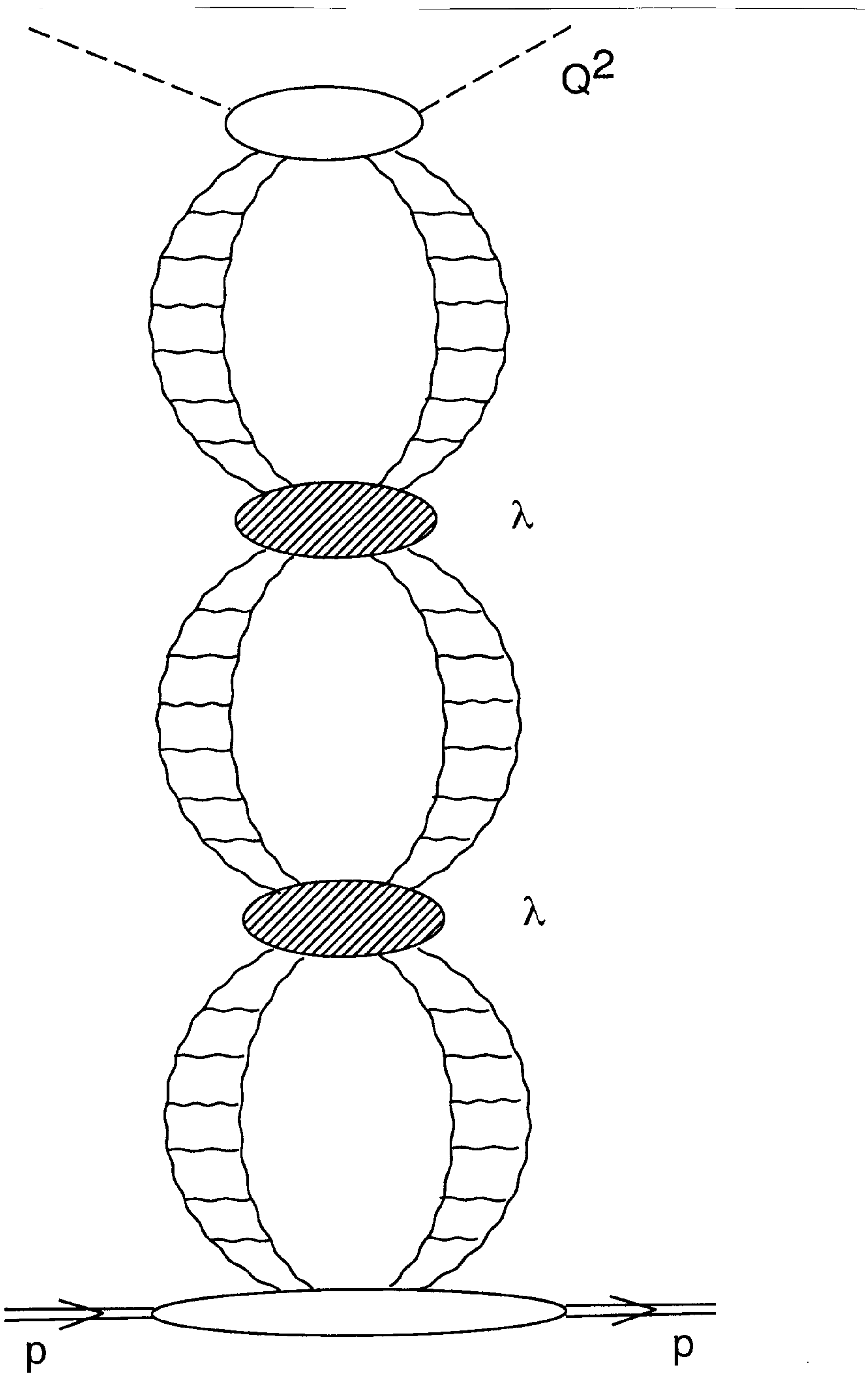

Fig. 1

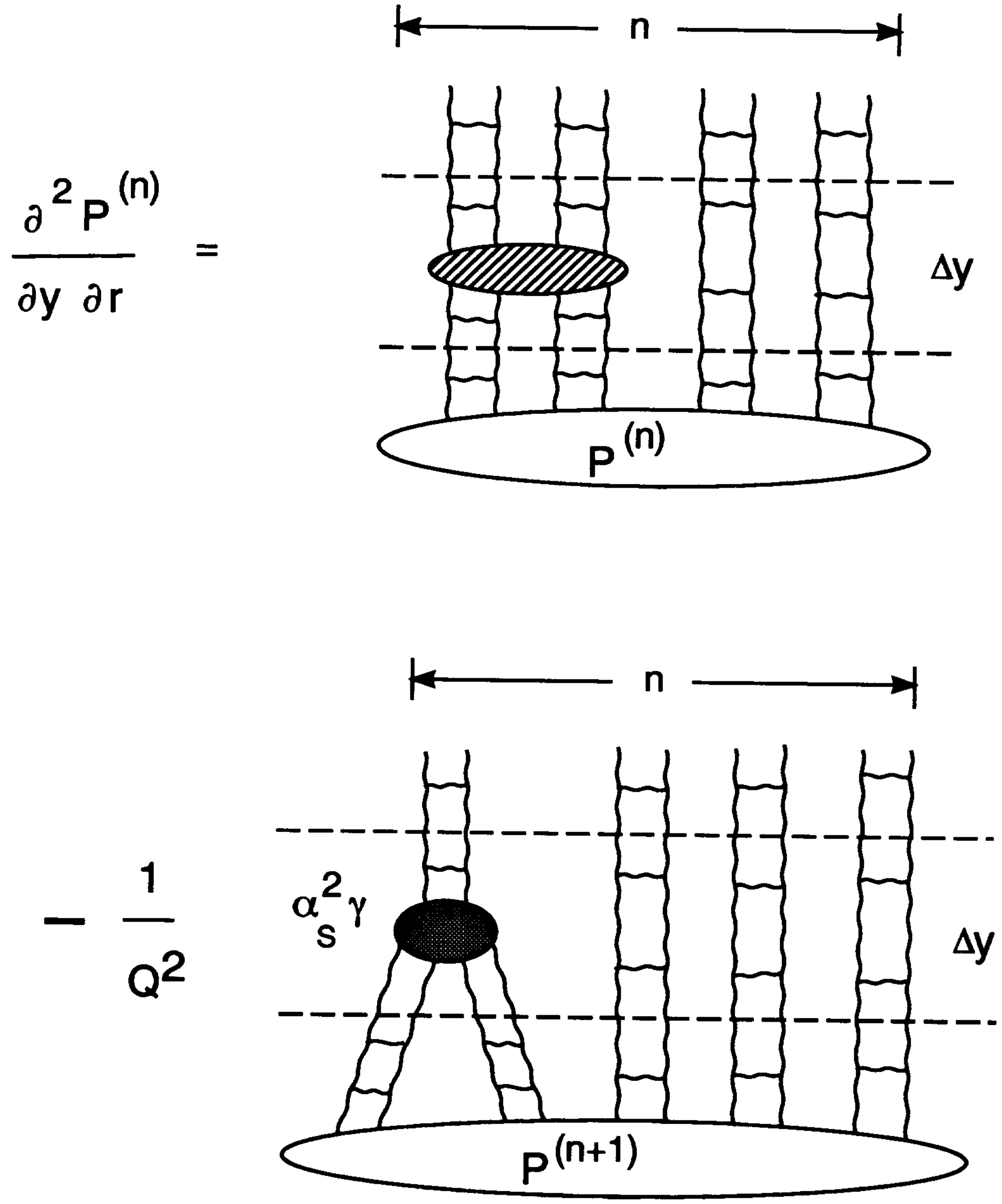

Fig. 2

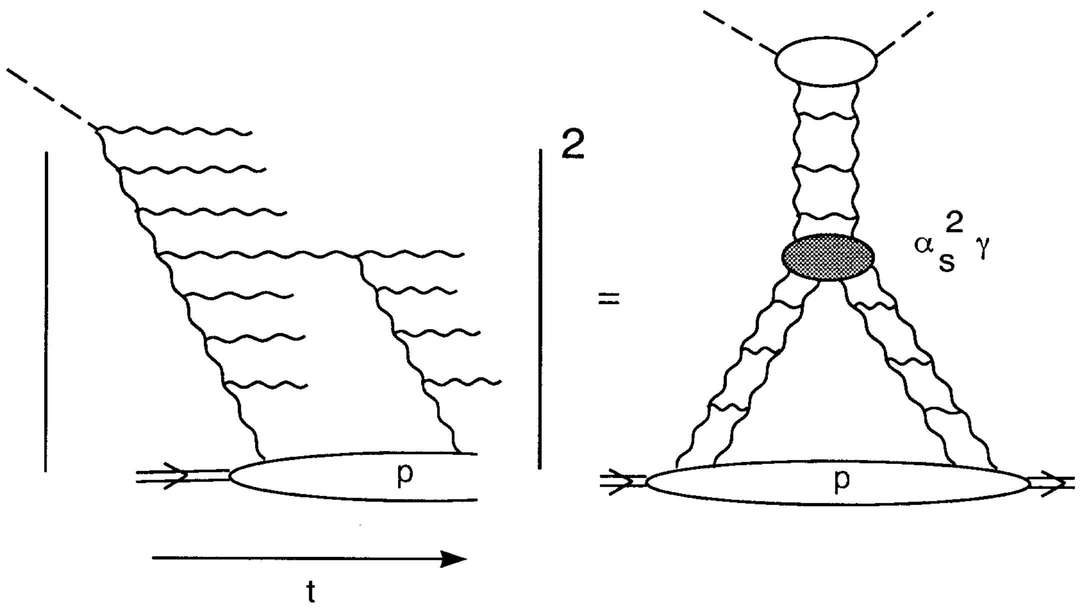

Fig. 3

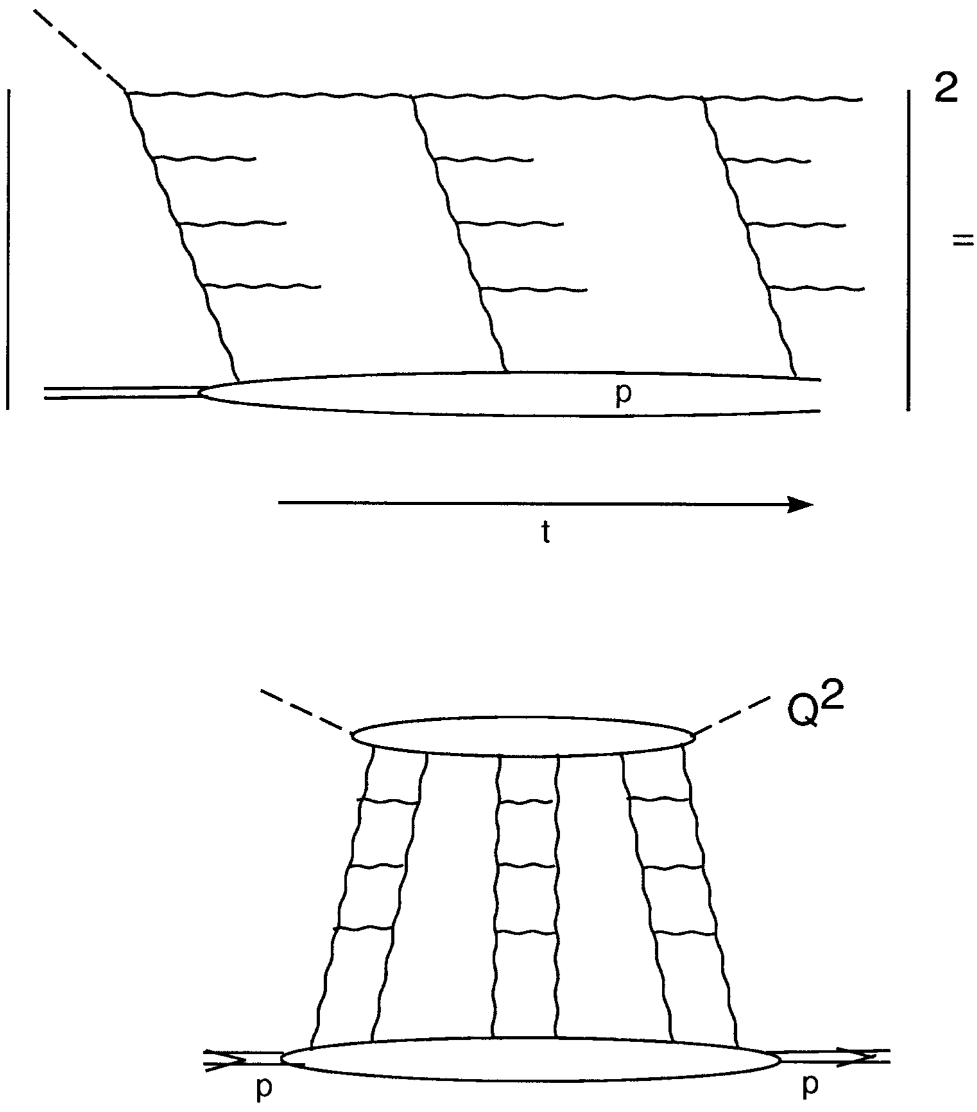

Fig. 4

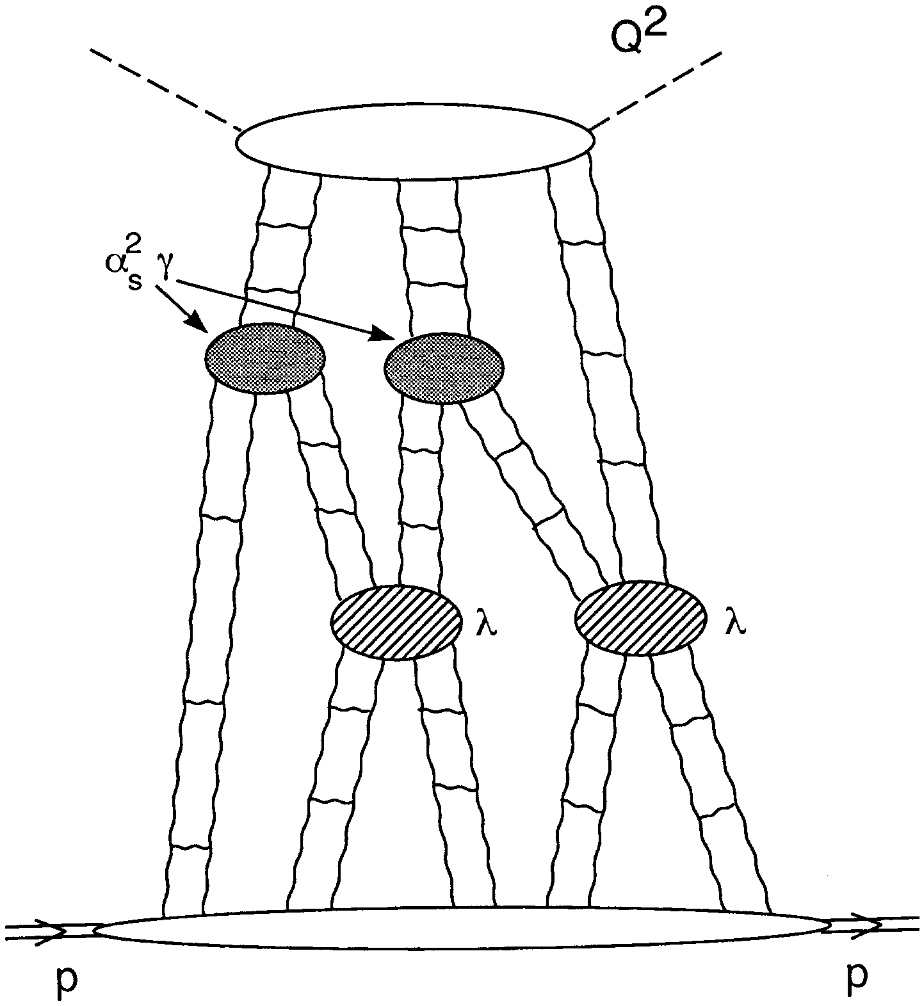

Fig. 5

x	$\Delta F^{\rm EKL}(num)$	$\Delta F^{\rm EKL}(pole)$	$\Delta F^{\rm EKL}(saddle)$
10^{-2}	$-3.26 \cdot 10^{-2}$	$-3.41 \cdot 10^{-2}$	$8 \cdot 10^{-3}$
10^{-4}	-3.44	-3.41	$2 \cdot 10^{-2}$

Table 1. Comparison of various contributions in the EKL approximation. Here $Q^2 = 15$ GeV$^2/c^2$, $\alpha_s = 0.25$ and $R_N^2 = 5$ GeV^{-2}.

Ansatz	x	$\Delta(x_B G)/x_B G$	$\Delta(x_B G)/\Delta(x_B G)_{GLR}$
MRSD0'	10^{-2}	$-1.4 \cdot 10^{-3}$	0.11
	10^{-4}	$-8.9 \cdot 10^{-3}$	0.29
	10^{-5}	$-14.7 \cdot 10^{-3}$	0.31
MRSD-'	10^{-2}	$-1.2 \cdot 10^{-3}$	0.084
	10^{-4}	$-13.5 \cdot 10^{-3}$	0.1
	10^{-5}	$-40.5 \cdot 10^{-3}$	0.09

Table 2. Correction to the gluon distribution function for two different *ansätze*. Here $Q^2 = 15$ GeV$^2/c^2$, $\alpha_s \simeq 0.21$ and $R_N^2 = 5$ GeV^{-2}.

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