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by

Liacir S. Lucena*, Fernando D. Nobre*
Constantino Tsallis[1]* and Luciano R. da Silva[1]*

[1]Centro Brasileiro de Pesquisas Físicas - CBPF/CNPq
Rua Dr. Xavier Sigaud, 150
22290 - Rio de Janeiro, RJ - Brasil

*Departamento de Física
Universidade Federal do Rio Grande do Norte
59000 Natal, RN - Brasil

SIMPLE AND MULTIPLE SCATTERING IN DISORDERED ONE-DIMENSIONAL MEDIA: RENORMALISATION GROUP APPROACH

Liacir S. LUCENA, Fernando D. NOBRE*, Constantino TSALLIS[§*], and Luciano R. da SILVA[§*]*

*Departamento de Física, Universidade Federal do Rio Grande do Norte, 59000 Natal, RN, BRAZIL

[§]Permanent address: Centro Brasileiro de Pesquisas Físicas/CNPq Rua Dr. Xavier Sigaud, 150, 22290 Rio de Janeiro, RJ, BRAZIL

ABSTRACT

The exact characteristic penetration length ξ associated with both simple and multiple incoherent elastic scattering in semi-infinite one-dimensional disordered media is established as a function of p (concentration of scattering centers) and f_o (transmission coefficient of a single center). Then we exhibit how these phenomena can be seen as critical ones, and the corresponding ξ are reobtained within convenient real space renormalisation group frameworks. Finally we discuss a generalized model where the single center transmission coefficient f can randomly take two different values f_1 and f_2.

I INTRODUCTION

The basic scattering theory can be considered by now as a satisfactorily developed one (see, for instance, Refs. [1,2]). This is particularly true for ordered media; the problem for disordered media presents a higher degree of difficulty. The scope of the present work is to exhibit how this type of study can be usefully undertaken from the standpoint of the critical phenomena theory (see, for instance, Ref. [3]): no such trial has been attempeted before, although the corresponding treatment of diffusion is already available [4,5]. We illustrate the approach on simple one-dimensional models of incoherent simple and multiple elastic scattering.

In Section II we establish exact results concerning the dependence of intensity I on distance ℓ, and the associated penetration length ξ; in Section III we reobtain ξ within convenient real space renormalisation group (RG) frameworks; in Section IV we extend the discussion to generalized models, and we finally conclude in Section V.

II SCATTERING MODELS

Let us consider a semi-infinite regular linear chain (with crystalline parameter equal to unity) of elastic scattering centers which scatter *along* the chain; we note f_i the intensity transmission coefficient of the i-th center ($0 \leq f_i \leq 1$, $\forall i$;

$1 - f_i$ is the fraction reflected backwards).

By assuming *simple* scattering regime along the entire semi-space, the intensity ratio $I(\ell)/I_0$ after crossing the ℓ first centers (I_0 being the initial incident intensity of whatever is scattered) is given by

$$\frac{I(\ell)}{I_0} = \prod_{i=1}^{\ell} f_i \qquad \text{(simple)} \tag{1}$$

If the transmission coefficient is f_0, one and the same for all the centers, then Eq. (1) can be rewritten as follows:

$$I(\ell)/I_0 = e^{-\ell/\xi} \qquad \text{(simple)} \tag{2}$$

where

$$\xi = \frac{1}{\ell n \frac{1}{f_0}} \tag{3}$$

If we assume instead that the scattering regime is *multiple* along the entire semi-space, the establishment of the ratio $I(\ell)/I_0$ demands a small development. Consider the first two scattering centers, with coefficients f_1 and f_2; they constitute a composite scattering center with equivalent coefficient $f_e \equiv I(2)/I_0$ given by

$$f_e = f_1 f_2 \{1 + (1-f_1)(1-f_2) + [(1-f_1)(1-f_2)]^2 + \ldots\}$$

$$= \frac{f_1 f_2}{1-(1-f_1)(1-f_2)} \tag{4}$$

This expression can be conveniently rewritten as follows:

$$\frac{1-f_e}{f_e} = \frac{1-f_1}{f_1} + \frac{1-f_2}{f_2} \qquad (4')$$

The recursive use of Eq. (4') leads to the following general expression:

$$\frac{1-I(\ell)/I_0}{I(\ell)/I_0} = \sum_{i=1}^{\ell} \frac{1-f_i}{f_i} \qquad \text{(multiple)} \qquad (5)$$

If the transmission coefficient is f_0, one and the same for all the centers, then Eq. (5) provides

$$\frac{I(\ell)}{I_0} = \frac{1}{I+\ell/\xi} \qquad \text{(multiple)} \qquad (6)$$

where

$$\xi = \frac{f_0}{1-f_0} \qquad (7)$$

Note that the *multiple* process hypothesis leads, in the long distances limit, to a *power* law ($I(\ell)/I_0 \sim \xi/\ell$), in contrast to the *exponential* law corresponding to *simple* scattering.

Let us now consider the case where there is a *dilution* of scattering centers on the linear chain, i.e. to each center we associate the following distribution law:

$$P(f) = (1-p)\,\delta(f-1) + p\,\delta(f-f_0) \qquad (0 \leq p, f_0 \leq 1) \qquad (8)$$

This new situation is nothing but the previous one ($p=1$) with rescaled distances (note that the intensity $I(\ell)$ does not depend, for $\ell \gg 1$, on the order of centers and vacancies as long as their respective concentrations are preserved); consequently Eqs. (2) and (6) are still valid but ξ becomes ξ/p, i.e. Eqs. (3) and (7) are respectively extended into

$$\xi = \frac{1}{p\ell n\frac{1}{f_o}} \qquad \text{(simple)} \tag{9}$$

and

$$\xi = \frac{f_o}{p(1-f_o)} \qquad \text{(multiple)} \tag{10}$$

III RENORMALISATION GROUP APPROACH

Consider the case where all centers are identical ($f_i = f_o, \forall_i$). Following the standard RG procedures [3], we renormalise a "block" of b scattering centers into a smaller "block" b' centers; the penetration length ξ scales as follows

$$\xi'/b' = \xi/b \tag{11}$$

The question is whether this equation and those (to be established) renormalising f_o and p (into f_o' and p') enable us to calculate the function $\xi(f_o,p)$.

Let us first treat the pure case (p = 1). The RG recursive equation for f_o is given by

$$f_o'^{b'} = f_o^b \qquad \text{(simple)} \tag{12}$$

or

$$b' \frac{1-f_o'}{f_o'} = b \frac{1-f_o}{f_o} \qquad \text{(multiple)} \tag{13}$$

We immediately verify that Eqs. (11) and (12) (Eqs. (11) and

(13)) lead, *for all b and b'*, to Eq. (3) (Eq. (7)) excepting for an arbitrary multiplicative constant. Note also that both Eqs. (12) and (13) provide a stable (trivial) fixed point $f_o = 0$, as well as an unstable (*critical*) fixed point $f_o = 1$.

The RG discussion of the diluted case ($p \leq 1$) is less straightforward and unambiguous than the previous one ($p = 1$). The binary law (8) becomes, for a "block" of b scattering centers, a complex one, namely

$$P_b(f) = \sum_{i=o}^{b} \binom{b}{i} (1-p)^{b-i} p^i \delta(f-\bar{f}_i(f_o)) \tag{14}$$

where

$$\bar{f}_i(f_o) = \begin{cases} f_o^i & \text{(simple)} \qquad (15.a) \\ \dfrac{f_o}{f_o + i(1-f_o)} & \text{(multiple)} \qquad (15.b) \end{cases}$$

A fully satisfactory $b \to b'$ RG would demand to identify $P_b \equiv P_b(f;f_o,p)$ and $P'_b, \equiv P_{b'}(f;f'_o,p')$, which is of course impossible because the two distributions have different amounts of δ-functions. Consequently, if we do not intend to follow the complete and rather complex evolution, under successive renormalisations, of the distribution law, an approximation has to be done. Our present choice will be to approach the actual renormalised distribution by

$$P'(f) = (1-p')\delta(f-1) + p'\delta(f-f'_o) \tag{16}$$

In other words, we replace the *impossible* identification of

P_b and $P'_{b'}$, by the *possible* identification of P and P', while preserving as many moments as we can (*two* in our case, because we have only *two* parameters, namely p and f_o). Then the recursive relations for p and f_o are constructed by imposing

$$<g(f)>_{P'_{b'}} = <g(f)>_{P_b} \tag{17}$$

and

$$<[g(f)]^2>_{P'_{b'}} = <[g(f)]^2>_{P_b} \tag{18}$$

where g(f) is an arbitrary (in principle) function to be chosen. By using Eq. (14), we may rewrite Eqs. (17) and (18) as follows:

$$\sum_{i=0}^{b'} \binom{b'}{i} (1-p')^{b'-i} p'^{i} g(\bar{f}_i(f'_o))$$

$$= \sum_{i=o}^{b} \binom{b}{i} (1-p)^{b-i} p^i g(\bar{f}_i(f_o)) \tag{19}$$

and

$$\sum_{i=0}^{b'} \binom{b'}{i} (1-p')^{b'-i} p'^{i} [g(\bar{f}_i(f'_o))]^2$$

$$= \sum_{i=0}^{b} \binom{b}{i} (1-p)^{b-i} p^i [g(\bar{f}_i(f_o))]^2 \tag{20}$$

These equations determine (at least implicitely) $p' = p'(p,f_o)$ and $f'_o = f'_o(p,f_o)$, as soon as we have chosen g(f). The simplest choice one can think of is g(f) = f (we denote by f-RG the corresponding recursive relations): it yields quite reasonable results over the entire (p,f_o)-space (see Fig. 1), as well as the *exact* "critical" behaviour (namely $\xi \propto 1/p$ if $p \to 0$ and $0 < f_o < 1$,

and $\xi \propto 1/(1-f_0)$ if $f_0 \to 1$ and $0 < p \leq 1$) for both simple and multiple scattering models. However the simplest choice is not the best one: if we choose $g(f) = s(f)$ (we denote by s-RG the corresponding recursive relations), where

$$s(f) = \begin{cases} \ell n \frac{1}{f} & \text{(simple)} \qquad (21.a) \\ \frac{1-f}{f} & \text{(multiple)} \qquad (21.b) \end{cases}$$

we verify that $g(\bar{f}_i(f_0)) = i\,g(f_0)$ for *both* simple and multiple scatterings. By using this property, Eqs. (19) and (20) can be straightforwardly summed up, and we obtain

$$b'p's(f_0') = b\,p\,s(f_0) \qquad (22)$$

and

$$b'p'(1-p')[s(f_0')]^2 = b\,p\,(1-p)[s(f_0)]^2 \qquad (23)$$

Equation (22) together with Eq. (11) immediately yield (excepting for an arbitrary multiplicative constant)

$$\xi = \frac{1}{p\,s(f_0)} \qquad (24)$$

which is the *exact* answer for both types of scattering. In a certain sense it is astonishing to verify that a relatively simple choice of g(f) exists, which exactly compensates the errors introduced by our rough (in principle) approximation, namely the replacement of a rather *complex* distribution law by a simple *binary* one. Finally it is worthy to note that Eqs. (22)

and (23) provide a f_o-independent recursive relation for p, namely

$$b' \frac{p'}{1-p'} = b \frac{p}{1-p} \qquad (25)$$

which presents the fixed points p = 0 (unstable) and p = 1 (stable). This equation is quite different from the typical percolation one, namely $q'^{b'} = q^b$, where we have introduced the variable $q \equiv 1 - p$ corresponding to the concentration of "holes" of scattering centers.

IV GENERALIZED MODELS

Let us now extend the scattering models we have been considering by associating, with each scattering center, the distribution law

$$P(f) = (1-p)\delta(f-f_1) + p\delta(f-f_2) \qquad (0 \leq f_1, f_2 \leq 1) \qquad (26)$$

which, for $f_1 = 1$ and $f_2 = f_o$, recovers Eq. (8). Eqs. (2) and (6) still hold, with expressions (3) and (7) generralized into

$$\xi = \frac{1}{(1-p)s(f_1) + ps(f_2)} \qquad (27)$$

with s(f) given by Eq. (21) (for an arbitrary distribution P(f), Eq. (27) would become $\xi = 1/\langle s(f) \rangle_p$).

If we want to approach the present extended model whithin

a RG framework, we can follow along the lines of Section III, and introduce the renormalized distribution law

$$P'(f) = (1-p')\delta(f-f_1') + p'\delta(f-f_2') \tag{28}$$

which generalizes Eq. (16). We have now a three-dimensional parameter-space (namely p, f_1 and f_2); consequently, to Eqs. (17) and (18) we add

$$\langle [g(f)]^3 \rangle_{P_{b'}'} = \langle [g(f)]^3 \rangle_{P_b} \tag{29}$$

P_b (and analogously $P_{b'}'$) is now given by

$$P_b(f) = \sum_{i=o}^{b} \binom{b}{i} (1-p)^{b-i} p^i \delta(f-\bar{f}_i(f_1,f_2)) \tag{30}$$

where

$$\bar{f}_i(f_1,f_2) = \begin{cases} f_1^{b-i} f_2^i & \text{(simple)} \quad (31.a) \\ \dfrac{1}{1+(b-i)\dfrac{1-f_1}{f_1}+i\dfrac{1-f_2}{f_2}} & \text{(multiple)} \quad (31.b) \end{cases}$$

If we choose g(f) = s(f) given by Eq. (21) (noted s-RG), we verify that $g(\bar{f}_i(f_1,f_2)) = (b-i)g(f_1) + ig(f_2)$ for *both* simple and multiple scatterings. By using this property, Eqs. (17), (18) and (29) provide

$$b'[(1-p')s(f_1') + p's(f_2')] = b[(1-p)s(f_1) + ps(f_2)] \tag{32}$$

$$b'p'(1-p')[s(f_2') - s(f_1')]^2 = bp(1-p)[s(f_2) - s(f_1)]^2 \qquad (33)$$

$$b'p'(1-p')(1-2p')[s(f_2') - s(f_1')]^3 = bp(1-p)(1-2p)[s(f_2) - s(f_1)]^3 \qquad (34)$$

Eq. (32) together with Eq. (11) provide (excepting for an arbitrary multiplicative constant) the *exact* answer, namely Eq. (27).

V CONCLUSION

We have discussed simple and multiple incoherent elastic (one-dimensional) scattering in pure, diluted and center-mixed (binary distribution which recovers the pure and diluted cases as particular ones) semi-infinite linear chain. This model is equivalent to a situation in which the beam which is being scattered has a normal incidence into a stratified semi-infinite medium composed by scattering slices. The main exact results we have established (re-established for the most trivial among them) are:

i) The dependence of the intensity on distance for scattering centers of arbitrary transmission coefficients (Eqs. (1) and (5)); in the case of identical centers, the laws are respectively exponential (Eqs. (2) and (3)) and power-like (Eqs. (6) and (7)) for simple and multiple scatterings.

ii) The extension of point (i) to the diluted (Eqs. (9) and (10) for the simple and multiple models respectively) and

center-mixed (Eq. (27) together with Eq. (21)) linear chains.

We have also argued that these phenomena can fruitfully be seen as "critical" ones, and as such, should be tractable within renormalisation group frameworks. We have exhibited how this can be done in the one-dimensional systems mentioned above by using standard real space renormalisation techniques. In this sense our main results are:

i) By renormalising "blocks" of b centers into "blocks" of b' centers (b' < b), we have obtained, *for all values of b and b'*, the exact penetration length ξ for the pure linear chain.

ii) In the case of the diluted and center-mixed linear chains, we approximate the complex distribution (of the transmission coefficient f) which emerges under successive renormalisations, by a binary one. There is no unique way to do this: we choose to preserve as many first momenta (two in the diluted case, three in the center-mixed one) of g(f) as we can, where g(f) is an arbitrary function. The simplest choice g (f) = f is already quite satisfactory (see Figs. 1.b and 1.c). However the best choice is g(f) = s(f) where s(f) is *additive* (in the sense that s (i centers) = i s(one center); see Eq. (21)): for this choice the exact penetration length ξ is reobtained, *for all values of b and b'*, for both the diluted and center-mixed chains, and both simple (exponential law) and multiple (power law) scattering models. In a certain sense, it is surprising that a function s(f) does exist which exactly compensates the error involved in our approximation where only a *finite* (instead of an *infinite*) number of momenta is

preserved. However such convenient functions have already been pointed out for a variety of systems, such as the Ising [6], Potts [7,8], Z(N) [9] and resistor [10] models.

iii) The geometrical nature of the present scattering problem in the diluted chain is *different* from that of (standard) percolation, as can be seen through the way the concentration p renormalises (see Eq. (25)).

The extension of the present ideas to more complex types of scattering (D-dimensional, anisotropic, inelastic, coherent) in d-dimensional systems would be very welcome. We are presently working in some of these extensions.

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CAPTION FOR FIGURE

FIG. 1 - (a) Indicative RG flow, in the p (concentration) $-f_o$ (transmission coefficient) space, for both f-RG and s-RG for simple as well as multiple scattering diluted models (• denotes fixed points; the line $p = 0$ and $0 \leqslant f_o \leqslant 1$ is a line of fixed points). Typical iso-ξ lines (the value of the penetration length ξ is indicated on the curves) corresponding to simple (b) and multiple (c) scattering models: full lines are the *exact* result (reobtained by the s-RG, $\forall(b',b)$, and dashed lines correspond to the f-RG with $b' = 1$ and $b = 2$ (note the improvement when ξ increases: for $\xi = 100$ no graphical discrepancy exists).

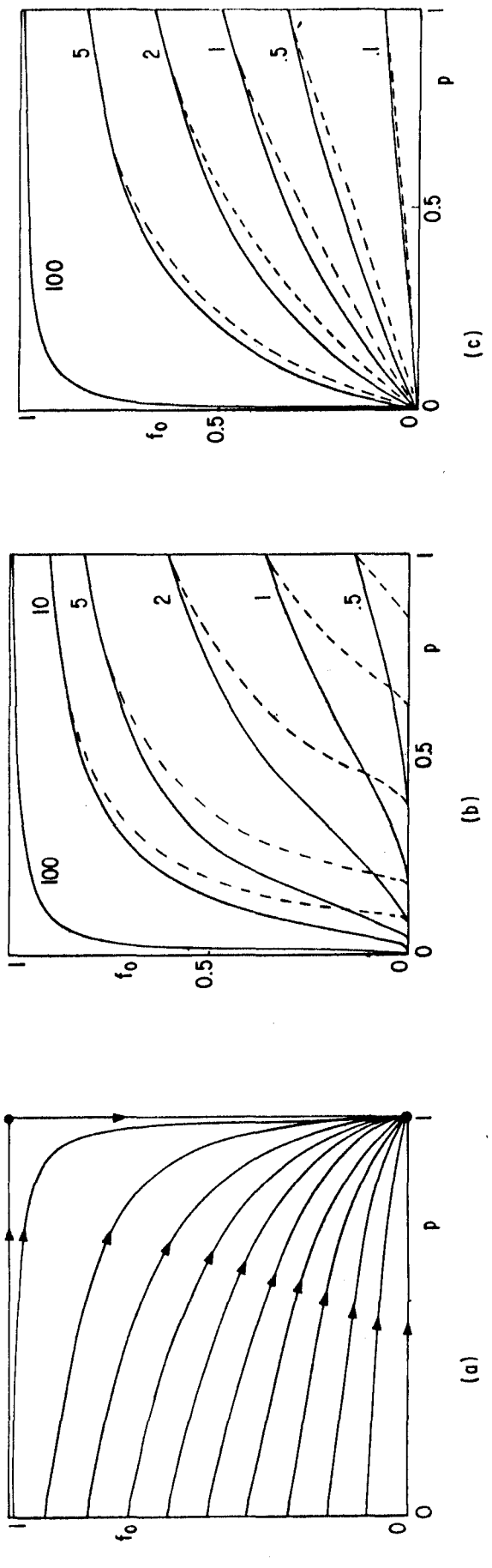

FIG.1