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ON THE STABILITY OF THE EINSTEIN UNIVERSE

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ABSTRACT

It is shown that the Einstein Universe is stable by a large class of exact perturbations, which are made starting from a detailed exam of the topology of the model, and which include perturbations of the type considered by Lemaître. The problem is reduced to the one-dimensional motion of a particle, in a potential well whose minimum corresponds to the configuration of the Einstein Universe.

A meaningful concept of stability of a Cosmological Model can be satisfactorily introduced in the context of the Perturbation Theory of Cosmological Models. In this way, Lifshitz [1], and Lifshitz and Khalatnikov [2] analysed the stability of Cosmological solutions of Einstein equations by developing a general method of treating first-order perturbations of isotropic models. Hawking [3] also treated perturbations of cosmological models by the use of the quasi-Maxwellian formulation of Einstein field equations. A final and complete review of the subject was recently given by Novello, Salim and Heintzmann [4].

In the same scheme of first-order perturbations, the original idea of stability seems however to be due to Lemaître [5], who examined the Einstein universe and showed that it was unstable — actually Einstein equations imply that the amplitude of the initial first-order perturbations of the matter density of the model increases exponentially with time, the rate of increasing depending only on the matter density present in the original model. In all the papers cited above perturbations are nevertheless first-order, that is, only linear terms in the perturbations are kept and the dynamics is given by the linearized field equations over the unperturbed background.

In the present paper we consider a class of exact perturbations of the Einstein Universe, which are made starting from an analysis of the global (topological) structure of the model. We show that the Einstein model is stable with respect to this class of exact perturbations (which include perturbations of the type considered by Lemaître). This example motivates not only a program of making exact perturbations in cosmological models, generating by this procedure new stable structures, but also the reexam of previous results of perturbation theory in the light of exact equations.

The procedure to make these perturbations involves a detailed exam of the topological structure of the model to be perturbed. To this end we characterize here the Einstein Universe as the simply connected Lie group $R \times S^3$ on which we introduce a left-invariant Lorentzian metric, which is solution of Einstein equations. S^3 is the topological 3-sphere, which is a Lie group acting on itself by left multiplication [6], [7]. Introducing on S^3 the Euler coordinates (χ, θ, ϕ), with $0 < \theta < \pi$, $0 \leq \chi$, $\phi \leq 2\pi$, the left invariant vector fields on S^3 can be expressed

$$X_1 = \frac{\partial}{\partial \chi}$$

$$X_2 = \cos\chi \frac{\partial}{\partial \theta} + \frac{\sin\chi}{\sin\theta} \frac{\partial}{\partial \phi} - \mathrm{cotg}\,\theta \sin\chi \frac{\partial}{\partial \chi}$$

$$X_3 = -\sin\chi \frac{\partial}{\partial \theta} + \frac{\cos\chi}{\sin\theta} \frac{\partial}{\partial \phi} - \mathrm{cotg}\,\theta \cos\chi \frac{\partial}{\partial \chi}$$

with corresponding left-invariant dual 1-forms

$$\omega^1 = d\chi + \cos\theta \, d\phi$$

$$\omega^2 = \cos\chi \, d\theta + \sin\theta \sin\chi \, d\phi$$

$$\omega^3 = -\sin\chi \, d\theta + \sin\theta \cos\chi \, d\phi$$

which satisfy the algebra of S^3,

$$[X_i, X_j] = \varepsilon_{ijk} X_k \; , \quad d\omega^i = -\varepsilon^{ijk} \omega^j \wedge \omega^k \tag{1}$$

On the manifold R we introduce the coordinate $t(-\infty<t<\infty)$ with vector field $X_0 = \partial/\partial t$ and dual 1-form dt, and satisfying obviously

$$[X_0, X_i] = 0 \quad , \quad i = 1,2,3 \qquad (2)$$

The manifold $R \times S^3$ is the covering group of the algebra (1),(2), and (X_0, X_1, X_2, X_3) and $(dt, \omega^1, \omega^2, \omega^3)$ constitute bases respectively for vector fields and 1-forms on $R \times S^3$. The Einstein model is obtained by introducing on $R \times S^3$ the left invariant Lorentzian metric

$$g(X_a, X_b) = \mathrm{diag}(1, -\lambda^2, -\lambda^2, -\lambda^2) \qquad a,b = 0,1,2,3 \qquad (3)$$

or equivalently

$$ds^2 = dt^2 - \lambda^2 [(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2] \qquad (4)$$

where λ^2 is a constant parameter. The geometry (3) or (4) is solution of Einstein equations with the cosmological constant term, for $k\rho = -2\Lambda = \frac{1}{2\lambda^2}$, where ρ is the mass density of the pressureless fluid, as measured locally by the comoving observers with four--velocity field $\partial/\partial t$. By construction the space-like sections t=const have the topology of S^3. Now S^3 has the structure of a fiber bundle, with base space S^2 and fiber homeomorphic to S^1 [8]. In other words S^3 has the local decomposition $S^1 \times S^2$, this decomposition being realized by splitting the tangent vector spaces of S^3 with respect to the vector field $X_1 = \partial/\partial\chi$. We have a vertical space V spanned by the vector X_1 with corresponding dual 1-form ω^1; and the horizontal space H orthogonal to X_1 spanned by the vectors $Y_2 = \partial/\partial\theta$ and $Y_3 = \partial/\partial\phi - \cos\theta\, \partial/\partial\chi$, with corresponding dual 1-forms $\sigma^2 = d\theta$ and $\sigma^3 = d\phi$. The metric on V is expressed as $g_V(X_1, X_1) = 1$, and gives the geometry of S^1; the metric on H is expressed as g_H: $g_H(Y_2, Y_2) = 1$, $g_H(Y_3, Y_3) = \sin^2\theta$, other components zero, and gives the geometry of S^2. The geometry of the Einstein Universe is then split into

$$ds^2 = dt^2 - \lambda^2 \{g_V(X_1, X_1)(\omega^1)^2 + g_H(X_m, X_n)\sigma^m\sigma^n\} \tag{5}$$

according to the fibering $R \times S^1 \times S^2$, where $m, n = 2, 3$.

Now starting from the geometry (5) we make the following perturbation: the radius of the 2-sphere S^2 is made time-dependent. We then obtain a new manifold with the same topology $R \times S^3$, and time-dependent geometry by given

$$ds^2 = dt^2 - \{\lambda^2 g_V(X_1, X_1)(\omega^1)^2 + B^2(t)\, g_H(X_m, X_n)\sigma^m\sigma^n\} \tag{6}$$

The spatial sections $t = \text{const.}$ have the topology of S^3, analogous to the static Einstein Universe. The dynamics of the perturbed models are described by Einstein equations with the cosmological constant term. We take for the matter content of the model a perfect fluid, with matter-energy density ρ and pressure π, as measured by the comoving observers with four-velocity field $\partial/\partial t$. We distinguish from (6)

1) Einstein Universe

$B = B_E = \text{const.}$

and Einstein equations imply

$$k\rho_E = -2\Lambda = \frac{1}{2\lambda^2} \quad , \quad B_E{}^2 = \lambda^2 \tag{7}$$

2) Einstein Perturbed Universes

Einstein equations for (6) reduce to three independent differential equations. Two of them define ρ and π, and the third one yields the differential equation for B(t)

$$\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{B^2} - \frac{\lambda^2}{B^4} = 0 \tag{8}$$

where a dot denotes t-derivative. Equation (8) has the first integral

$$\dot{B}^2 = -1 + \frac{2\lambda^2 \ell n B}{B^2} + \frac{C}{B^2} \tag{9}$$

where C is an integration constant. Introducing the new variable $q = B^2(t)$, eqs. (8) and (9) can be rewritten as

$$\ddot{q} = -2 + \frac{2\lambda^2}{q} \tag{10}$$

$$\frac{1}{2}\dot{q}^2 = -2q + 2\lambda^2 \ell n\, q + 2C \tag{11}$$

The dynamics of the models as given by eqs. (10) and (11) can be reduced to the 1-dim motion of a particle in a potential well, described by the Lagrangean

$$L = \frac{1}{2} \dot{q}^2 - V(q) \tag{12}$$

where

$$V(q) = 2q - 2\lambda^2 \ell n\, q \tag{13}$$

The graph of the potential (13) is depicted in Fig. 1. The minimum of the potential occurs for $q_E = \lambda^2 = B_E^{\ 2}$, that is, the configuration of the Einstein universe is a point of stability of the class of models (6).

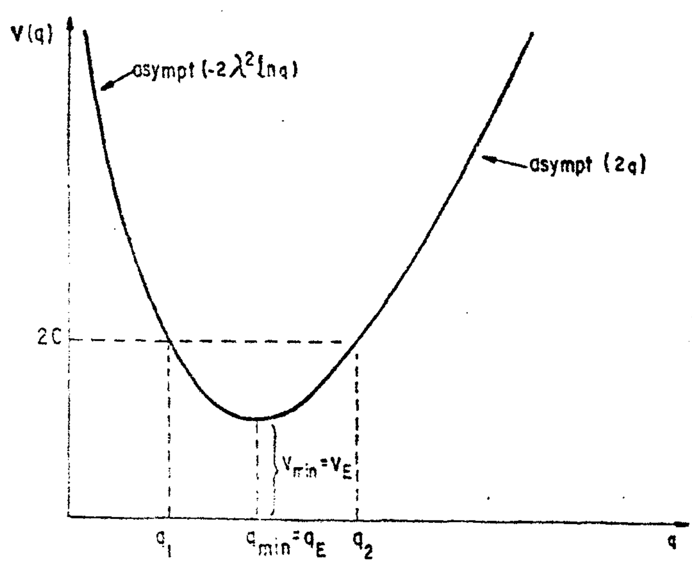

Fig. 1. The graph of the potential V(q), with the asymptotic behaviour for small and large q.

The value of V_{min} is given by

$$V_{min} = V_E = 2\lambda^2 (1-\ell n\lambda^2)$$

Introducing the canonical momentum $p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$, the Hamiltonian of the system is a constant of motion given by

$$H = \frac{1}{2} p^2 + V(q) = 2C \tag{14}$$

The trajectories of the system in the phase plane (q,p) are closed, the turning points given by q_1 and q_2 (cf. Fig.1). In fact from the autonomous system of equations of motion

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -2 + \frac{2\lambda^2}{q} \end{aligned} \tag{15}$$

the trajectories of the system can be drawn as in Fig. 2. The arrow describes the direction of increasing time.

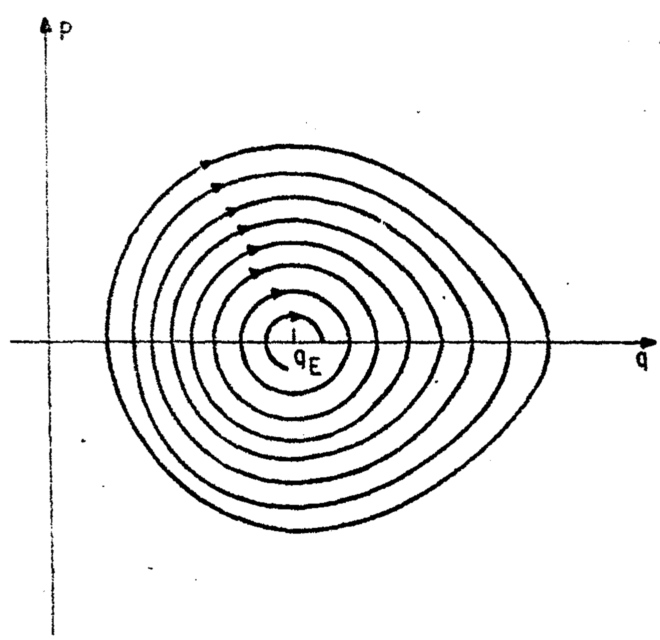

Fig. 2. The trajectories of the system in the (q,p) - phase plane.

At this point we can discuss the meaning of exact perturbations and stability of the Einstein universe: each trajectory of Fig.2- - characterized by the "energy" parameter 2C - corresponds to a model which is an exact perturbation of the Einstein geometry, and is stable in the sense that its phase-plane (q,p) amplitude about the stability point $(q=q_E, p=0)$ is always bounded. The point $(q=q_E, p=0)$ - which corresponds to the configuration of the Einstein universe - is a stability point of the system. By decreasing the value of the "energy" 2C to the value $V_{min}=V_E$ (cf. also Fig. 1) we can confine the trajectory to any neighborhood about the stability point $(q=q_E, p=0)$. On the other hand any infinitesimal perturbation of the Einstein model obtained by taking the "energy" $2C=V_E+\varepsilon$ (ε positive and infinitesimal) is stable and corresponds to an exact bounded trajectory about the stability point, and infinitesimally close to it.

Expanding $V(q)$ about $V_{min}=V_E$ we obtain

$$V(q) = V_E + \frac{1}{\lambda^2}(q-q_E)^2 + \ldots \qquad (16)$$

For 2C close to V_E we obviously have that $q-q_E$ is small and the motion is a sinusoidal oscilation about q_E described by

$$q(t) - q_E = \ell \sin\nu_0 t$$

where $\nu_0^{-1}=\lambda/\sqrt{2}$, and the amplitude $\ell=\lambda\sqrt{2C-V_E}$. For oscilations about $(q=q_E, p=0)$ with large amplitudes, we use action-angle variables in the Hamilton-Jacobi formulation of the system and we can express $q(t)$ as a Fourier series in the fundamental frequency given by

$$\nu_0^{-1} = \int_{q_1}^{q_2} \frac{dq}{\sqrt{-4q+4\lambda^2 \ell n q+4C}} \tag{17}$$

Using (10) and (11) we express the matter-energy density ρ and pressure π as

$$k\rho = \frac{2\lambda^2 \ell n\, q+2C-\lambda^2/2}{2q^2} + \Lambda$$
$$k\pi = \frac{2\lambda^2 \ell n\, q+2C-5\lambda^2/2}{2q^2} - \Lambda \tag{18}$$

For $q-q_E$ infinitesimal, namely $2C = V_E + \varepsilon$ with ε infinitesimal we obviously have that $\rho = \rho_E + \delta\rho$, $\pi = 0 + \delta\pi$, where $\delta\rho$ and $\delta\pi$ are infinitesimal perturbations to the corresponding values of the Einstein model, and bounded for all t. Nevertheless for each trajectory q(t) we must garantee that ρ and π satisfy the energy conditions [9] $\rho>0$, $|\pi|\leq\rho$ and $|\frac{d\pi}{d\rho}|<1$ for all t. In the scheme of perfect fluid it is easy to verify that these conditions are always satisfied for a large range of the "energy" 2C. We note, for small oscilations about $(q = q_E, p = 0)$, that we must change the value of Λ in case we also demand that $\delta\pi>0$, for all t.

We make some final comments. Each trajectory of Fig. 2 - which can be obtained by continuously increasing the "energy" parameter 2C, starting from $V_{min} = V_E$ —— corresponds to an exact solution of Einstein equations, with ρ and π given by (18). In the perfect fluid scheme such models satisfy reasonable physical conditions for a large range of the "energy" 2C. The effect of introduction of viscosity terms in the behaviour of the fluid must be examined. If by some physical process the geometry could fluctuate and the curvature of the 2-sphere changed sign (in other words if the topology of S^2 changed to give a 2-dim open space) we can show that the future of the system would be to expand indefinitely to $q \to \infty$, in which the quantity (17) would be a measure of a "relaxation time" of the system.

The class of perturbations we have considered are spatially homogeneous, or have infinite wave-length (in the language of Refs. [1] and [2]), but at least the small exact time-dependent perturbation $q - q_E = \ell \sin \nu_0 t$ can be properly localized. Our result poses the striking question that the program of the theory of perturbations of cosmological models, and its eventual applications to the theory of galaxy formation, should then be reexamined in the light of the exact dynamics of the perturbations.

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