THE DYNAMICS OF A BAR IN THE PRESENCE OF OBSTACLES*

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ABSTRACT

The dynamic evolution of visco-elastic and purely elastic bars hitting rigid and elastic obstacles are studied, from either the theoretical, numerical or computational point of view.

1. INTRODUCTION

Many interesting problems in the Engineering Sciences reduce themselves to the study of the quasi-static or dynamic

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evolution of a continuous medium in the presence of obstacles, the constraints imposed by those obstacles being of unilateral type. For equilibrium problems, and for the quasi-static evolution, there are adequate results available. However, in many concrete situations, it is necessary to take into account the dynamic character of the evolution. This is the case, for example, of an elastic body hitting a rigid or elastic obstacle in the course of its evolution: a dynamic version of Signorini's problem, where solutions with "shocks" should be expected.

That is the question we shall discuss in the present paper, with the restriction of a unique space dimension and the assumption of a visco-elastic behavior for the body.

More complex examples, but realistic, appear in the structural analysis of nuclear power plants. For the so called "pipe whip problem", explained in [5], the plastic behavior of the material ought to be considered.

In the case of an elastic obstacle, the problem does not pose any major difficulty. It deserves however being mentioned, in view of the fact that the rigid obstacle case is treated as a limit case of elastic obstacle, when rigidity tends to infinity.

In the case of a rigid obstacle, the force law associated to the constraint can be written in the form

$$-R \in \partial\varphi(U),$$

φ being the indicator function of $\mathbb{R}^+$, and U the displacement of the particle getting in contact with the obstacle.

Problems of this nature have appeared in the literature since long ago (J.L. Lions [4], H. Brezis [1]), and recently M. Schatzman [7] treated the problem in a finite dimensional setting. From the point of view of Mechanics, this corresponds to the problem of motion of a rigid solid in the presence of a rigid obstacle. Passage to an infinite dimensional situation, to be pursued in this piece, may present new difficulties, and being inside less severe hypothesis than in [7] is advisable. We shall note that the presence of viscosity terms will be valuable for the mathematical analysis. The solution, as in [7], will exhibit a bounded measure to represent the rigid obstacle reaction. Through the computational simulations, based on the elastic obstacle approximation (rigidity as parameter), we will be able to approach the purely elastic body case.

We shall also observe that the problem can be formulated in terms of a variational inequality, when the solution presents some regularity.

The plan of the article is the following:

2. The physical problem.
3. Basic theoretical results.
4. Proof of Theorem 3.1.
5. Proof of Theorem 3.3.
6. A variational inequality.
7. A numerical scheme. Convergence.
8. Proof of Theorem 7.1.
9. Results of some numerical simulations.

2. THE PHYSICAL PROBLEM

The bar occupies, in its natural state, the domain $\Omega = (0,L)$; x denotes a generic point of Ω. The displacements do occur along the bar axis, and are characterized by a unique scalar function $\{u(x),\ x \in \Omega\}$. Naturally, u is also a function of time t in the dynamic problems to be considered in the sequel.

The density ρ is taken to be equal to 1 in the theoretical analysis. The stress field inside the bar is a scalar denoted by σ.

The bar is supposed to present a visco-elastic behaviour, so that the constitutive equation is written

$$\sigma = aE + b\dot{E}, \qquad a > 0, \quad b > 0, \tag{2.1}$$

with

$$\begin{cases} E = u' = \dfrac{\partial u}{\partial x} \\ \dot{E} = \dfrac{\partial E}{\partial t} \end{cases} . \quad (1) \tag{2.2}$$

The situation of an elastic material is the limit case $b = 0$.

The object of this study is to analyse the evolution of the bar, under the action of a certain load, when its motion is constrained by the presence of an exterior obstacle at $x = 0$. The obstacle will be elastic or rigid. We put

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(1) In what follows the dot will denote partial differentiation in time and the prime differentiation with respect to x.

(2.3) $$U(t) = u(0,t),$$

and

(2.4) $$R = -\sigma(0),$$

which is the support reaction on the bar. Besides R the only external action to be considered will be a body force

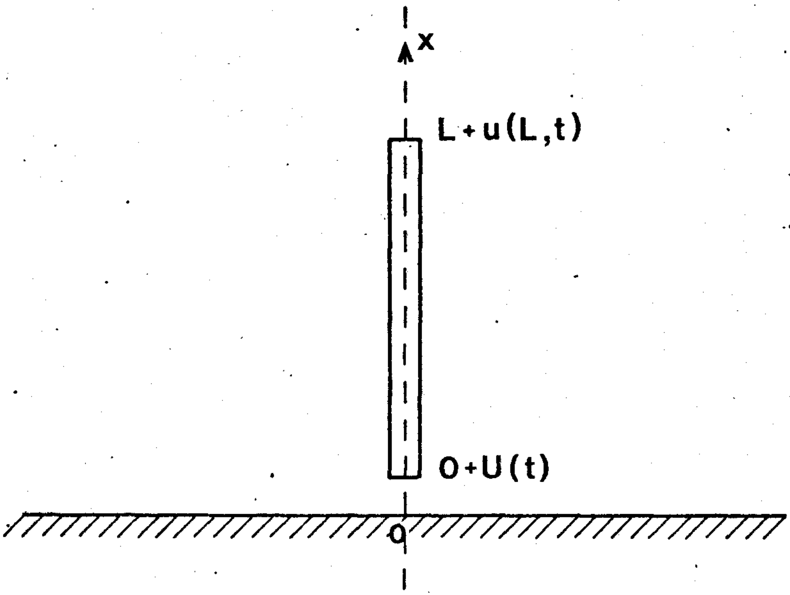

Figure 1

described by a density $f = f(x,t)$. In particular, particle $x = L$ is supposed free, that is

(2.5) $$\sigma(L) = 0, \quad \forall\ t,$$

and the evolution isothermal.

The formulation of the problem depends on the situation to be considered: in the elastic obstacle case, it is legitimate to search a motion in which the velocities are continuous.

In the case of the rigid obstacle, the possibility of shock waves ought to be allowed (at least for elastic bars), and a weak formulation in time is indicated, even though parasite solutions could be introduced, as observed by P.D. Lax [3].

The following notation will be used:

$$(2.6)\qquad \begin{cases} a(u,v) = \displaystyle\int_0^L a\,E(u)\,E(v)\,dx, \\ b(\dot u,v) = \displaystyle\int_0^L b\,E(\dot u)\,E(v)\,dx, \\ (f,v) = \displaystyle\int_0^L f\,v\,dx. \end{cases}$$

And the initial data $u_o(x) = u(x,0)$, $\dot u_o(x) = \dot u(x,0)$ will be supposed to satisfy

$$(2.7)\qquad U(0) \geq 0.$$

In the case of an elastic obstacle, with rigidity k, the force law associated to the constraint is written

$$R = 0 \quad \text{if} \quad U \geq 0,$$
$$R = -kU \quad \text{if} \quad U \leq 0,$$

that is, denoting by φ^+ (resp. φ^-) the positive part (resp. negative) of φ,

$$(2.8)\qquad R = kU^-.$$

The theorem of the virtual powers implies

$$(2.9)\qquad (\ddot u(t),v) + a(u(t),v) + b(\dot u(t),v) = (f(t),v) + kU^-(t)V, \quad \forall\, v,$$

where $V = v(0)$. The functional framework for relation (2.9) will be made precise in the variational formulation of the problem (Theorem 3.1).

In the case of the rigid obstacle, the force law associated to the constraint is written

$$(2.10)\qquad U \geq 0, \quad R \geq 0, \quad RU = 0,$$

which is equivalent to

$$(2.11)\qquad U \geq 0, \quad R(W-U) \geq 0, \quad \forall\ W \geq 0.$$

The weak formulation in time of the theorem of the virtual powers then leads to

$$(2.12)\qquad \int_0^T \{-(\dot{u}(t),\dot{v}(t)) + a(u(t),v(t)) + b(\dot{u}(t),v(t))\}dt + (\dot{u}_o,v(0)) = \int_0^T \{(f(t),v(t)) + R(t)V(t)\}dt,$$

$$\forall\ \{v(t),\ t \in (0,T)\},\quad v(T) = 0,$$

with

$$(2.13)\qquad u(0) = u_o,$$

and

$$(2.14)\qquad \int_0^T R(t)(W(t)-U(t))dt \geq 0, \qquad \forall\ W \geq 0 \quad \text{on} \quad (0,T).$$

For this problem, the functional framework will be made precise in Theorem 3.3.

3. BASIC THEORETICAL RESULTS

We first introduce the notation to be used. We put $\mathcal{H} = L^2(\Omega)$, and denote by $(\cdot,\cdot)$ the usual scalar product on $\mathcal{H}$, with $|\cdot|$ the associated norm. On the other hand, $\mathcal{V}$ will stand for $H^1(\Omega)$ and its usual norm denoted by $\|\cdot\|$. The bilinear forms $a(u,v)$ and $b(u,v)$, defined in (2.6), are continuous on $\mathcal{V}\times\mathcal{V}$. Furthermore, given $\lambda > 0$, there exists $\alpha > 0$ (resp. $\beta > 0$) such that

$$(3.1)\qquad \begin{cases} a(v,v) + \lambda|v|^2 \geq \alpha\|v\|^2, & \forall\ v \in \mathcal{V} \\ (\text{resp. } b(v,v) + \lambda|v|^2 \geq \beta\|v\|^2, & \forall\ v \in \mathcal{V}). \end{cases}$$

The result describing the mathematical properties of the problem with an elastic obstacle (2.8)-(2.9) is the following.

<u>Theorem 3.1</u> - Let $k > 0$, $T > 0$, $u_o \in \mathcal{V}$, $\dot{u}_o \in \mathcal{H}$ and $f \in L^2(0,T;\mathcal{H})$ be given. Then there exists a unique function u such that

$$(3.2)\qquad u \in L^\infty(0,T;\mathcal{V}),$$

$$(3.3)\qquad \dot{u} \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}),$$

$$(3.4)\qquad u(0) = u_o,$$

$$(3.5)\qquad \dot{u}(0) = \dot{u}_o,$$

which verifies relation (2.9) for every $v \in \mathcal{V}$ and for almost every $t \in (0,T)$.

<u>Remark 3.2</u> - The solution put in evidence by this theorem depends, naturally, on k. It will be noted u_k from now on.

Problem (2.10)-(2.14), which corresponds to the rigid obstacle, will be treated as a limit case, $k \to \infty$.

The result relative to this problem is the following.

Theorem 3.3 - The data are $T > 0$, $u_0 \in \mathcal{V}$, with $U_0 > 0$, $\dot{u}_0 \in \mathcal{H}$ and $f \in L^2(0,T;\mathcal{H})$. Then there exists a function u, and a bounded measure R on $[0,T]$, such that:

$$\text{(3.6)} \qquad u \in L^\infty(0,T;\mathcal{V}), \quad \dot{u} \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \quad U \in C^0([0,T]),$$

$$\text{(3.7)} \qquad u(0) = u_0,$$

$$\text{(3.8)} \qquad U \geq 0,$$

$$\text{(3.9)} \qquad \int_0^T \{-(\dot{u},\dot{v}) + a(u,v) + b(\dot{u},v)\}dt + (\dot{u}_0, v(0)) = \int_0^T (f,v)dt + \langle R,V\rangle,$$
$$\forall\, v \in L^2(0,T;\mathcal{V}), \text{ with } \dot{v} \in L^1(0,T;\mathcal{H}),$$
$$V \in C^0([0,T]), \quad v(T) = 0,$$

$$\text{(3.10)} \qquad \langle R, W-U\rangle \geq 0, \qquad \forall\, W \in C^0([0,T]), \qquad W \geq 0.$$

Remark 3.4 - The support reaction appears here as a measure on $[0,T]$. Relation (3.10) is a weakened form of (2.11) (or (2.14)).

Remark 3.5 - $\langle\ ,\ \rangle$ is the duality between the space of bounded measures on $[0,T]$ and $C^0([0,T])$.

4. PROOF OF THEOREM 3.1

This theorem is probably classical, or almost, the demonstration being built through standard arguments.

For the sake of simplicity we take $k = 1$.

Uniqueness - Let u_1 and u_2 be two solutions, and put $u = u_1 - u_2$. For $\lambda > 0$ fixed, define

$$z_\alpha(t) = e^{-\lambda t} u_\alpha(t), \qquad \alpha = 1,2,$$
$$z(t) = e^{-\lambda t} u(t).$$

Writing (2.9) successively for u_1, u_2, taking the difference and choosing $v = \dot z$, we obtain

$$(\ddot z + 2\lambda \dot z + \lambda^2 z, \dot z) + a(z,\dot z) + b(\dot z + \lambda z, \dot z) = (Z_1^- - Z_2^-)\dot Z,$$

that is,

$$\frac{1}{2}\frac{d}{dt}\{|\dot z|^2 + \lambda^2 |z|^2 + a(z) + \lambda\, b(z)\} + 2\lambda|\dot z|^2 + b(\dot z) \le$$
$$\le |z|\,|\dot z| \quad (*),$$

observing that

$$|Z_1^- - Z_2^-| \le |Z_1 - Z_2|.$$

It follows then, using $z(0) = 0$, $\dot z(0) = 0$ and (3.1),

$$(4.1) \qquad |\dot z|^2 + \alpha\|z\|^2 + \lambda b(\dot z) + \beta\int_0^t \|\dot z(\tau)\|^2\, d\tau$$
$$\le C\Big(\int_0^t \|z\|^2\, d\tau\Big)^{1/2} \Big(\int_0^t \|\dot z\|^2\, d\tau\Big)^{1/2}.$$

Since λ is positive, all the terms in the left hand side have the same property. In particular

$$\beta\int_0^t \|\dot z\|^2\, d\tau \le C\Big[\int_0^t \|z\|^2\, d\tau\Big]^{1/2}\Big[\int_0^t \|\dot z\|^2\, d\tau\Big]^{1/2},$$

(*) We put $a(v,v) = a(v)$, $b(v,v) = b(v)$.

that is,

$$\beta\left[\int_0^t \|\dot z\|^2 \, d\tau\right]^{1/2} \leq C\left[\int_0^t \|z\|^2 \, d\tau\right]^{1/2}.$$

Hence the second member of (4.1) is bounded by

$$C_1 \int_0^t \|z\|^2 \, d\tau .$$

We have then, from (4.1),

$$\alpha\|z\|^2 \leq C\int_0^t \|z\|^2 \, d\tau ,$$

from where we get $z = 0$, and the uniqueness result.

Existence - It is based on the Faedo-Galerkin method, the limit in the non-linear term taken through compacity arguments.

Let $\{w_i, \, i \geq 1\}$ be a sequence of independent vectors in $\mathcal{V}$, with linear combinations dense in $\mathcal{V}$. The initial data u_o and $\dot u_o$ are approximated by u_{om} and $\dot u_{om}$: u_{om} (resp. $\dot u_{om}$) is a linear combination of the $\{w_i, \, 1 \leq i \leq m\}$ and $u_{om} \to u_o$ in $\mathcal{V}$ (resp. $\dot u_{om} \to \dot u_o$ in $\mathcal{H}$) as $m \to \infty$.

Let now

$$u_m(t) = \sum_1^m g_{jm}(t) w_j$$

be the solution of the problem

(4.2) $$(\ddot u_m(t), w_j) + a(u_m(t), w_j) + b(\dot u_m(t), w_j)$$
$$= (f(t), w_j) + U_m^-(t) W_j, \quad 1 \leq j \leq m,$$

(4.3) $$u_m(0) = u_{om},$$

(4.4) $$\dot u_m(0) = \dot u_{om}.$$

We first establish a priori estimations for u_m. We multiply (4.2) by $\dot g_{jm}(t)$, sum on j, and integrate from

0 to t, to obtain:

$$\frac{1}{2}\{|\dot{u}_m(t)|^2 + a(u_m(t)) + [U_m^-(t)]^2\} + \int_0^t b(\dot{u}_m(\tau))d\tau = \tag{4.5}$$

$$= \int_0^t (f(\tau),\dot{u}_m(\tau))d\tau + $$

$$+ \frac{1}{2}\{|\dot{u}_{om}|^2 + a(u_{om}) + [U_m^-(0)]^2\}.$$

Since $u_{om} \to u_o$ in $\mathcal{V}$, $U_m(0) \to U(0)$ in $\mathbb{R}$; hence, the majoration

$$U_m^-(0) \le |U_m(0)|$$

assures that the terms in the right hand side of (4.5) corresponding to $t = 0$ are bounded independently of m. On the other hand we have

$$\left|\int_0^t (f(\tau),\dot{u}_m(\tau))d\tau\right| \le \frac{1}{2}|f|^2_{L^2(0,T;\mathcal{H})} + \tag{4.6}$$

$$+\frac{1}{2}\int_0^t |\dot{u}_m(\tau)|^2 d\tau .$$

Relation (4.5) then fournishes

$$|\dot{u}_m(t)|^2 \le C + \int_0^t |\dot{u}_m(\tau)|^2 d\tau ,$$

so that

$$|\dot{u}_m(t)| \le C_1 , \tag{4.7}$$

where C_1 = constant in $t \in (0,T)$ and $m \ge 1$. Further, $a(u_m(t))$ and $\int_0^T b(\dot{u}_m(\tau))d\tau$ are bounded, and then, by (3.1):

(4.8) $$\|u_m(t)\| \leq C_2,$$

(4.9) $$\int_0^T \|\dot{u}_m(\tau)\|^2 \, d\tau \leq C_3.$$

Finally,

(4.10) $$U_m^-(t) \leq C_4,$$

with all the constants being independents of $t \in (0,T)$ and $m \geq 1$.

From estimates (4.7), (4.8), (4.9) one can deduce the existence of a function $u \in L^\infty(0,t;\mathcal{V})$, with $\dot{u} \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V})$, such that, at least for a sub-sequence,

(4.11) $u_m \to u$ weak* in $L^\infty(0,T;\mathcal{V})$,

(4.12) $\dot{u}_m \to \dot{u}$ weak* in $L^\infty(0,T;\mathcal{H})$ and weak in $L^2(0,T;\mathcal{V})$.

On the other side, it results from (4.8) and (4.9) that

(4.13) $$|U_m(t)| \leq C_4,$$

(4.14) $$\int_0^T \dot{U}_m^2(\tau)\,d\tau \leq C_5.$$

Hence

$$U_m \to U \quad \text{weak* in} \quad L^\infty(0,T),$$

$$\dot{U}_m \to \dot{U} \quad \text{weak in} \quad L^2(0,T),$$

which implies

$$U_m \to U \quad \text{weak in} \quad H^1(0,T).$$

But in one dimension,

$$H^1(0,T) \subset C^0([0,T]),$$

with compact injection. Then

(4.15) $\quad U_m \to U \quad$ strongly in $\quad C^0([0,T])$.

From this we deduce

(4.16) $\quad U_m^- \to U^- \quad$ strongly in $\quad C^0([0,T])$.

Convergences (4.11), (4.12) and (4.16) imply, for j fixed, that

$$a(u_m, w_j) \to a(u, w_j) \quad \text{weak* in} \quad L^\infty(0,T),$$

$$b(\dot{u}_m, w_j) \to b(\dot{u}, w_j) \quad \text{weak in} \quad L^2(0,T),$$

$$U_m^- W_j \to U^- W_j \quad \text{in} \quad C^0([0,T]),$$

$$(\ddot{u}_m, w_j) \to \frac{d^2}{dt^2}(u, w_j) \quad \text{in} \quad \mathcal{D}'(0,T).$$

Taking the limit $m \to \infty$ in (4.2), we obtain

$$(\ddot{u}, w_j) + a(u, w_j) + b(\dot{u}, w_j) = (f, w_j) + U^- W_j,$$

$$j \geq 1, \quad \text{a.e.} \quad t \in (0,T).$$

Hence, by the density in $\mathcal{V}$ of the linear combinations of $\{w_j, j \geq 1\}$, we deduce (2.9) valid for every $v \in \mathcal{V}$ and almost every $t \in (0,T)$.

By classical arguments we can also get (3.4) and (3.5). Then u is the desired solution.

5. PROOF OF THEOREM 3.3

The proof consists in showing the existence of a solution of problem (3.6)-(3.10) which is the limit, as $k \to \infty$, of solutions of problem (3.2)-(3.5), (2.9). Here the notation of last section will be modified: the solution generated by Theorem 3.1 is indicated by u_k (Cf. Remark 3.2).

<u>A priori estimates I</u> - We choose $v = \dot{u}_k(t)$ in (2.9) to get

$$\frac{1}{2}\frac{d}{dt}\{|\dot{u}_k(t)|^2 + a(u_k(t)) + k(U_k^-(t))^2\} + \\ + b(\dot{u}_k(t)) = (f(t),\dot{u}_k(t)).$$

Integrating this relation over $(0,t)$ and taking (2.7), (3.4), (3.5) into account $(U_k^-(0) = 0)$:

$$(5.1) \qquad |\dot{u}_k(t)|^2 + a(u_k(t)) + k(U_k^-(t))^2 + 2\int_0^t b(\dot{u}_k(\tau))d\tau = \\ = 2\int_0^t (f(\tau),\dot{u}_k(\tau))d\tau + |\dot{u}_o|^2 + a(u_o).$$

The terms in the left hand side being positives, we explore this relation in the following way. First deduce

$$|\dot{u}_k(t)|^2 \leq C_1 + 2\int_0^t |f(\tau)||\dot{u}_k(\tau)|d\tau \\ \leq C_2 + \int_0^t |\dot{u}_k(\tau)|^2 d\tau,$$

which implies, by Gronwall's lemma,

$$(5.2) \qquad |\dot{u}_k(t)| \leq C_3 \qquad \text{(constant in } k \text{ and } t).$$

Now the right hand side of (5.1) is bounded, implying

$$(5.3) \qquad k(U_k^-(t))^2 \leq C_4,$$

$$(5.4) \qquad \int_0^t b(\dot{u}_k(\tau))d\tau \leq \frac{1}{2} C_4,$$

$$(5.5) \qquad a(u_k(t)) \leq C_4.$$

After (5.2), $|u_k(t)|$ is also bounded independently of k and t, and hence, by properties (3.1),

(5.6) $$\|u_k(t)\| \le c_5,$$

(5.7) $$\int_0^T \|\dot{u}_k(t)\|^2 \, dt \le c_6,$$

with constants in k and t. Those estimates, together with the trace theorem, yield

(5.8) $$|U_k(t)| \le c_7$$

(5.9) $$\int_0^T \dot{U}^2(t)dt \le c_8.$$

A priori estimates II - We choose in (2.9), for the virtual motion v, a translation in the direction $x < 0$, that is, $v(x) = V < 0$, V constant. We get

$$-V \int_0^T k \, U_k^-(t)dt = \int_0^T (f(t),v)dt - $$

$$- (\dot{u}_k(T),v) + (\dot{u}_o,v).$$

After (5.2):

(5.10) $$\int_0^T k \, U_k^-(t)dt \le c_9.$$

The above collection of estimates imply that:

$$
(5.11)\left\{\begin{array}{llllllll}
u_k & \text{remain in a bounded set of} & & & & & & L^\infty(0,T;\mathcal{V}), \\
\dot{u}_k & '' & '' & '' & '' & '' & '' & L^\infty(0,T;\mathcal{H}), \\
\dot{u}_k & '' & '' & '' & '' & '' & '' & L^\infty(0,T;\mathcal{V}), \\
\sqrt{k}\; U_k^- & '' & '' & '' & '' & '' & '' & L^\infty(0,T), \\
U_k & '' & '' & '' & '' & '' & '' & L^\infty(0,T), \\
\dot{U}_k & '' & '' & '' & '' & '' & '' & L^2(0,T), \\
k\; U_k^- & '' & '' & '' & '' & '' & '' & L^1(0,T).
\end{array}\right.
$$

Therefore we can conclude the existence of an element

$$(5.12)\qquad u \in L^\infty(0,T;\mathcal{V}),$$

with

$$(5.13)\qquad \dot{u} \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}),$$

$$(5.14)\qquad U \in L^\infty(0,T),$$

$$(5.15)\qquad \dot{U} \in L^2(0,T),$$

and a bounded measure R, such that, at least for a subsequence,

$$
(5.16)\left\{\begin{array}{lll}
u_k \to u & \text{weak* in} & L^\infty(0,T;\mathcal{V}), \\
\dot{u}_k \to \dot{u} & \text{weak* in} & L^\infty(0,T;\mathcal{H}), \text{ and} \\
 & \text{weak in} & L^2(0,T;\mathcal{V}), \\
U_k \to U & \text{weak* in} & L^\infty(0,T), \\
\dot{U}_k \to \dot{U} & \text{weak in} & L^2(0,T), \\
kU_k^- \to R & \multicolumn{2}{l}{\text{vaguely in the space of bounded measures}} \\
 & \multicolumn{2}{l}{\text{on } [0,T].}
\end{array}\right.
$$

Those properties guarantee, in particular (J. Necas, [6]), that

(5.17) $\quad U_k \to U$ strongly in $C^o([0,T])$,

and $U \geq 0$, due to (5.3).

Let now v be a function in $L^2(0,T;\mathcal{V})$, with $\dot{v} \in L^1(0,T;\mathcal{H})$, $v(T) = 0$ and $V \in C^o([0,T])$. We test (2.9) with the virtual motion $v(t)$ and integrate on $(0,T)$, to get

$$\int_0^T \{-(\dot{u}_k(t),\dot{v}(t)) + a(u_k(t),v(t)) + b(\dot{u}_k(t),v(t))\}dt$$
$$= \int_0^T (f(t),v(t))dt - (\dot{u}_o,v(0)) +$$
$$+ \int_0^T k\, U_k^-(t)\, V(t)dt.$$

Now we take the limit in this expression, using (5.16), to obtain

$$\int_0^T \{-(\dot{u},\dot{v}) + a(u,v) + b(\dot{u},v)\}dt = \tag{5.19}$$
$$= \int_0^T (f,v)dt - (\dot{u}_o,v(0)) + \langle R,V\rangle .$$

On the other hand, (3.7) is satisfied, since $u_k(0) = u_o$ for all k.

Let now W be taken in $C^o([0,T])$. The function

$$\varphi(\xi) = \begin{cases} \dfrac{\xi^2}{2} & \text{if } \xi \leq 0 \\ 0 & \text{if } \xi \geq 0, \end{cases}$$

is convex, hence

$$\frac{1}{2}[(W^-)^2 - (U_k^-)^2] \geq -U_k^-(W - U_k), \quad \text{on } [0,T];$$

in particular, if we choose $W \geq 0$ on $[0,T]$,

$$U_k^-(W-U_k) \geq 0, \quad \text{on} \quad [0,T],$$

so that

$$\int_0^T k\, U_k^-(W-U_k)dt \geq 0.$$

If $k \to \infty$, we deduce from the last relation in (5.16) and from (5.17) that

$$\langle R, W-U\rangle \geq 0, \quad \forall\, W \in C^0([0,T]), \quad W \geq 0.$$

6. A VARIATIONAL INEQUALITY

We go back to the problem of the rigid obstacle described in section 2 for some general coments.

Let u be the configuration of the system at time t; the set of the virtual motions compatibles with the imposed constraint, denoted by $\mathbb{K}(u)$, depends on the actual configuration, that is, on the displacement field u, existing at time t. More precisely, the compatibility of a virtual motion (that is of a virtual velocity field) is described in the following way:

$$v \in \mathbb{K}(u)$$

if and only if

$$(6.1) \qquad \begin{cases} U \geq 0, \\ V \geq 0 \quad \text{if} \quad U = 0, \\ V \in \mathbb{R} \quad \text{if} \quad U > 0. \end{cases}$$

Observe that those conditions imply

$$\mathbb{K}(u) = \emptyset \quad \text{if} \quad U < 0.$$

Naturally, for every u, $\mathbb{K}(u)$ is a convex set.

The force law associated to the constraint imposed by the presence of the rigid obstacle is described in (2.10) or (2.11); $R(t)$ is the obstacle reaction at time t. We have the following

<u>Lemma 6.1</u> - Let R be a function a.e. defined on $(0,T)$ and u a.e. differentiable. Then, to say that u and R are associated by the force law (2.11), a.e. on $(0,T)$, is equivalent to

$$(6.2) \qquad \begin{cases} \dot{u}(t) \in \mathbb{K}(u(t)) \\ R(t)\ (V-\dot{U}(t)) \geq 0, \end{cases}$$

$$\forall\ v \in \mathbb{K}(u(t)),$$

$$\text{a.e. on } (0,T).$$

<u>Proof</u>: 1) Let us assume (2.11), and let t be a point where u is differentiable. We test (2.11) with $U(t+h)$ and $U(t-h)$, $h > 0$; dividing by h each relation, we get

$$R(t)\ \dot{U}(t) = 0.$$

Let now $v \in \mathbb{K}(u(t))$; we have

$$R(t)(V-\dot{U}(t)) = R(t)V \geq 0,$$

which is (6.2).

2) Conversely, if (6.2) is assumed, then $U(t) \geq 0$, after (6.1), since $\mathbb{K}(u(t)) \neq \emptyset$. Furthermore, in view that $\mathbb{K}(u(t))$ is a cone, the second relation (6.2) implies $R\dot{U} = 0$; it is equivalent then to

$$R(t)\dot{V} \geq 0, \qquad \forall\, v \in \mathbb{K}(u(t)).$$

Hence, if $U(t) > 0$, V can have any sign and $R = 0$; if $U(t) = 0$, V can take only non-negative values and $R \geq 0$. We obtain then (2.10), which implies relation (2.11).

In view of this lemma the problem of the rigid obstacle can be formulated in a different way. We have the following

Corollary 6.2 - Let R be a function a.e. defined on $(0,T)$ and u a function a.e. differentiable on $(0,T)$. The couple (u,R) is a solution of problem (2.11), (2.12), (2.13) if and only if,

$$\dot{u}(t) \in \mathbb{K}(u(t)), \tag{6.3}$$

$$u(0) = u_o, \tag{6.4}$$

$$\int_0^T \{-(\dot{u},v-\dot{u}) + a(u,v-\dot{u}) + b(\dot{u},v-\dot{u}) - (f,v-\dot{u})\}dt + (\dot{u}_o,v(0)-\dot{u}_o) \geq 0, \tag{6.5}$$

$$\forall\, v, \text{ such that } v(t) \in \mathbb{K}(u(t)), \text{ a.e. on } (0,T).$$

Remark 6.3 - In the framework defined by Corollary 6.2, problem (2.11), (2.12), (2.13) is posed in terms of a quasi-variational inequality, to use the terminology of J.L. Lions [2].

Remark 6.4 - The solution of problem (2.11), (2.12), (2.13) exhibited by Theorem 3.3 is not related to the above corollary: the support reaction appears as a measure on the interval $(0,T)$, and not as a function.

7. A NUMERICAL SCHEME. CONVERGENCE

The process itself of proving existence of a solution for problem (3.6)-(3.10) suggests a natural scheme for the computation of that solution: to interpret $\{u_k(t),\ k\, u_k^-(0,t)\}$ as a regularization of the solution $\{u(t), R(t)\}$, and make discretizations in (2.9) to define the approximated solution $\{u^k_{h,\Delta t}(t),\ R^k_{h,\Delta t}(t)\}$. The "convergence" in the regularization parameter k $(\to\infty)$ would be guaranteed by the argument of Theorem 3.3.

The discretization of (2.9) is done in the following way. At first we make a change of variables, $z(t) \doteq e^{-\lambda t} u(t)$, $\lambda > 0$, transforming (2.9) into the problem

$$
(7.1)\begin{cases}
\text{(i)} & (\ddot z(t),v) + \lambda^2(z(t),v) + a(z(t),v) + \lambda b(z(t),v) \\
& \quad + 2\lambda(\dot z(t),v) + b(\dot z(t),v) + \frac{k}{2}\, z(0,t)v(0) = \\
& \quad = (\bar f(t),v) + \frac{k}{2}\,|z(0,t)|v(0), \quad \forall\, v \in \mathcal{V}, \\
\text{(ii)} & z(0) = u_o, \\
\text{(iii)} & \dot z(0) = \dot u_o - \lambda u_o,
\end{cases}
$$

where $\bar f(t) = e^{-\lambda t} f(t)$. After that we define

$$x_j = jh, \quad 0 \le j \le M, \quad h = \frac{L}{M},$$

$$t_n = n\,\Delta t, \quad 0 \le n \le N, \quad \Delta t = \frac{T}{N},$$

$$\theta_n(t) = \begin{cases} 1 & \text{if} \quad t \in [t_n, t_{n+1}), \\ 0 & \text{otherwise,} \end{cases}$$

$$V_h = \{v \in C^o(0,L) \mid v|_{[x_j, x_{j+1}]} \text{ is a polynomial of degree } \le 1, \ 0 \le j \le M-1\},$$

and, as the approximation to $z(t)$, the function

$$(7.2) \qquad \zeta(t) = \sum_{n=0}^{N-1} \{\zeta^{n,\alpha}(x) + (t-t_n)\delta_t \zeta^n(x)\}\theta_n(t),$$

where $\{\zeta^n(x)\}_o^N$ satisfy

$$(7.3)\begin{cases} \text{(i)} & \zeta^n \in V_h, \quad 0 \le n \le N, \\ \text{(ii)} & \zeta^o \in V_h \text{ such that } \zeta^o(x_j) = u_o(x_j), \ 0 \le j \le M, \\ \text{(iii)} & \zeta^1 \in V_h \text{ such that } \zeta^1(x_j) = u_o(x_j) + \Delta t \ \dot{u}_o(x_j), \\ & \qquad 0 \le j \le M, \\ \text{(iv)} & (\partial_t^2 \zeta^n, v) + \lambda^2(\zeta^{n,\alpha}, v) + a(\zeta^{n,\alpha}, v) + \\ & + \lambda b(\zeta^{n,\alpha}, v) + 2\lambda(\delta_t \zeta^n, v) + \lambda b(\delta_t \zeta^n, v) \\ & + \frac{k}{2}\zeta^{n,\alpha}(0)v(0) \\ & \qquad = (\bar{f}^{n,\alpha}, v) + \frac{k}{2}|\zeta^n(0)|v(0), \quad \forall\, v \in V_h. \end{cases}$$

In the above equations, the notation used is

$$\delta_t \zeta^n = \frac{\zeta^{n+1} - \zeta^{n-1}}{2\Delta t},$$

$$\partial_t^2 \zeta^n = \frac{\zeta^{n+1} - 2\zeta^n + \zeta^{n-1}}{(\Delta t)^2},$$

$$\zeta^{n,\alpha} = \alpha\,\zeta^{n+1} + (1-2\alpha)\zeta^n + \alpha\,\zeta^{n-1}, \quad 0 \le \alpha \le \frac{1}{2},$$

$$\partial_t \zeta^n = \frac{\zeta^{n+1} - \zeta^n}{\Delta t},$$

$$\bar{f}^n = \bar{f}(t_n).$$

The approximation (7.2) is well defined by conditions (7.3), the implied algorithm being unconditionally stable and convergent. We have the following

Theorem 7.1 - Assuming the same hypothesis as in Theorem 3.1, and defining

$$(7.4) \qquad \partial_t \zeta(t) = \sum_{n=0}^{N-1} \partial_t \zeta^n \theta_n(t),$$

$$(7.5) \qquad \delta_t \zeta(t) = \sum_{n=0}^{N-1} \delta_t \zeta^n \theta_n(t),$$

we have that, for $0 < \alpha \leq 1/2$,

$$(7.6) \qquad \zeta \to \bar{z} \quad \text{weak* in} \quad L^\infty(0,T;\mathcal{V}),$$

$$(7.7) \qquad \partial_t\zeta \to \dot{z} \quad \text{weak* in} \quad L^\infty(0,T;\mathcal{H}),$$

$$(7.8) \qquad \delta_t\zeta \to \dot{z} \quad \text{weak in} \quad L^2(0,T;\mathcal{V}),$$

$$(7.9) \qquad k\,\zeta^-(0,\cdot) \to k z^-(0,\cdot) \quad \text{strong in} \quad C^0([0,T]),$$

independently of the manner as h and Δt go to zero.

Remark 7.2 - We observe that the approximation to $u(t)$ is $u^k_{h,\Delta t}(t) = e^{\lambda t}\,\zeta(t)$, and that to $R(t)$ is $R^k_{h,\Delta t}(t) = k\, e^{\lambda t}\,\zeta^-(0,t)$.

8. PROOF OF THEOREM 7.1

The first step in proving (7.6)-(7.9) is to obtain basic a priori estimates. For this we test equation (7.3)(iv) at $v = \delta_t \zeta^n$, getting

$$(8.1) \qquad \frac{1}{2\Delta t}\{|\partial_t\zeta^n|^2 - |\partial_t\zeta^{n-1}|^2\} +$$

$$\frac{\alpha}{2\Delta t}\{[\lambda^2\ |\zeta^{n+1}|^2 + a(\zeta^{n+1}) +$$

$$+ \lambda\ b(\zeta^{n+1})] - [\lambda^2\ |\zeta^{n-1}|^2 + a(\zeta^{n-1}) +$$

$$+ \lambda\ b(\zeta^{n-1})]\} +$$

$$(1-2\alpha)\{\lambda^2(\zeta^n,\delta_t\zeta^n)+a(\zeta^n,\delta_t\zeta^n)+\lambda b(\zeta^n,\delta_t\zeta^n)\}+2\lambda|\delta_t\zeta^n|^2+\lambda b(\delta_t\zeta^n) +$$

$$+ \frac{k\alpha}{2\cdot 2\Delta t}\{[\zeta^{n+1}(0)]^2 - [\zeta^{n-1}(0)]^2\}$$

$$+ (1-2\alpha)\ \frac{k}{2}\ \zeta^n(0)\ \delta_t\ \zeta^n(0) =$$

$$= (\bar{f}^{n,\alpha},\ \delta_t\zeta^n) + \frac{k}{2}\ |\zeta^n(0)|\ \delta_t\zeta^n(0),$$

because of the identities

$$\partial_t^2 a^n = \frac{\partial_t a^n - \partial_t a^{n-1}}{\Delta t},$$

$$\delta_t a^n = \frac{\partial_t a^n + \partial_t a^{n-1}}{2},$$

$$A(a^{n,\alpha};\delta_t a^n) = \frac{\alpha}{2\Delta t}[A(a^{n+1}) - A(a^{n-1})] + (1-2\alpha)A(a^n;\delta_t a^n),$$

valid for any sequence $\{a^n\}$ and every bilinear form $A(u;v)$.

Now assuming $0 < \alpha \le 1/2$, multiplying (8.1) by $2\Delta t$ and summing from $n = 1$ to $n = m$, we obtain

$$(8.2) \qquad |\partial_t\zeta^m|^2 + \alpha\lambda^2|\zeta^{m+1}|^2 + \alpha a(\zeta^{m+1}) + \alpha\lambda b(\zeta^{m+1})$$

$$+ 4\lambda \sum_{j=1}^{m} \Delta t|\delta_t\zeta^j|^2 + 2\lambda \sum_{j=1}^{m} \Delta t\ b(\delta_t\zeta^j)$$

$$+ \frac{k\alpha}{2}[\zeta^{m+1}(0)]^2 \le C_1 + C_2 \sum_{j=1}^{m} \Delta t\|\zeta^j\|^2$$

$$+ \epsilon \sum_{j=1}^{m} \Delta t\ \|\delta_t\zeta^j\|^2,$$

where $\epsilon > 0$ is arbitrary and the constants depend on the initial data, f, α, k and λ. To reach inequality (8.2), one should observe, the trace theorem for $H^1(\Omega)$ was used, as well as the standard estimation $a \cdot b \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$, $\epsilon > 0$.

Choosing an appropriate value for ϵ in (8.2), it yields

$$|\partial_t \zeta^{m+1}|^2 + \|\zeta^{m+1}\|^2 + c_3 \sum_{j=1}^{m} \Delta t \, \|\delta_t \zeta^j\|^2$$
$$\leq c_4 + c_5 \sum_{j=1}^{m} \Delta t \, \|\zeta^j\|^2, \quad c_i > 0,$$

which implies, coupled with the discrete form of Gronwall's lemma, the basic a priori estimate

$$(8.3) \qquad |\partial_t \zeta^n|^2 + \|\zeta^n\|^2 + c_6 \sum_{j=1}^{n} \Delta t \, \|\delta_t \zeta^j\|^2 \leq c_7, \quad c_i > 0,$$

for $n = 1,2,\ldots,N$. In view of (7.2), (7.4) and (7.5), (8.3) can be presented as

$$(8.4) \qquad |\partial_t \zeta|_{L^\infty(0,T;\mathcal{H})} \leq \text{const.},$$

$$(8.5) \qquad |\zeta|_{L^\infty(0,T;\mathcal{V})} \leq \text{const.},$$

$$(8.6) \qquad |\delta_t \zeta|_{L^2(0,T;\mathcal{V})} \leq \text{const.} \, .$$

The second step in the proof is to extract a convergent sub-sequence of $\{\zeta = \zeta(\Delta t, h) \mid \Delta t > 0, \ h > 0\}$. Estimates (8.4)-(8.6) imply the existence of $\tilde{z} \in L^\infty(0,T;\mathcal{V})$, $\tilde{z}_1 \in L^\infty(0,T;\mathcal{H})$ and $\tilde{z}_2 \in L^2(0,T;\mathcal{V})$, such that

$$(8.7) \qquad \begin{cases} \zeta \to \tilde{z} & \text{weak* in } L^\infty(0,T;\mathcal{V}), \\ \partial_t \zeta \to \tilde{z}_1 & \text{weak* in } L^\infty(0,T;\mathcal{H}), \\ \delta_t \zeta \to \tilde{z}_2 & \text{weak in } L^2(0,T;\mathcal{V}), \end{cases}$$

when h and Δt approach zero.

We claim that $\tilde{z}_1 = \tilde{z}_2 = \dot{\tilde{z}}$. Indeed, since

$$\mathcal{D}(0,T) = \{\psi \in C^\infty(0,T) \mid \exists\, \delta = \delta(\psi) > 0 \text{ such that } \psi|_{\complement[\delta,T-\delta]} = 0\},$$

we can choose Δt small enough so that, by summation by parts,

$$\begin{aligned}\langle \partial_t \zeta, \psi\rangle &= \int_0^T \partial_t\, \zeta(t)\, \psi(t)dt = \\ &= \sum_{n=0}^{N-1} \partial_t\, \zeta^n(x) \int_{t_n}^{t_{n+1}} \psi(t)dt \\ &= -\sum_{n=1}^{N} \zeta^n(x) \left\{\int_{t_n}^{t_{n+1}} \psi(t)dt - \int_{t_{n-1}}^{t_n} \psi(t)dt\right\}/\Delta t \\ &= -\langle \zeta, \partial_t \psi\rangle, \qquad \forall\, \psi \in \mathcal{D}(0,T).\end{aligned}$$

But $\partial_t \psi \to \dot{\psi}$ strongly in $L^1(0,T)$, hence

$$\langle \partial_t \zeta, \psi\rangle \to -\langle \tilde{z}, \dot{\psi}\rangle = \langle \dot{\tilde{z}}, \psi\rangle,$$

that is $\tilde{z}_1 = \dot{\tilde{z}}$.

A similar reasoning implies $\tilde{z}_2 = \dot{\tilde{z}}$.

On the other hand it follows from Sobolev's embedding theorem and (8.5)-(8.6),

$$(8.8) \qquad |\zeta(0,t)| \le \text{const.},$$

$$(8.9) \qquad \int_0^T (\delta_t \zeta)^2\, (0,\tau)d\tau \le \text{const.}\,.$$

Hence, as $h, \Delta t \to 0$,

$$\begin{aligned}\zeta(0,\cdot) &\to \tilde{z}(0,\cdot) \quad \text{weak* in } L^\infty(0,T), \\ \delta_t \zeta(0,\cdot) &\to \dot{\tilde{z}}(0,\cdot) \quad \text{weak in } L^2(0,T),\end{aligned}$$

which implies

$$\zeta(0,\cdot) \to \tilde{z}(0,\cdot) \quad \text{weak in} \quad H^1(0,T).$$

Since $H^1(0,T) \subset C^0([0,T])$, with compact injection, then

$$\zeta(0,\cdot) \to \tilde{z}(0,\cdot) \quad \text{strong in} \quad C^0([0,T]),$$

that is,

(8.10) $$\zeta^-(0,\cdot) \to \tilde{z}^-(0,\cdot) \quad \text{strong in} \quad C^0([0,T]).$$

The last step in the proof consists in showing that $\tilde{z}$ is a solution of (7.1). Because of the uniqueness property we would have then $\tilde{z} = z$, so that (8.7) and (8.10) would imply the theorem.

To check this fact, test equation (7.3)(iv) at he interpolant $v_h \in V_h$ of a given $v \in \mathcal{V}$, multiply by $\theta_n(t)$, and add from 0 to N-1. After that take the limit h, $\Delta t \to 0$. Since

$$\sum_{n=0}^{N-1} (t-t_n)\, \delta_t\, \zeta^n(x)\, \theta_n(t) \to 0$$

pointwise,

$$v_h \to v \quad \text{strong in} \quad \mathcal{V},$$

and

$$\lambda^2(\zeta,v_h) + a(\zeta,v_h) + \lambda\, b(\zeta,v_h) \to \lambda^2(\tilde{z},v) +$$

$$+ a(\tilde{z},v) + \lambda\, b(\tilde{z},v) \quad \text{weak* in} \quad L^\infty(0,T),$$

$$2\lambda(\delta_t\zeta,v_h) + \lambda\, b(\delta_t\zeta,v_h) \to 2\lambda(\dot{\tilde{z}},v) + \lambda\, b(\dot{\tilde{z}},v)$$

$$\text{weak in} \quad L^2(0,T),$$

$$\frac{k}{2}\,\zeta(0,\cdot)v_h(0) - \frac{k}{2}\sum_{n=0}^{N-1} |\zeta^n(0)|\theta_n(\cdot)v_h(0)$$

$$\to \; -k\, \tilde{z}^-(0,\cdot)v(0) \quad \text{strong in} \quad C^0([0,T]),$$

$$(\partial_t^2 \zeta, v_h) \to \frac{d^2}{dt^2}(\tilde{z}, v) \quad \text{in} \quad \mathcal{D}'(0,T),$$

are implied by (8.7) and (8.10), it is clear that $\tilde{z}$ satisfies (7.1).

9. RESULTS OF SOME NUMERICAL SIMULATIONS

In this section we present two results obtained by implementing algorithm (7.3) as a finite element code. In both examples a 100-element regular partition was used, and the calculations were performed on the IBM 370/145 at CBPF with double precision. The parameter α was chosen as 1/4 and $\Delta t = 0.01$. The values $\lambda = 0$ and $\lambda = 0.25$ were tested in both cases, with no distinction between the corresponding solutions appearing at the adopted scale.

Example 1

We simulate an elastic bar with $a = 1$, $b = 0$, $\rho = 1$ and $L = 1$. The external body force f was taken zero and the obstacle spring constant $k = 100$. The problem consists in computing the motion of the bar when it is submitted to the initial conditions $u_o(x) \equiv 0$ and $\dot{u}_o(x) = -1$. The exact solution of this problem is known, given by (see Fig. 2)

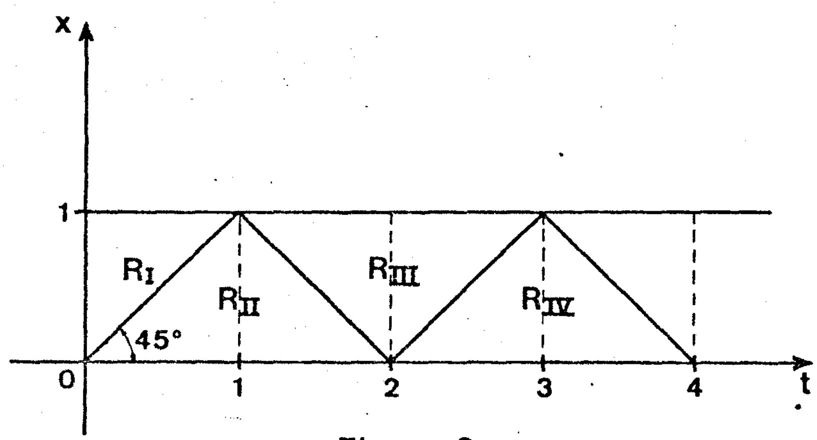

Figure 2

$$
(9.1)\quad \begin{cases} u(x,t) = -t \quad \text{if} \quad (x,t) \in R_I, \\ u(x,t) = -x + \frac{1}{k}[e^{k(x-t)}-1] \quad \text{if} \quad (x,t) \in R_{II}, \\ u(x,t) = t + \frac{1}{k}[e^{k(2-x-t)}+e^{k(x-t)}]-2(1+\frac{1}{k}), \\ \qquad\qquad \text{if} \quad (x,t) \in R_{III}, \\ \text{etc....} \end{cases}
$$

This example is to be understood as a test case for validating the code. In Figure 3 we have plotted the exact

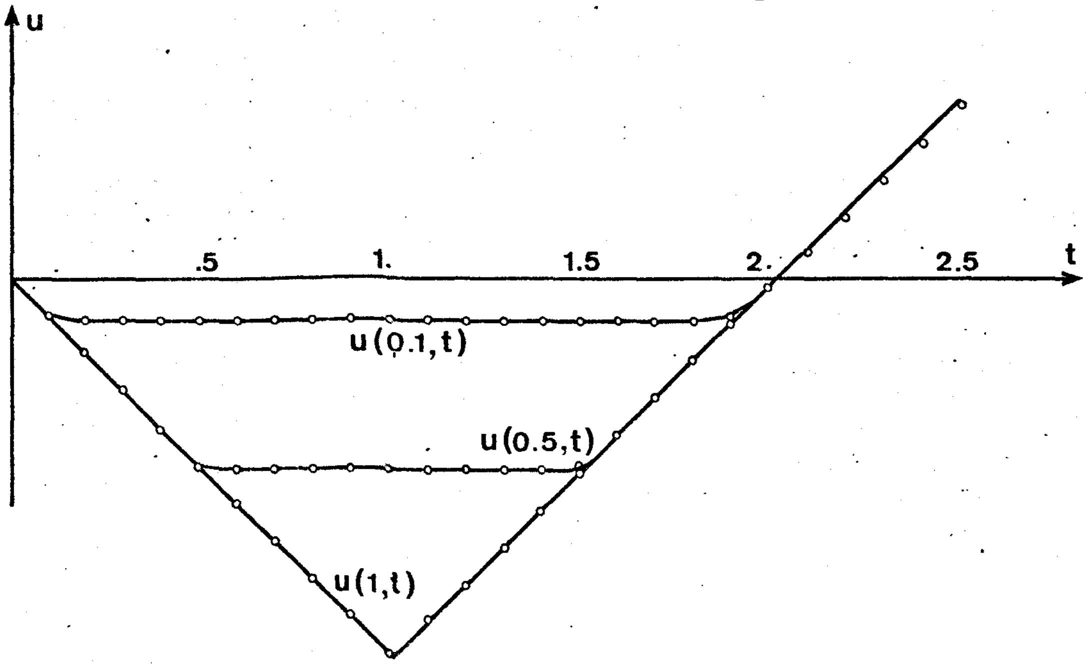

Figure 3

solution (continuous line) as well as the computed points.

Example 2

The same as before, with only one difference: the action of an external field $f = -1$ is taken into account. In this situation no exact solution is known in analytic form. As we can observe in Figure 4, the bar, after separating from the obstacle, comes back again to hit it for the second time, due to the effect of the field.

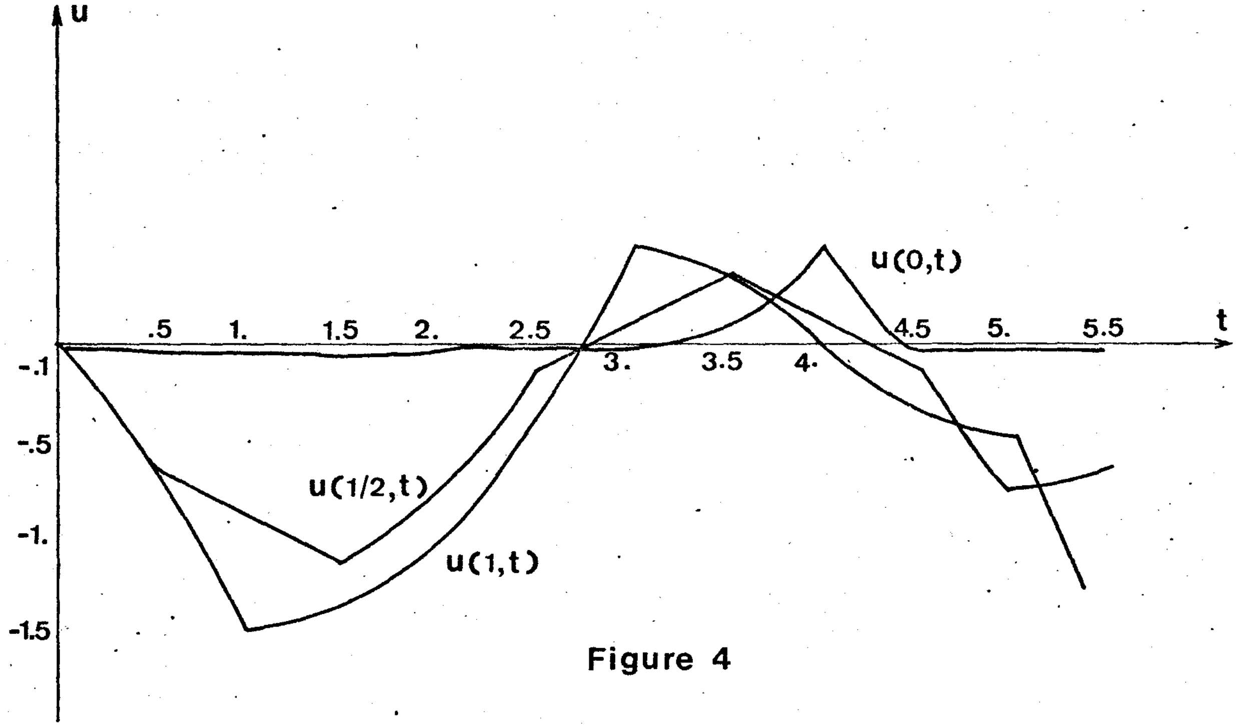

Figure 4

One last remark: the consideration of a viscosity $b > 0$ in the bar implies a smoothing of the curve corners.

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