NOTAS DE FÍSICA

VOLUME XVI

Nº 12

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RIO DE JANEIRO - BRAZIL

1970

Notas de Física - Volume XVI - Nº 12

SPINOR SOLUTIONS TO THE SCHWARZSCHILD METRIC

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(Received October 7th, 1970)

ABSTRACT

The class of solutions represented by Hermitian two-by-two matrices associated to time independent metrics possessing a symmetry Euclidian Killing vector are derived. This class of spinor fields are solutions of the gravitational spinor equation proposed by Sachs. More general radially symmetric spinor fields, as for instance the class of spinor fields associated to the gravitational field of a charged massive particle, are obtained directly from the spinor class associated to the Schwarzschild field by using the property that the Hermitian two-by-two matrices $\sigma_{\mu}^{K\dot{M}}$, describing the field, form a basys in the space of the complex two-by-two matrices. The properties which appear due to the covariance of the theory under the group SU_2 are analyzed.

1. INTRODUCTION

In this paper we treat the problem of determination of spinor solutions which are associated to radially symmetric time independent metrics in general relativity. This problem was already treated in the literature [1], however, presently we use a general approach for deriving these solutions. As

* Partial support from the National Research Council of Brazil is acknowledged.

result the whole class of solutions is consistently obtained. Since the three dimensional spherical symmetry is a metrical symmetry we have to start with a given general radially symmetric metric field, and look for the 2 × 2 Hermitian $\sigma_\mu^{K\dot{M}}$ matrices corresponding to this metric. As it is well known, there exists a whole class of such matrices, since given a metric tensor we will get a class of spinor solutions. The elements of this class being related by internal unimodular transformations under which the metric components are invariant. Thus, we may consider these internal transformations as a group of gauge transformations, the $g_{\mu\nu}$ being gauge invariant. It is also known that the interaction of gravitation with fermion fields is described by the σ_μ, not by the $g_{\mu\nu}$, as consequence it may happen that differents elements of the class of σ_μ corresponding to a given metric, conduct to inequivalent results which eventually may be observed.

The method presently used for deriving these spinor solutions is the three dimensional group of motions generated by the Euclidian Killing vector of the rotations in three-dimensions. This is done in the Sect. 2. In the Sect. 3 it is shown that the general class of spinor solutions conducting to a radially symmetric $g_{\mu\nu}$, it is solution of the field equations of Sachs [2]. In Sect. 4 two different solutions of these field equations are presented, and the connection between them is stablished by means of a pseudo-orthogonal internal matrix. In Sect. 5 a general procedure is stablished for the obtention of more general radially symmetric metrics, in terms of the associated spinor fields. Presently we generalize the Schwarzschild metric by allowing the source particle to be charged.

The notation used is the following: the local Minkowskian metric is taken with signature -2. For convenience, in Sect. 2 we start with the tetrad field of basys vectors which are associated to the Hermitian matrices σ_μ. The reason is that it looks easier to work out the Killing equation first in terms of the tetrads and at the end to translate the results for the σ_μ. The notation used for the tetrads is that usually found in the literature, that is, the internal degrees of freedom are denoted by letters inside of a bracket. Greek letters run from 0 to 3, and Latin letters from 1 to 3.

2. TETRADS CORRESPONDING TO RADIALLY SYMMETRIC METRICS

Starting with the equation of the tetrad calculus,

$$g_{\mu\nu} = h_{\mu(\alpha)}\, h_{\nu(\beta)}\, \eta^{\alpha\beta} \tag{1}$$

we impose invariance of $g_{\mu\nu}$ under the infinitesimal transformations

$$x'^{o} = x^{o}$$

$$x'^{i} = x^{i} + \varepsilon^{ik} x^{k},\ \varepsilon^{ik} = -\varepsilon^{ki}$$

which gives rise to the Killing equation

$$\bar{\delta} g_{\mu\nu}(x) = 0 \tag{2}$$

for the Euclidian Killing vector $\xi^{\mu} = (0, \varepsilon^{ik} x^{k})$. Since the $\eta^{\alpha\beta}$ are coordinate scalars with constant numerical value, we have

$$\bar{\delta}\eta_{\alpha\beta} = 0 .$$

Using that

$$\bar{\delta} h_{(\beta)\nu} = -\xi^{\lambda}_{,\nu}\, h_{\lambda(\beta)} - h_{(\beta)\nu,\lambda}\, \xi^{\lambda}$$

we get for Eqs. (1) and (2)

$$(h^{2}_{o(o)} + h_{o(i)}\, h^{(i)}_{o})_{,k}\, \xi^{k} = 0 \tag{3}$$

$$(h_{o(s)}\, h^{(s)}_{i})_{,k}\, \xi^{k} + h_{o(s)}\, h^{(s)}_{k}\, \xi^{k}_{,i} = 0 \tag{4}$$

$$(h^{(s)}_{i}\, h_{(s)j})_{,k}\, \xi^{k} + h^{(s)}_{i}\, h_{(s)k}\, \xi^{k}_{,j} + h_{j(s)}\, h^{(s)}_{k}\, \xi^{k}_{,i} = 0 \tag{5}$$

For obtaining these relations in this simple form we have made use of the condition

$$h_{(o)i} = 0 \,. \tag{6}$$

This condition imposes no restriction on the metric components, and is usually used in the literature. [3] In this paper we will use this possibility of simplifying the calculations by means of (6). However, the choice (6) implies that the degrees of freedom for taking rotations in the "planes (o)(i)" are dropped, that is, the tetrad fields satisfying (6) contain only three internal degrees of freedom, corresponding to rotations in the "planes (i)(j)". Also as consequence of (6) we have

$$h^{o}_{(i)} = 0 \,. \tag{7}$$

The solutions of the Eqs. (3), (4) and (5) are

$$h^2_{o(o)} + h_{o(i)}\, h^{(i)}_{o} = f(r) \tag{8}$$

$$h_{o(s)}\, h^{(s)}_{i} = g(r)\, x^i/r \tag{9}$$

$$h^{(s)}_{i}\, h_{(s)j} = -p(r)\, \delta_{ij} + q(r) x^i x^j/_{r^2} \,. \tag{10}$$

We can still perform coordinate transformation which keep unchanged the spherical symmetry of $g_{\mu\nu}$, they have the form

$$x^s = F_1(r', x'^o)x'^s$$

$$x^o = F_2(r', x'^o)$$

The functions F_1 and F_2 can be chosen such that in the new frame

$$g = 0$$

$$p = 1$$

That is, we may write the relations (8), (9) and (10) as

$$h_{(o)o} = (f(r))^{1/2} \tag{11}$$

$$h_{(s)o} = 0 \tag{12}$$

$$h_i^{(s)} h_{(s)j} = -\delta_{ij} + q(r)\, x^i x^j / r^2 \tag{13}$$

The relation (13) is satisfied by the tetrad field

$$h_i^{(s)} = \delta_i^s + \phi(r)\, x^s x^i / r^2 \tag{14a}$$

$$h_{(s)i} = \eta_{si} + \eta_{sm}\, \phi(r)\, x^m x^i / r^2 \tag{14b}$$

where $\eta_{si} = -\delta_{si}$, and we call attention to the fact that in this section we work in cartesian coordinates. Thus, the distance r here means the Euclidian distance $(x^s x^s)^{1/2}$.

The tetrad components defined by (11) through (14), give according to (1) the correct metric components [4]

$$g_{oo} = h^2_{(o)o} = f(r) \tag{15a}$$

$$g_{os} = 0 \tag{15b}$$

$$g_{ij} = h_i^{(s)} h_{(s)j} = -\delta_{ij} + q(r) x^i x^j / r^2 \tag{15c}$$

However, the tetrad vectors of (14) are not the unique possible solutions of the equation (13), or equivalently, are not the unique solutions for the three dimensional metric g_{ij}. Indeed, the whole class of tetrad fields defined by

$$\tilde{h}_i^{(s)} = M^{(s)}{}_{(n)}\, h_i^{(n)}$$

with M an arbitrary point dependent matrix satisfying

$$M^T \eta M = \eta$$

will be also solutions for the same metric g_{ij} [5]. Since $\eta = -1$, the matrices M will be orthogonal. Therefore, the solutions corresponding to radially symmetric time independent metric fields, as given by the relations (15), will contain three arbitrary functions given by the independent matrix elements of M.

The only condition which has to be satisfied is the asymptotic limit of a flat space. This is obtained by imposing that for large values of r,

$$f(r) \to 1$$
$$\phi(r) \to 0$$
$$M^{(s)}_{(n)}(r) \to R^{s}_{n}$$

where R is a rotation matrix belonging to the sub-group of the Lorentz group which is associated to SU_2 in special relativity.

The contravariant tetrad vectors take the form

$$h^{i}{}_{(k)} = \delta^{i}_{k} - x^{i} x^{k}/r^{2}\ \phi(1+\phi)^{-1} \tag{16a}$$

$$h^{o}{}_{(o)} = 1/h_{(o)o} \tag{16b}$$

$$h^{i}{}_{(o)} = 0 \tag{16c}$$

and we also have the class $\bar{h}^{i}{}_{(k)}$ defined similarly to before. Given the solutions (11), (12) and (14) for the $h_{(\mu)\nu}$ and the solutions (16) for the $h^{(\mu)\nu}$ we can write down the correspondent Hermitian matrices in the two-dimensional complex internal vector space

$$\sigma_{o} = (f(r))^{1/2}\ \mathring{\sigma}_{o} \tag{17a}$$

$$\sigma_{i} = \mathring{\sigma}_{i} + x^{i}/r^{2}\ \phi(r)x^{k}\ \mathring{\sigma}_{k} \tag{17b}$$

and

$$\sigma^{o} = 1/(f(r))^{1/2}\ \mathring{\sigma}_{o} \tag{18a}$$

$$\sigma^{i} = \mathring{\sigma}^{i} - x^{i}/r^{2}\phi(1+\phi)^{-1}\ x^{k}\ \mathring{\sigma}^{k} \tag{18b}$$

where the $\mathring{\sigma}_{o}$ is the two-by-two identity matrix and $\mathring{\sigma}_{i}$ are the Pauli matrices [6]. Introducing the matrices

$$\mathring{\tau}_{\mu} = \varepsilon\ \mathring{\sigma}^{*}_{\mu}\ \varepsilon,\ \varepsilon = i\ \mathring{\sigma}^{2}$$

which will give $\mathring{\tau}_{o} = -\mathring{\sigma}_{o}$ and $\mathring{\tau}^{i} = \mathring{\sigma}^{i}$, we can write for the metric tensor the expression

$$-2g_{\mu\nu} = Tr(\sigma_\mu \tau_\nu) \tag{19}$$

By the same type of conclusion as was obtained before for the tetrad field, it is possible to show that for a given $g_{\mu\nu}$ there exists a whole family of σ_μ (and τ_ν) which satisfy the relation (19). Since for our case the only arbitrary degrees of freedom belong to the spatial indices, this class of Hermitian matrices satisfying (19) will be the family of Hermitian matrices which are transformed one in the other by a unimodular unitary matrix which depends on the coordinates

$$\tilde{\sigma}_i = N\, \sigma_i\, N^+,\ |N| = 1,\ N^+ = N^{-1}$$

In this situation there will be no distinction between the σ_i and τ_i, that is, both will transform similarly, and we may as well write the Eq. (19) for $\mu = i$, $\nu = j$ only with σ matrices standing on the right hand side.

It can also be shown that the group SU_2 acting on the complex two-dimensional vector space is directly related to the group of local rotations of the fourlegs.

3. THE FIELD EQUATIONS IN SPINOR FORM FOR THE EXTERIOR SOLUTIONS

The matrices σ_μ (and σ^μ) corresponding to a possible solution for a time independent radially symmetric metric, depend on two arbitrary functions of r. In this section we will obtain a definite value for these functions by using the field equations. As it is known, the Lagrangian density for the gravitational field may be written in terms of the σ_μ and τ_μ matrices and their derivatives [7]. The field equations which follow from variations in τ_μ (or σ_μ) have the form,

$$\frac{\delta \mathcal{L}}{\delta \tau_\mu} = \frac{1}{4} (\tau^\alpha P^+_{\mu\alpha} + P_{\mu\alpha} \tau^\alpha) = 0 \tag{20}$$

(where we have set equal to zero the Ricci scalar since we treat with the exterior field).

The expression of the Ricci tensor written in term of the spinor variables is,

$$R_{\mu\beta} = \frac{1}{4} \mathrm{Tr}\, (P^{+}_{\lambda\mu}(\sigma^{\lambda}\, \tau_{\beta} - \sigma_{\beta}\, \tau^{\lambda}) + (\tau_{\beta}\, \sigma^{\lambda} - \tau^{\lambda}\, \sigma_{\beta})P_{\lambda\mu}) \tag{21}$$

Using the property that the σ_{μ} and τ_{μ} are Hermitian and $P_{\lambda\mu}$ is antisymmetric we may write this as

$$R_{\mu\beta} = \frac{1}{2} \mathrm{Tr} \left(\sigma_{\beta} \frac{\delta \mathscr{L}}{\delta \tau_{\mu}} \right) \tag{22}$$

Since the $\delta\mathscr{L}/\delta\tau_{\mu}$ as given by Eq. (20) is Hermitian, the right hand side of (22) is symmetric over the indices μ, β as it should be since the Ricci tensor is symmetric. Thus, we may use as the field equations for the exterior field the relations

$$(\tau^{\alpha}\, P^{+}_{\mu\alpha} + P_{\mu\alpha}\, \tau^{\alpha}) = 0 \tag{23}$$

In this section we show that the σ_{μ} of (17) are solutions of these equations, the resulting differential equations for the two functions $f(r)$ and $\phi(r)$ being the differential equations for the Schwarzschild radial functions.

Transforming to the spherical coordinate system, we get for the σ_{μ}.

$$\begin{cases} \sigma'_0 = \sigma_0 & (24a) \\ \sigma'_1 = -\,(1+\phi)\, \vec{\overset{\circ}{\sigma}} \cdot \frac{\vec{r}}{r} & (24b) \\ \sigma'_2 = -\, r\, \vec{\overset{\circ}{\sigma}} \cdot \vec{e}_{\theta} & (24c) \\ \sigma'_3 = -\, r \sin\Theta\, \vec{\overset{\circ}{\sigma}} \cdot \vec{e}_{\phi} & (24d) \end{cases}$$

where $\vec{\overset{\circ}{\sigma}} = (\overset{\circ}{\sigma}{}^1, \overset{\circ}{\sigma}{}^2, \overset{\circ}{\sigma}{}^3)$. For our practical purposes it will be more convenient to use instead of (23) the equation

$$P^{+}_{\mu\alpha}\,\sigma^{\alpha} + \sigma^{\alpha}\,P_{\mu\alpha} = 0 \qquad (25)$$

which follows from variations in the σ_{α}. The spinor curvature $P_{\mu\alpha}$ being defined by

$$P_{\mu\alpha} = \Gamma_{\mu,\alpha} - \Gamma_{\alpha,\mu} + \Gamma_{\alpha}\Gamma_{\mu} - \Gamma_{\mu}\Gamma_{\alpha}$$

$$\Gamma_{\mu} = -\frac{1}{4}\left[\tau_{\rho}\,\sigma^{\rho}_{\ ,\mu} + \{\mu\nu,\rho\}\,\tau^{\rho}\,\sigma^{\nu}\right]$$

Writing the relations (24) as

$$\begin{cases} \sigma'_0 = e^{\nu/2}\,\overset{o}{\sigma}_0 & (26a) \\ \sigma'_1 = -\,e^{\lambda/2}\,\vec{\overset{o}{\sigma}}\cdot\vec{r}/r & (26b) \\ \sigma'_2 = -\,r\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\theta} & (26c) \\ \sigma'_3 = -\,r\,\sin\theta\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\phi} & (26d) \end{cases}$$

and the corresponding components for the matrices τ'_{μ} , σ'^{μ} and τ'^{μ},

$$\begin{cases} \tau'_0 = -\,e^{\nu/2}\,\overset{o}{\sigma}_0 \\ \tau'_1 = -\,e^{\lambda/2}\,\vec{\overset{o}{\sigma}}\,\vec{r}/r \\ \tau_2 = -\,r\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\theta} \\ \tau_3 = -\,r\,\sin\theta\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\phi} \end{cases} \qquad \begin{cases} \sigma'^0 = e^{-\nu/2}\,\overset{o}{\sigma}_0 \\ \sigma'^1 = e^{-\lambda/2}\,\vec{\overset{o}{\sigma}}\cdot\vec{r}/r \\ \sigma'^2 = \frac{1}{r}\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\theta} \\ \sigma'^3 = \frac{1}{r\sin^2\theta}\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\phi} \end{cases} \qquad \begin{cases} \tau'^0 = -e^{-\nu/2}\,\overset{o}{\sigma}_0 \\ \tau'^1 = e^{-\lambda/2}\,\vec{\overset{o}{\sigma}}\cdot\vec{r}/r \\ \tau'^2 = \frac{1}{r}\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\theta} \\ \tau'^3 = \frac{1}{r\sin^2\theta}\,\vec{\overset{o}{\sigma}}\cdot\vec{e}_{\phi} \end{cases}$$

We find after a straightforward calculation that the field equations (25) give the two differential equations for the functions ν and λ,

$$\nu' + \lambda' = 0$$

$$\lambda' - \frac{1 - e^{\lambda}}{r} = 0$$

which are the equations for the Schwarzschild problem, the primes indicating differentiation with respect to r.

4. THE INTERNAL DEGREES OF FREEDOM FOR THE SOLUTIONS OF THE FIELD EQUATIONS IN SPINOR FORM

The field equations (25) are covariant with respect to the arbitrary group of coordinate transformations of general relativity, but they are also covariant with respect to the local unimodular group SL_2. For the problem presently studied this group was reduced to the group SU_2 by means of the condition (6). Thus, we should expect that several solutions of (25) do exist for any given system of coordinates. These several solutions are then related one to the other by unitary unimodular coordinate dependent matrices in the two-dimensional complex vector space. In this section we present two of such possible solutions. For convenience we work in spherical coordinates. These internal degrees of freedom will be presented here not directly in terms of the elements of SU_2, but in terms of the "rotation" matrices for the associated tetrad vectors. As was said, these "rotation" matrices are directly related to the matrices of SU_2, case where they are really matrices of rotation, that is orthogonal four-by-four matrices.

Another possible solutions of (25) for the Schwarzschild problem are given by

$$\begin{cases} \sigma_0 = e^{\nu/2} \overset{\circ}{\sigma}_0 & (27a) \\ \sigma_1 = e^{\lambda/2} \overset{\circ}{\sigma}_1 = e^{-\nu/2} \overset{\circ}{\sigma}_1 & (27b) \\ \sigma_2 = r \overset{\circ}{\sigma}_2 & (27c) \\ \sigma_3 = r \sin\theta \overset{\circ}{\sigma}_3 & (27d) \end{cases}$$

The tetrad components associated to the solutions given by (26) and (27) respectivelly, are

$$h^{(\alpha)}{}_{\mu} = \begin{pmatrix} e^{\nu/2} & 0 & 0 & 0 \\ 0 & e^{\lambda/2}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{\lambda/2}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{\lambda/2}\cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$k^{(\alpha)}{}_{\mu} = \begin{pmatrix} e^{\nu/2} & 0 & 0 & 0 \\ 0 & e^{\lambda/2} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}$$

These two matrices are connected by means of a pseudo-orthogonal internal matrix,

$$h^{(\alpha)}{}_{\mu} = M^{(\alpha)}{}_{(\beta)}\, k^{(\beta)}{}_{\mu}$$

$$M^{T}\,\eta\, M = \eta$$

Its value being

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

As it is easily seen, the sub-matrix corresponding to the transformation in the spatial indices (i) =((1), (2), (3)) is orthogonal, this sub-matrix corresponds to one element of SU_2 in the complex two-dimensional internal space.

5. GENERALIZATION OF THE SPINOR CLASS OF SOLUTIONS ASSOCIATED TO THE SCHWARZSCHILD PROBLEM

The property that the Hermitian two-by-two matrices σ_μ, which describe the gravitational field in the spinor formulation, form a basys in the internal space of the complex two-by-two matrices allow us to obtain more general spinor class of solutions. A general solution is obtained as a linear combination of the σ_μ which correspond to the Schwarzschild problem. In this linear combination the coefficients contain the new information of the general field. Presently we determine the gravitational field generated by a charged massive particle [8] in terms of the Schwarzschild field, by this process. This method, in principle, may be further extended so as to take into account other physical properties besides the charge. Nevertheless, these further generalizations are not of use presently since we do not know how other coupling constants, besides the charge, are sources of gravitation.

The process outlined in this section may be carried out in any system of coordinates since it has nothing to do with coordinate transformations. However, the use of spherical coordinates will simplify greatly the results.

Given the σ_μ corresponding to the Schwarzschild problem (always understood that this σ_μ is an element of the class $\tilde{\sigma}_\mu$), we form the Σ_μ belonging to the class associated to the Nordstrom field by writing

$$\Sigma_\mu = \alpha^\lambda{}_\mu \, \sigma_\lambda \tag{28}$$

For σ_μ we take the simple form given by (27), where we have,

$$e^\nu = 1 - 2Gm/c^2r = 1 - 2a/r$$

G being the gravitational constant, and m the mass of the source. The matrix $\alpha^\lambda{}_\mu$ which generates according to (28) the solution for the Nordstrom's field is,

$$\alpha = \begin{pmatrix} \left(\dfrac{1-\frac{2a}{r}+\frac{b}{r^2}}{1-\frac{2a}{r}}\right)^{1/2} & 0 & 0 & 0 \\ 0 & \left(\dfrac{1-2a/r}{1-2a/r+b/r^2}\right)^{1/2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since we get a result similar to (27) for Σ_μ, with a new function ν_N given by

$$e^{\nu_N} = 1-2a/r + b/r^2 \ , \quad b = Ge^2/c^4$$

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REFERENCES

1. B. Franck, Nuovo Cimento, 61, A, 263 (1969).
2. M. Sachs, Nuovo Cimento, 47A, 759 (1967).
3. C. G. Oliveira, J. Tiomno, Nuovo Cimento, 24, 672 (1962).
   P. A. M. Dirac, Recent Developments in General Relativity, Pergamon Press, page 191, (1962).
4. J. L. Anderson, Principles of Relativity Physics, Acad. Press (1967).
5. The η in this equation is three-dimensional, that is, η_{ij}.
6. However, it will turn out better to take the $\overset{o}{\sigma}{}^i$ as the Pauli matrices, the $\overset{o}{\sigma}_i$ will then be the same matrices with a minus sign in front.
7. C. Marcio do Amaral, C. G. Oliveira, Nuovo Cimento, 63B, 318 (1969).
8. G. Nordstrom, Akad. Wet. 20, 10 (1918).