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ON THE TOPOLOGY OF THE SPACE OF ALL HOLOMORPHIC FUNCTIONS ON A GIVEN OPEN SUBSET

by

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ON THE TOPOLOGY OF THE SPACE OF ALL HOLOMORPHIC FUNCTIONS ON A GIVEN OPEN SUBSET *

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Let E and F be two complex Banach spaces, $\mathcal{P}(^mE; F)$ the Banach space of all continuous m-homogeneous polynomials from E to F (m = 0, 1, ...), U a non-void open subset of E, $\mathcal{H}(U; F)$ the vector space of all holomorphic F-valued functions on U, and $\hat{d}^m$ the differentiation mapping from $\mathcal{H}(U; \mathcal{F})$ into $\mathcal{H}(U; \mathcal{P}(^mE, F))$ of order m = 0, 1, ...

A seminorm p on $\mathcal{H}(U; F)$ is said to be ported by a compact subset K of U if to every open subset V of U containing K there corresponds a real number $c(V) > 0$ such that

$$p(f) \leqslant c(V).\sup_{x \in V} \| f(x) \|$$

for every $f \in \mathcal{H}(U;F)$. The topology $\mathcal{T}_\omega$ on $\mathcal{H}(U;F)$ is defined by all such seminorms ported by compact subsets of U. Each of the

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following conditions is necessary and sufficient for p to be ported by K:

(1) Corresponding to every real number $\varepsilon > 0$ there is a real number $c(\varepsilon) > 0$ such that, for every $f \in \mathcal{H}(U; F)$,

$$p(f) \leqslant c(\varepsilon) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\| .$$

(2) Corresponding to every real number $\varepsilon > 0$ and open subset V of U containing K, there is a real number $c(\varepsilon, V) > 0$ such that, for every $f \in \mathcal{H}(U; F)$,

$$p(f) \leqslant c(\varepsilon, V) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\| .$$

U is ξ-equilibrated, where $\xi \in U$, if $(1-\lambda)\xi + \lambda x \in U$ for any $x \in U$, $\lambda \in C, |\lambda| \leqslant 1$. If U is ξ-equilibrated, the Taylor series at ξ of any $f \in \mathcal{H}(U; F)$ converges to f in the sense of $\mathcal{T}_\omega$. If U is ξ-equilibrated, the following condition is necessary and sufficient for p to be ported by K: corresponding to every open subset V of U containing K, there is a real number $c(V) > 0$ such that, for every $f \in \mathcal{H}(U;F)$,

$$p(f) \leqslant c(V) \, . \sum_{m=0}^{\infty} \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(\xi).(x-\xi) \right\| .$$

The compact-open topology on the vector space $\mathcal{C}(U; F)$ of all continuous F-valued functions on U induces a topology $\mathcal{T}_0$ on $\mathcal{H}(U; F)$. We have $\mathcal{T}_0 \subset \mathcal{T}_\omega$; $\mathcal{T}_0 = \mathcal{T}_\omega$ if and only if $\dim E < \infty$, or $F = 0$. Each $\hat{d}^m$ is continuous for the corresponding topologies $\mathcal{T}_\omega$; continuity of $\hat{d}^m$ for some $m \geq 1$ and the corresponding topologies $\mathcal{T}_0$ requires $\dim E < \infty$, or $F = 0$. However a subset $\mathfrak{X}$ of $\mathcal{H}(U; F)$ is bounded for $\mathcal{T}_\omega$ if and only if it is bounded for $\mathcal{T}_0$. Each of the following conditions is necessary and sufficient for $\mathfrak{X}$ to be bounded for $\mathcal{T}_\omega$:

(1) Corresponding to every compact subset K of U, there is a real number $C > 0$ such that $\| f(x) \| \leq C$ for every $f \in \mathfrak{X}$ and $x \in K$.

(2) Corresponding to every compact subset K of U, there are a real number $C > 0$ and an open subset V of U containing K such that $\|f(x)\| \leq C$ for every $f \in \mathfrak{X}$ and $x \in V$.

(1') Corresponding to every $\xi \in U$, there are real numbers $C > 0$ and $c > 0$ such that, for every $m = 0, 1, \ldots$ and $f \in \mathfrak{X}$,

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\| \leq C \cdot c^m .$$

(2') Corresponding to every compact subset K of U, there are real numbers $C > 0$ and $c > 0$ such that, for every $m = 0, 1, \ldots, f \in \mathfrak{X}$ and $x \in K$,

$$\left\| \frac{1}{m!} \hat{d}^m f(x) \right\| \leq C . c^m .$$

(3') Corresponding to every compact subset K of U, there are real numbers $C > 0$ and $c > 0$, and an open subset V of U containing K, such that, for every $m = 0, 1, \ldots, f \in \mathfrak{X}$ and $x \in V$,

$$\left\| \frac{1}{m!} \hat{d}^m f(x) \right\| \leq C . c^m .$$

Let $X \subset U$ be fixed, and suppose X meets every connected component of U. Then $\mathfrak{X}$ is bounded for $\mathcal{T}_\omega$ if and only if $\mathfrak{X}$ is equicontinuous on U and $\sup\{\|f(x)\| \mid f \in \mathfrak{X}\} < \infty$ for every $x \in X$. Denote by $\mathcal{T}_{\infty,X}$ the topology on $\mathcal{H}(U;F)$ defined by the family of seminorms $f \to \|\hat{d}^m f(x)\|$ for $m = 0, 1, \ldots$ and $x \in X$. If $\mathfrak{X}$ is $\mathcal{T}_\omega$-bounded, $\mathcal{T}_\omega$ and $\mathcal{T}_{\infty,X}$ induce the same topology on $\mathfrak{X}$; also the uniform structures associated with $\mathcal{T}_\omega$ and $\mathcal{T}_{\infty,X}$ induce the same uniform structure on $\mathfrak{X}$. If f, f_l $\mathcal{H}(U;F)$ for $l = 0, 1, \ldots$, then $f_l \to f$ in the sense of $\mathcal{T}_\omega$ as $l \to \infty$ if and only if $\{f_l\}$ is $\mathcal{T}_\omega$-bounded and $\hat{d}^m f_l(x) \to \hat{d}^m f(x)$ in $\mathcal{P}(^mE; F)$ as $l \to \infty$ for every $m = 0, 1, \ldots$ and $x \in X$. Also $\mathfrak{X}$ is $\mathcal{T}_\omega$-relatively compact if and only if $\mathfrak{X}$ is $\mathcal{T}_\omega$-bounded and $\{\hat{d}^m f(x) \mid f \in \mathfrak{X}\}$ is relatively compact in $\mathcal{P}(^mE; F)$ for every $m = 0, 1, \ldots$ and $x \in X$.

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