NOTAS DE FÍSICA

VOLUME II

Nº 12

THE FIRST EXCITED STATES OF THE C^{13} - N^{13} MIRROR PAIR

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1956

Notas de Física Vol.II- Nº 12

THE FIRST EXCITED STATES OF THE C^{13} – N^{13} MIRROR PAIR

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Centro Brasileiro de Pesquisas Físicas

Rio de Janeiro, D. F.

(December 15, 1955)

1. INTRODUCTION

The difference between ground states of mirror nuclei can be explained as due to the Coulomb forces between the protons apart from the neutron-hydrogen mass difference. This can be understood with the assumption that the force between a pair of protons is the same as that between a pair of neutrons. It is natural to assume that there will be an analogous correspondence between the excited states.

The levels of C^{13} – N^{13} have been treated by several authors using reaction theory (see Ehrman[1], Peaslee[2], Thomas[3, 4]). It seemed worthwhile to try some simple model following the line of Bethe[5] for the ground states, to see if reasonable results could be obtained.

We assumed a core with a spherical charge distribution

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of radius $R = 1.18 (A - 1)^{1/3} \times 10^{-13}$ cm; A - 1 being the mass number of the core, and the extra particle moving in the nuclear potential of the core and having the total spin of the nucleus. The difference between the energy levels of the two mirror nuclei will be

$$\Delta E_{st} = \int \rho V dv$$

where ρ is the charge distribution of the extra particle and V the electrostatic potential of the sphere. In order to calculate ρ we assumed first that the nuclear field was due to a square well. The parameters of the well being chosen so that we got reasonable values of the binding energies for the first two excited states, ^2S and ^2P.

In a second calculation we assumed that the nuclear potential was that of a harmonic oscillator. We fixed the parameter of the oscillator in order to fit the correct Coulomb energy difference between the two ground states.

Finally, we tried a model inspired by the α-particle model, as well as a slight modification of the second model, assuming that the charge of the core, instead of being a sphere, was distributed according to a Gaussian law.

In figure 1 the experimentally determined low lying levels of the mirror pair are shown with spin and parity assignments (from Ajzenberg[6]).

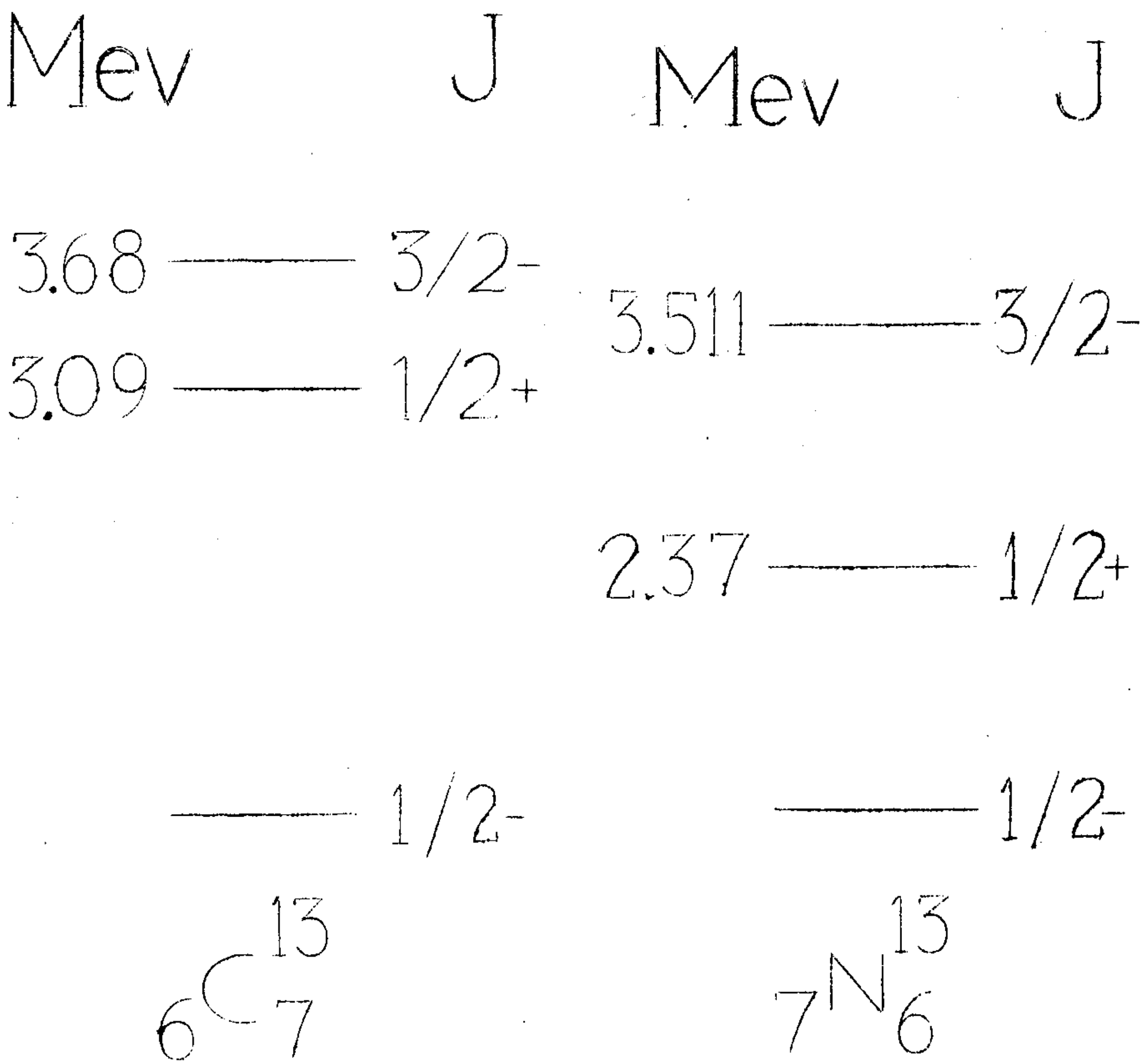

Figure 1

The two lowest excited states of the C^{13} - N^{13} mirror pair, from Ajezenberg [6].

II. THE SQUARE WELL

The wave functions are:

$$\Psi_{n,\ell} = A_{n,\ell}\, j_\ell\,(ar)\, Y_\ell^m(\theta,\varphi) \qquad r < R$$

$$= A_{n,\ell}\, \frac{j_1(aR)}{k_1(bR)}\, k_\ell\,(br)\, Y_\ell^m\,(\theta,\varphi) \qquad r > R$$

with

$$a^2 = 2M(D - |E_1|)/\hbar^2$$

$$b^2 = 2M\,|E_1|\,/\,\hbar^2$$

M the nucleon mass, D the well depth, $j_\ell\,(x)$ is the usual spherical Bessel function of order ℓ, while $k_\ell\,(x)$ is a spherical Bessel function of purely imaginary argument or

$$k_\ell\,(x) = \sqrt{2/\pi x}\;\; K_{\ell+1/2}(x)$$

where $K_n(x)$ is defined by Watson[7]. Then the Coulomb energy of the extra particle is

$$\Delta E_\ell = \int \rho_\ell V dv = (Z-1)\; e^2\, |A_\ell|^2 \left\{ \frac{1}{2R^3} \int_0^R (3R^2 - r^2)\, j_\ell^2\,(ar)\, r^2 dr + \left[j_\ell^2\,(aR)/k_1^2(bR) \right] \int_R^\infty r b_\ell^2(br)\, dr \right\} =$$

$$= (Z - 1)\, e^2\, |A_\ell|^2 \left\{ \frac{3R^2}{4} \left[j_\ell^2(aR) - j_{\ell-1}(aR)\, j_{\ell+1}(ar) \right] - \right.$$

$$- (1/2R^3a^5) \int_0^{aR} j_\ell^2(z)\, z^4\, dz +$$

$$\left. + \left[j_\ell^2(aR)/b^2 k_\ell^2(bR) \right] \int_{bR}^{\infty} k_\ell^2(z)\, z\, dz \right\} .$$

The result for $\ell = 1$ (the ground and second excited states of the A = 13 pair) is

$$\Delta E_1 = (Z - 1)\, e^2 |A_1|^2 \left\{ \frac{3R^2}{4} \left[j_1^2(aR) - j_0(aR)\, j_2(aR) \right] - \right.$$

$$- (1/2R^3a^5) \left[\frac{3aR}{4} \cos 2aR + \left(\frac{(aR)^2}{4} - 5/8 \right) \sin 2aR + \right.$$

$$\left. + (aR)^3/6 + aR/2 \right] + \left[j^2(aR)/b^2 k^2(bR) \right] - \left[(2bR + 1) \times \right.$$

$$\left. \left. \times\ e^{-bR}/2(bR)^2 - \int_{bR}^{\infty} (e^{-2z}/z)\, dz \right] \right\}$$

where

$$|A_1|^2 = -2b^2/(a^2 + b^2)\ R^3\ j_0(aR)\, j_2(aR).$$

The first excited state is an S state, yielding

$$\Delta E_o = (Z-1)\, e^2 |A_o|^2 \left\{ \frac{3}{4a^2} - (3/8a^3R)\sin 2aR - \right.$$

$$-(1/2R^3a^5)\left[(aR)^3/6 - \left(\frac{(aR)^2}{4} - 1/8\right)\sin 2aR - \right.$$

$$\left. \left. -(aR/4)\cos 2aR \right] + \left[j_o^2(aR)/b^2k_o^2(bR) \right] \int_{bR}^{\infty} (e^{-2z}/z)\,dz \right\}$$

with

$$|A_o|^2 = 2ab^2/(a^2+b^2)R^2 \left[aRj_1^2(aR) - j_o(aR)j_1(aR) \right] .$$

To obtain the well constants it was first observed that for a well radius of the form given by Peaslee[2]; i.e.,

$R = 1.18\, A_e^{1/3} \times 10^{-13}$ cm, $A_e = A - 1$, no ^2S state was bound for $D < 63.2$ Mev, at which depth, $|E_1^1| \approx 22$ Mev. The values of D and R were finally picked such that $|E_o^2| = 5$ Mev, $|E_1^1| = 8$ Mev then $R = 10^{-12}$ cm $= 4.36\, A_e^{1/3}$ with $D = 11.33$ Mev giving $|E_o^1| = 9.73$ Mev $|E_1^2| = 2.21$ Mev and no $\ell = 2$ state is bound. Using These well parameters

$\Delta E_1^1 = 1.091$ Mev

$\Delta E_o^2 = 1.048$ Mev $\qquad \Delta E_1^1 - \Delta E_o^2 = 43$ Kev

$\Delta E_1^2 = 1.036$ Mev $\qquad \Delta E_1^1 - \Delta E_1^2 = 55$ Kev.

The experimental difference to be explained is 720 Kev.

III. THE OSCILLATOR POTENTIAL

The wave functions in this case are

$$\Psi(r, \theta, \varphi) = \frac{1}{r} R_{n\ell}(r) \; Y_\ell^m(\theta, \varphi)$$

$$R_{n,\ell} = N_{n,\ell} r^{\ell+1} v_{n,\ell}(r) \exp(-\alpha^2 r^2/2).$$

For n = 0

$$v_{o\ell} = 1, \qquad N_\ell^2 = \left(\alpha^{2\ell+3} \; 2^{\ell+2}\right) / \left[\sqrt{\pi}\; 1.3\ldots(2\ell+1)\right],$$

n = 1

$$v_{1,\ell} = 1 - 2\alpha r^2/(2\alpha + 3)$$

$$N_\ell^2 = \alpha^{2\ell+3} \; 2^{\ell+1} (2\ell+3) / \sqrt{\pi}\; 1.3\ldots(2\ell+1).$$

We need integrals of the type

$$I_{n,\ell} = \int R_{n,\ell}^2 \, V(r) \, dr$$

for which there is a recurrence relation (Talmi[8])

$$I_{1,\ell} = \frac{2\ell+3}{2} I_{0,\ell} - (2\ell+3) I_{0,\ell+1} + \frac{2\ell+5}{2} I_{0,\ell+2}.$$

We have in particular

$$\Delta E(^1P) = I_{o,1}$$

$$\Delta E(^2S) = I_{1,0} = \frac{3}{2} I_{0,0} - 3 I_{0,1} + \frac{5}{2} I_{0,2}$$

$$\Delta E(^2P) = I_{1,1} = \frac{5}{2} I_{0,1} - 5 I_{0,2} + \frac{7}{2} I_{0,3}.$$

For which we obtained

$$I_{0,0} = \frac{(Z-1)e^2}{R} \left\{ \operatorname{erf}(\alpha R) \left[\frac{3}{2} - \frac{3}{4\alpha^2 R^2} \right] + \frac{e^{-\alpha^2 R^2}}{\sqrt{\pi}} \cdot \frac{3}{2\alpha R} \right\}$$

$$I_{0,1} = \frac{(Z-1)e^2}{R} \left\{ \operatorname{erf}(\alpha R) \left[\frac{3}{2} - \frac{5}{4\alpha^2 R^2} \right] + \frac{e^{-\alpha^2 R^2}}{\sqrt{\pi}} \cdot \frac{5}{2\alpha R} \right\}$$

$$I_{0,2} = \frac{(Z-1)e^2}{R} \left\{ \operatorname{erf}(\alpha R) \left[\frac{3}{2} - \frac{7}{4\alpha^2 R^2} \right] + \frac{e^{-\alpha^2 R^2}}{\sqrt{\pi}} \left[\frac{7}{2\alpha R} + \right. \right.$$

$$\left. \left. + \frac{2}{5} \alpha R \right] \right\}$$

$$I_{0,3} = \frac{(Z-1)e^2}{R} \left\{ \operatorname{erf}(\alpha R) \left[\frac{3}{2} - \frac{9}{4\alpha^2 R^2} \right] + \right.$$

$$\left. + \frac{e^{-\alpha^2 R^2}}{\sqrt{\pi}} \left[\frac{9}{2\alpha R} + \frac{32}{35} \alpha R + \frac{12}{105} (\alpha R)^3 \right] \right\}$$

and finally,

$$\Delta E(^1P) = \frac{(Z-1)e^2}{R} \left\{ \left[\frac{3}{2} - \frac{5}{4\alpha^2 R^2} \right] \operatorname{erf}(\alpha R) + 5e^{-\alpha^2 R^2} / 2\sqrt{\pi}\, \alpha R \right\}$$

$$\Delta E(^2S) = \frac{(Z-1)e^2}{R} \left\{ \left[\frac{3}{2} - \frac{7}{4\alpha^2 R^2} \right] \operatorname{erf}(\alpha R) + \frac{e^{-\alpha^2 R^2}}{\sqrt{\pi}} \left(\alpha R + \frac{7}{2\alpha R} \right) \right\}$$

$$\Delta E(^2P) = \frac{(Z-1)e^2}{R} \left\{ \left[\frac{3}{2} - \frac{9}{4\alpha^2R^2} \right] \mathrm{erf}(\alpha R) + \frac{e^{-\alpha^2R^2}}{\sqrt{\pi}} \left[\frac{2}{5} (\alpha R)^3 + \right.\right.$$

$$\left.\left. + \frac{6}{5} \alpha R + \frac{9}{2\alpha R} \right] \right\} .$$

By setting $\Delta E(^1P) = 3.00$ Mev, one obtains $\alpha R = 1.35$, $R = 1.18\, A_e^{1/3} \times 10^{-13}$ cm, then

$$\Delta E(^2S) = 2.70 \text{ Mev}$$
$$\Delta E(^2P) = 2.60 \text{ Mev}$$

and

$$\Delta E(^1P) - \Delta E(^2S) = 300 \text{ Kev}$$
$$\Delta E(^1P) - \Delta E(^2P) = 400 \text{ Kev}$$

and the spacing between the unperturbed levels will be

$$E(^2S) - E(\ P\ ) = 10.3 \text{ Mev.}$$

Finally, we repeated the calculation assuming that the charge is uniformly distributed in a shell, in order to have less charge in the center of the sphere. The results gave

$$\Delta E(^1P) = \frac{(Z-1)e^2}{R} \left[\mathrm{erf}(\alpha R) - 2\alpha R e^{-\alpha^2R^2} / 3\sqrt{\pi} \right]$$

$$\Delta E(^2S) = \frac{(Z-1)e^2}{R} \left[\mathrm{erf}(\alpha R) - e^{-\alpha^2R^2} \left(2(\alpha R)^3 + \alpha R \right) / 3\sqrt{\pi} \right]$$

$$\Delta E(^2P) = \frac{(Z-1)e^2}{R} \left[\mathrm{erf}(\alpha R) - 2e^{-\alpha^2R^2} \left(2(\alpha R)^5 + (\alpha R)^3 + \right.\right.$$

$$\left.\left. + 6\alpha R \right) / 15\sqrt{\pi} \right] .$$

The results can be expressed more simply if we assume that the distribution of the charge in the core is Gaussian. The potential of such a distribution is

$$V(r) = \frac{(Z-1)e}{R} \operatorname{erf}(\alpha R).$$

We must now calculate the integrals $I_{0,0}, I_{0,1}, I_{0,2}, I_{0,3}$ using this potential. We need the following integrals, whose calculation is explained in the appendix

$$\int_0^\infty \operatorname{erf}(\alpha R)\, e^{-\alpha^2 R^2}\, r^n\, dr \qquad \text{for } n = 1, 3, 5, 7.$$

We got

$$I_{0,0} = 4\ K$$

$$I_{0,1} = 10K/3$$

$$I_{0,2} = 43K/15$$

$$I_{0,3} = 177K/70$$

where

$$K = (Z-1)e^2\ \alpha/2\sqrt{2\pi}$$

Using the recurrence relations the final result was

$$\Delta E(^1P) = 5.72\alpha 10^{-13} \text{ Mev}$$

$$\Delta E(^2S) = 5.50\alpha 10^{-13} \text{ Mev}$$

$$\Delta E(^2P) = 4.82\alpha 10^{-13} \text{ Mev}$$

Again adjusting α so that $\Delta E(^1P) = 3.00$ Mev we got

$$\Delta E(^2S) = 2.88 \text{ Mev}$$

$$\Delta E(^2P) = 2.53 \text{ Mev.}$$

IV. CONCLUSION

For the first excited state, the difference to be explained is 720 Kev. We always got the displacement with the correct sign, but for the square well potential the results are too far from the experimental values. Also the values of the well parameters are much too unrealistic. With regard to the oscillator, the results are much better.

For the second excited states we have the following situation, calling

$$\delta_1 = \Delta E(^1P) - \Delta E(^2S)$$
$$\delta_2 = \Delta E(^1P) - \Delta E(^2P)$$

the experimental results show that $\delta_1 > \delta_2$ while we get $\delta_1 < \delta_2$ for all models considered. This result no doubt flows from the fact that the first excited state in N^{13} is unstable against the emission of a proton, implying that it is improper to consider the charge effect as a small perturbation (cf. Blatt[9]).

APPENDIX

Let

$$\int_0^{\infty} \operatorname{erf}(A\,r)e^{-Br^2}\, r\, dr = \frac{A}{2B} \cdot \frac{1}{\sqrt{A^2+B}}$$

By successive derivation with respect to B we get

$$\int_0^{\infty} \mathrm{erf}(Ar)\, e^{-Br^2}\, r^3\, dr = \frac{A}{2B^2}\, \frac{1}{\sqrt{A^2+B}} + \frac{1}{2}\, \frac{A}{2B}\, \frac{1}{(A^2+B)^{3/2}}$$

$$\int_0^{\infty} \mathrm{erf}(Ar)\, e^{-Br^2}\, r^5\, dr = \frac{1}{2} A \left[\frac{2}{B^3\sqrt{A^2+B}} + \frac{1}{B^2(A^2+B)^{3/2}} + \frac{3}{4B(A^2+B)^{5/2}} \right]$$

$$\int_0^{\infty} \mathrm{erf}(Ar) e^{-Br^2}\, r^7\, dr = \frac{1}{2} A \left[\frac{6}{B^4 \sqrt{A^2+B}} + \frac{3}{B^3(A^2+B)^{3/2}} + \frac{9}{4B^2(A^2+B)^{5/2}} + \frac{15}{8B(A^2+B)^{7/2}} \right].$$

Finally, by making $A = \sqrt{B} = \alpha$ we get the desired integrals.

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I N D I C E

N O T A S   D E   F Í S I C A

V O L. I I   (1954-1956)