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Supersymmetry and Integrability in Planar Mechanical Systems

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Abstract

We present an $N = 2$-supersymmetric mechanical system whose bosonic sector, with two degrees of freedom, stems from the reduction of an $SU(2)$ Yang-Mills theory with the assumption of spatially homogeneous field configurations and a particular ansatz imposed on the gauge potentials in the dimensional reduction procedure. The Painlevé test is adopted to discuss integrability and we focus on the rôle of supersymmetry and parity invariance in two space dimensions for the attainment of integrable or chaotic models. Our conclusion is that the relationships among the parameters imposed by supersymmetry seem to drastically reduce the number of possibilities for integrable interaction potentials of the mechanical system under consideration.

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1 Introduction

The study of integrability in classical and quantum field theories has been developed for quite a time, actually since the beginning of the eighties, with relevant results that contributed a great deal for the understanding of these theories and, moreover, allowed the improvement of non-perturbative techniques[1]–[2]. On the other hand, a number of streams of investigation on chaos has been pushed forward, mainly considering spatially homogeneous field solutions and by performing calculations in the framework of lattice field theory[3]–[11]. These studies revealed the existence of chaotic solutions in a considerably vast class of gauge theories and, more recently, also in the context of superstrings and supermembrane theories[12]–[15].

Up to now, a detailed analysis relating supersymmetry and chaos, in much the same way as chaos is studied in field theories, is lacking in the literature. Close to this issue, we should mention a number of attempts to discuss stability and chaos in the framework of brane theories, by concentrating on their bosonic sector[14]–[15]. Nevertheless, even in this context, one should put more emphasis on the specific rôle of supersymmetry in the determination of stability and chaos.

A similar situation is observed in connection with the investigation of integrable supersymmetric theories, where the integrable or non-integrable character is ascertained, without however highlighting the mechanisms or those specific properties of supersymmetry which work in favour, or against, integrability[16]–[20].

Our work sets out to tackle this issue, that we believe should be more manifestly worked out. To pursue an investigation focusing on the rôle of supersymmetry in connection with integrability and chaos, we propose to start off from a supersymmetric mechanical system, rather than a field-theoretic model. The system we choose to work with is built up as the $N = 2$-extended supersymmetric version of a dimensionally reduced $SU(2)$ Yang-Mills theory that arises when spatially homogeneous fields are considered and a particular ansatz on the gauge potentials is adopted in the dimensional reduction scheme so that only two degrees of freedom survive[21] in the mechanical limit. We also devote special attention to the rôle of parity symmetry, since we assume the latter is an invariance of the interactions involved in the systems we shall be considering. Our analysis of integrability shall therefore rely on our considerations on supersymmetry and parity invariance. They dictate special conditions in the space of parameters so that, instead of having to take by decree special choices of these parameters, as it is usually done, we invoke these two invariances to naturally restrict and select possibilities in parameter space. As a matter of fact, we anticipate that parity may appear in two versions for planar systems, and this point shall be suitably taken care of here.

Our paper is organised as follows. In Section 2, we propose a general 2-dimensional purely bosonic model with parity symmetry and we identify the cases of integrability. Next, the $N = 2$-supersymmetric extension of the model is written down in Section 3. The complete bosonic sector, now enlarged by the presence of two supersymmetries, is discussed in full details in Section 4, where we pay due attention to the rôle of parity and we pick out Painlevé test as a criterium to infer about integrability. In Section 5, we reassess the question of the integrability for the bosonic sector of our $N = 2$-model, but now taking into account the constraints dictated by parity whenever it is imposed also to the fermionic interactions. A very restrictive class of potentials comes out that fulfills integrability. Finally, in Section 6, we present our Final Discussions and we draw our General Conclusions.

2 The ordinary bosonic model with considerations on parity symmetry

We assume the most general fourth-order polynomial potential for two degrees of freedom described by the variables x and y:

$$V = C_1x^4+C_2y^4+C_3x^3y+C_4xy^3+C_5x^2y^2+C_6x^3+C_7y^3+C_8x^2y+C_9xy^2+C_{10}x^2+C_{11}y^2+C_{12}xy. \quad (1)$$

It may be considered as a sort of protopotential used to build up a general non-supersymmetric polynomial potential up to fourth order. We are bound to fourth order because we have in mind mechanical models derived from Yang-Mills theories and these, as we know, display self-interaction vertices for three and four potentials. Since we are interested in realistic models, we impose parity symmetry which is respected by mechanical and electromagnetic models. We shall not be dealing with models coming from chiral gauge theories.

To implement parity in the model, we have to consider that there are two possibilities, since we are in a 2-dimensional space:

$$x\text{-parity}: \begin{array}{c} x \to -x \\ y \to y \end{array} \tag{2}$$

or

$$y\text{-parity}: \begin{array}{c} x \to x \\ y \to -y \end{array} \tag{3}$$

In the first case, the resulting potential is

$$V = C_1 x^4 + C_2 y^4 + C_5 x^2 y^2 + C_7 y^3 + C_8 x^2 y + C_{10} x^2 + C_{11} y^2. \tag{4}$$

This potential looks like the sum of two well-known potentials:
a quartic potential (Yang-Mills-type)

$$V_{YM} = Ax^2 + By^2 + ax^4 + by^4 + dx^2 y^2, \tag{5}$$

which is known to be integrable in the following cases[22]:

a) $A = B, \quad a = b, \quad d = 6a.$	*that in our case is* $C_{10} = C_{11}, C_1 = C_2, C_5 = 6C_1.$
b) $A, \quad B, \quad a = b, \quad d = 2a.$	*that in our case is* $C_{10}, C_{11}, C_1 = C_2, C_5 = 2C_1.$
c) $A = 4B, \quad a = 16b, \quad d = 12a.$	*that in our case is* $C_{10} = 4C_{11}, C_1 = 16C_2, C_5 = 12C_1$
d) $A = 4B, \quad a = 8b, \quad d = 6b.$	*that in our case is* $C_{10} = C_{11}, C_1 = C_2, C_5 = 6C_1.$
e) $d = 0$ *(trivial)*	*that in our case is* $C_5 = 0.$

and the Henon-Heiles potential:

$$V_{HH} = \frac{1}{2}\left(Ax^2 + By^2\right) + ax^2 y - \frac{1}{3} by^3, \tag{6}$$

that exhibits well-known integrable cases[22]:

a) $A = B, \quad a = -b.$	*that in our case is* $C_{10} = C_{11}, \quad C_7 = \frac{1}{3} C_8.$
b) $A, \quad B, \quad 6a = -b.$	*that in our case is* $C_{10}, \; C_{11}, \quad C_7 = 2C_8.$
c) $16A, \quad B, \quad 16a = -b.$	*that in our case is* $C_{10} = 16C_{11}, \quad C_7 = \frac{16}{3} C_8.$
d) $a = 0$ *(trivial)*	*that in our case is* $C_8 = 0$

3 The supersymmetric model

Now, we shall consider an $N = 2$ supersymmetric mechanical model[23], defined as follows. The two (complex) Grassmannian parameters of the superspace will be denoted by θ and $\overline{\theta}$. The two real coordinates of a planar particle, x and y, are the bosonic components of the superfields coordinates, which are given by

$$X(t, \theta, \overline{\theta}) = x(t) + \Theta^\dagger \gamma_1 \Lambda(t) + \Lambda^\dagger(t) \gamma_1 \Theta - \frac{1}{2} \Theta^\dagger \gamma_3 \Theta f_1(t), \tag{7}$$

and

$$Y(t, \theta, \overline{\theta}) = y(t) + \Theta^\dagger \gamma_2 \Xi(t) + \Xi^\dagger(t) \gamma_2 \Theta - \frac{1}{2} \Theta^\dagger \gamma_3 \Theta f_2(t), \tag{8}$$

with:

$$\Theta \equiv \begin{pmatrix} \theta \\ \overline{\theta} \end{pmatrix}, \quad \Lambda \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \Xi \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{9}$$

where all the λ's and ξ's are Grassmannian variables. The γ_j's are the Dirac matrices corresponding to the two-dimensional Euclidean space under consideration and they may be chosen so as to coincide with the Pauli matrices: $\gamma_i \equiv \sigma_i$ and $\gamma_3 \equiv -i\gamma_1\gamma_2 = \sigma_3$. Θ is Majorana spinor, which, in this particular representation of the γ-matrices, takes the form given in (9), where the "bar" stands for complex conjugation. On the other hand, Λ and Ξ are Dirac fermions. Therefore, Eqs. (7–8) yield:

$$X = x + \theta\left(\lambda_1 - \overline{\lambda}_2\right) - \overline{\theta}\left(\overline{\lambda}_1 - \lambda_2\right) + \theta\overline{\theta} f_1 \tag{10}$$

and

$$Y = y + i\theta\left(\xi_1 - \overline{\xi}_2\right) + i\overline{\theta}\left(\overline{\xi}_1 - \xi_2\right) + \theta\overline{\theta} f_2. \tag{11}$$

It is noteworthy to remark that it is precisely the combination $\left(\lambda_1 - \overline{\lambda}_2\right)$ the one that carries the fermionic degrees of freedom of X. On the other hand, as for Y, its spinorial degress of freedom are all located in $\left(\xi_1 - \overline{\xi}_2\right)$.

The supersymmetry covariant derivatives are as below:

$$D \equiv \partial_{\overline{\theta}} - i\theta\partial_t \tag{12}$$

$$\overline{D} \equiv \partial_\theta - i\overline{\theta}\partial_t, \tag{13}$$

they satisfy:

$$D^2 = 0 \tag{14}$$

$$\overline{D}^2 = 0 \tag{15}$$

$$\left\{D, \overline{D}\right\} = -2i\partial_t. \tag{16}$$

The super-action to be considered contains, besides the kinetic terms, the most general superpotential, up to third order in the superfield coordinates (this implies a fourth-order potential in terms of the physical coordinates)

$$S = \int dt d\theta d\overline{\theta}\, \left\{\frac{M}{2}\left[DX\overline{D}X + DY\overline{D}Y\right] + U(X,Y)\right\}, \tag{17}$$

where the first term gives rise to the kinetic terms and the superpotential $U(X,Y)$ is assumed to be given by:

$$U(X,Y) = k_1X^2Y + k_2XY^2 + k_3X^2 + k_4Y^2 + k_5XY + k_1'X^3 + k_2'Y^3, \tag{18}$$

the k's being arbitrary real constants. Since the term in XY may be canceled out by means of a proper linear transformation (a rotation in the X-Y plane), then the constant k_5 may be set as zero, $k_5 = 0$, without loss of generality. Similarly, the terms linear in X or in Y were not considered, since they may be eliminated by a translation redefinition, $X' = X + \text{const}$ and $Y' = Y + \text{const}'$. The equations of motion may be used to eliminate the non-dynamical degrees, of freedom f_j, and, thus, the super-action,

$S = \int dtL$, yields the following Lagrangian where quartic terms in the potential are present:

$$\begin{aligned}
L = & \frac{M\dot{\vec{x}}^2}{2} + iM\left(\overline{\lambda}_j\dot{\lambda}_j + \overline{\xi}_j\dot{\xi}_j - \overline{\lambda}_1\dot{\overline{\lambda}}_2 - \lambda_2\dot{\lambda}_1 - \overline{\xi}_1\dot{\overline{\xi}}_2 - \xi_2\dot{\xi}_1\right) - \frac{k_1^2 + 9{k'_1}^2}{2M}x^4 - \frac{k_2^2 + 9{k'_2}^2}{2M}y^4 + \\
& - \frac{6k_1k'_1 + 2k_1k_2}{M}x^3y - \frac{6k_2k'_2 + 2k_1k_2}{M}xy^3 + \\
& - \frac{2k_1^2 + 2k_2^2 + 3k'_1k_2 + 3k_1k'_2}{M}x^2y^2 - \frac{6k_3k'_1}{M}x^3 - \frac{6k_4k'_2}{M}y^3 + \\
& - \frac{4k_1k_3 + 2k_1k_4}{M}x^2y - \frac{4k_2k_4 + 2k_2k_3}{M}xy^2 - \frac{2k_3^2}{M}x^2 - \frac{2k_4^2}{M}y^2 + \\
& + \left[2ik_1\left(\lambda_1\overline{\xi}_1 - \lambda_1\xi_2 - \overline{\lambda}_2\overline{\xi}_1 + \overline{\lambda}_2\xi_2 + \overline{\lambda}_1\xi_1 - \overline{\lambda}_1\overline{\xi}_2 - \lambda_2\xi_1 + \lambda_2\overline{\xi}_2\right) + \right. \\
& \quad \left. - 2k_2\left(\xi_1\overline{\xi}_1 - \xi_1\xi_2 - \overline{\xi}_2\overline{\xi}_1 + \overline{\xi}_2\xi_2\right) - 6k'_1\left(\lambda_1\overline{\lambda}_1 - \lambda_1\lambda_2 - \overline{\lambda}_2\overline{\lambda}_1 + \overline{\lambda}_2\lambda_2\right)\right] x \\
& + \left[2ik_2\left(\overline{\lambda}_1\xi_1 - \overline{\lambda}_1\overline{\xi}_2 - \lambda_2\xi_1 + \lambda_2\overline{\xi}_2 + \lambda_1\overline{\xi}_1 - \lambda_1\xi_2 - \overline{\lambda}_2\overline{\xi}_1 + \overline{\lambda}_2\xi_2\right) + \right. \\
& \quad \left. - 2k_1\left(\lambda_1\overline{\lambda}_1 - \lambda_1\lambda_2 - \overline{\lambda}_2\overline{\lambda}_1 + \overline{\lambda}_2\lambda_2\right) - 6k'_2\left(\xi_1\overline{\xi}_1 - \xi_1\xi_2 - \overline{\xi}_2\overline{\xi}_1 + \overline{\xi}_2\xi_2\right)\right] y \\
& - 2k_3\left(\lambda_1\overline{\lambda}_1 - \lambda_1\lambda_2 - \overline{\lambda}_2\overline{\lambda}_1 + \overline{\lambda}_2\lambda_2\right) - 2k_4\left(\xi_1\overline{\xi}_1 - \xi_1\xi_2 - \overline{\xi}_2\overline{\xi}_1 + \overline{\xi}_2\xi_2\right).
\end{aligned} \tag{19}$$

In the next sections, the integrability conditions for this Lagrangian will be discussed, and the influence of supersymmetry and parity invariance shall be highlighted.

4 The bosonic sector and its integrability.

The direct application of the the Painlevé test (for a short review, see Appendix A) directly to the bosonic sector is not actually a good procedure, for the resolution of the systems that appear in the analysis becomes very complex.

In this section, we shall take into consideration the observation that the original model is not invariant under the two classes of parity transformations. This may set a more formal framework.

So, in a first attempt, we will impose parity symmetry, that is a discrete symmetry, only to the bosonic sector of the theory and after that we shall check how the constraints imposed by this invariance affects the integrability of the model.

Adopting invariance under x-parity, we have the following constraints on the coefficients of the potential:

$$C_3 = (6k_1k'_1 + 2k_1k_2) = 0, \tag{20}$$

$$C_4 = (6k_2k'_2 + 2k_1k_2) = 0, \tag{21}$$

$$C_6 = (6k_3k'_1) = 0, \tag{22}$$

$$C_9 = (4k_2k_4 + 2k_2k_3) = 0, \tag{23}$$

where the C's above are the coefficients of the bosonic sector of the original potential for which the parity symmetry is broken.

4.1 Parameters surviving the parity constraints

Solving the system of conditions for $k_1, k_2, k_1', k_2', k_3$ *e* k_4, we obtain as solution the following possibilities:

$$\begin{aligned}
&\{k_1' = k_1', k_2' = k_2', k_4 = k_4, k_3 = 0, k_2 = 0, k_1 = 0\}, \qquad (24)\\
&\{k_1 = k_1, k_2' = k_2', k_4 = k_4, k_3 = k_3, k_1' = 0, k_2 = 0\},\\
&\{k_2' = 0, k_1' = k_1', k_2 = k_2, k_3 = 0, k_1 = 0, k_4 = 0\},\\
&\{k_1' = k_1', k_1 = k_1, k_2 = -3k_1', k_2' = -1/3k_1, k_3 = 0, k_4 = 0\},\\
&\{k_2' = 0, k_2 = k_2, k_3 = k_3, k_1' = 0, k_4 = -1/2k_3, k_1 = 0\}.
\end{aligned}$$

To study the consequences of these solutions we shall present in the next subsection the Painlevé's test (see Appendix A) that have been very used in the search for integrable systems for being algorithm and with wide application.

4.2 Applying the Painlevé test

For the first case $\{k_1' = k_1', k_2' = k_2', k_4 = k_4, k_3 = 0, k_2 = 0, k_1 = 0\}$, we have the following potential:

$$Pot_1 = \frac{9}{2}\frac{{k_1'}^2}{M}x^4 + \frac{9}{2}\frac{{k_2'}^2}{M}y^4 + 6k_4\frac{k_2'}{M}y^3 + 2\frac{k_4^2}{M}y^2. \qquad (25)$$

Applying the Painlevé test, we obtain four branches referring to the uncoupled systems that survive the test.

For the second case $\{k_2 = 0, k_1' = 0, k_2' = k_2', k_4 = k_4, k_1 = k_1, k_3 = k_3\}$, we have the following potential:

$$Pot_2 = \frac{1}{2}\frac{k_1^2}{M}x^4 + \frac{9}{2}\frac{{k_2'}^2}{M}y^4 + \frac{(2k_1^2 + 3k_1k_2')}{M}x^2y^2 + \qquad (26)$$

$$+6k_4\frac{k_2'}{M}y^3 + \frac{(4k_1k_3 + 2k_1k_4)}{M}x^2y + 2\frac{k_3^2}{M}x^2 + 2\frac{k_4^2}{M}y^2, \qquad (27)$$

with dominant potencies:

$$\alpha_1 = -1, \alpha_2 = -1 \qquad (28)$$

and four branches with the following expressions for the resonances:

$$-1,\ 4,\ \frac{(2k_1 - 3k_2')}{k_1},\ \frac{(3k_2' + k_1)}{k_1}, \qquad (29)$$

that will show integer resonances if we set k_1, $k_2' = n\frac{1}{3}k_1$ where $n = \{-1, 0, 1, 2\}$.

For the case $n = -1$, it is not possible to determine the resonances.

For the case $n = 0$, we have the following potential:

$$Pot_3 = \frac{1}{2}\frac{k_1^2}{M}x^4 + 2\frac{k_1^2}{M}x^2y^2 + \frac{(4k_1k_3 + 2k_1k_4)}{M}x^2y + 2\frac{k_3^2}{M}x^2 + 2\frac{k_4^2}{M}y^2. \qquad (30)$$

It does not pass the Painlevé test because there appears a compatibility condition that cannot be fulfilled:

$$-4i\sqrt{2}(18k_1^2x_1^2 - 5k_4^2 - 4k_3k_4) = 0. \qquad (31)$$

This equality is indeed satisfied if k_1, k_3 *e* $k_4 = 0$, but this cancels out the potential.

For the case $n = 1$, we have the following potential:

$$Pot_4 = \frac{1}{2}\frac{k_1^2}{M}x^4 + \frac{1}{2}\frac{k_1^2}{M}y^4 + 3\frac{k_1^2}{M}x^2y^2 + 2k_4\frac{k_1}{M}y^3 + \frac{(4k_1k_3 + 2k_4k_1)}{M}x^2y + 2\frac{k_3^2}{M}x^2 + 2\frac{k_4^2}{M}y^2. \qquad (32)$$

And now, we obtain four branches with the following resonances:

$$-1, 1, 2, 4, \tag{33}$$

but with the following compatibility condition:

$$-2(-k_4 + k_3)M = 0, \tag{34}$$

to be verified in the resonance $j = 1$ of the first and of the second branch. Setting $k_3 = k_4$, the system becomes compatible and it passes the Painlevé test with only two branches and with the potential now written like bellow:

$$Pot_5 = \frac{1}{2}\frac{k_1^2}{M}x^4 + \frac{1}{2}\frac{k_1^2}{M}y^4 + 3\frac{k_1^2}{M}x^2y^2 + 2\frac{k_3k_1}{M}y^3 + 6k_3\frac{k_1}{M}x^2y + 2\frac{k_3^2}{M}x^2 + 2\frac{k_3^2}{M}y^2, \tag{35}$$

with dominant potencies:

$$\alpha_1 = -1, \alpha_2 = -1. \tag{36}$$

The values of the resonances for the two branches are:

$$-1, 1, 2, 4, \tag{37}$$

and for the first branch the coefficients of the dominant terms are:

$$x_0 = \frac{1}{2}\frac{iM}{k_1}, y_0 = \frac{1}{2}i\frac{M}{k_1}. \tag{38}$$

For the second branch, the coefficients read as follows:

$$x_0 = -\frac{1}{2}\frac{iM}{k_1}, y_0 = -\frac{1}{2}i\frac{M}{k_1}. \tag{39}$$

In the first branch, the arbitrary coefficients are:

$$y_1,\ y_2 \ \text{ and } \ y_4, \tag{40}$$

and the arbitrary coefficients of the second branch are:

$$y_1,\ \ y_2 \ \text{ and } \ x_4, \tag{41}$$

reminding that the variable t_0 is the fourth arbitrary quantity corresponding to the resonance -1.

So, the system is of fourth order and possesses four arbitrary coefficients; therefore, it is integrable.

For the case $n = 2$, we have the following potential:

$$Pot_6 = \frac{1}{2}\frac{k_1^2}{M}x^4 + 2\frac{k_1^2}{M}y^4 + 4\frac{k_1^2}{M}x^2y^2 + 4k_4\frac{k_1}{M}y^3 + \frac{(4k_1k_3 + 2k_4k_1)}{M}x^2y + 2\frac{k_3^2}{M}x^2 + 2\frac{k_4^2}{M}y^2. \tag{42}$$

It was not possible to determine the dominant terms.

For the third case $\{k_3 = 0, k_4 = 0, k_2 = k_2, k_1 = 0, k_1' = k_1', k_2' = 0\}$, we have the following potential:

$$Pot_7 = \frac{9}{2}\frac{{k_1'}^2}{M}x^4 + \frac{1}{2}\frac{k_2^2}{M}y^4 + \frac{(2k_2^2 + 3k_1'k_2)}{M}x^2y^2; \tag{43}$$

the expressions for resonances in this case are:

$$-1, 4, \frac{(3k_1' + k_2)}{k_2}, \ -\frac{(-2k_2 + 3k_1')}{k_2}, \tag{44}$$

that will show integer resonances if we set k_2, $k_1' = n\frac{1}{3}k_2$ where $n = \{-1, 0, 1, 2\}$.

For the case $n = -1$, the system passes the test with the following potential:

$$Pot_8 = \frac{1}{2}\frac{k_2^2}{M}x^4 + \frac{1}{2}\frac{k_2^2}{M}y^4 + \frac{k_2^2}{M}x^2y^2, \tag{45}$$

with dominant potencies:

$$\alpha_1 = -1, \alpha_2 = -1 \tag{46}$$

and the values of the resonances for the two branches:

$$0, -1, 3, 4. \tag{47}$$

For the first branch, the coefficients of the dominant terms are:

$$x_0 = \frac{\sqrt{(-M^2 - k_2^2 y_0^2)}}{k_2}, y_0 = y_0; \tag{48}$$

for the second branch, they are:

$$x_0 = -\frac{\sqrt{(-M^2 - k_2^2 y_0^2)}}{k_2}, y_0 = y_0. \tag{49}$$

In the first branch, the arbitrary coefficients are:

$$y_0, x_3 \quad \text{and} \quad y_4, \tag{50}$$

and the arbitrary coefficients of the second branch are:

$$y_0, \quad x_3 \quad \text{and} \quad y_4. \tag{51}$$

For the case $n = 0$, the system does not pass the test with the following potential:

$$Pot_9 = \frac{1}{2}\frac{k_2^2}{M}y^4 + 2\frac{k_2^2}{M}x^2y^2, \tag{52}$$

because there appears the following compatibility condition:

$$18ik_2^2y_1^2 = 0. \tag{53}$$

This equation is satisfied only if k_2 equal to zero, but in this case the potential vanishes, and this is not an interesting situation.

For the case $n = 1$, the system passes the test with the following potential:

$$Pot_{10} = \frac{1}{2}\frac{k_2^2}{M}x^4 + \frac{1}{2}\frac{k_2^2}{M}y^4 + 3\frac{k_2^2}{M}x^2y^2, \tag{54}$$

with the same dominant potencies:

$$\alpha_1 = -1, \alpha_2 = -1, \tag{55}$$

and resonances:

$$-1, 1, 2, 4 \tag{56}$$

in the two branches.

For the first branch, the coefficients of the dominant terms are:

$$x_0 = -\frac{1}{2}i\frac{M}{k_2}, y_0 = -\frac{1}{2}i\frac{M}{k_2}; \tag{57}$$

and, for the second branch, the coefficients appear as bellow:

$$x_0 = \frac{1}{2} i \frac{M}{k_2}, y_0 = \frac{1}{2} i \frac{M}{k_2}. \tag{58}$$

In the first branch, the arbitrary coefficients are:

$$x_1, \; x_2 \quad \text{and} \quad x_4 \tag{59}$$

and the arbitrary coefficients of the second branch are:

$$y_1, \quad y_2 \;\; \text{and} \;\; x_4. \tag{60}$$

For the case $n = 2$, the system does not pass the test with the following potential:

$$Pot_{11} = 2\frac{k_2^2}{M}x^4 + \frac{1}{2}\frac{k_2^2}{M}y^4 + 4\frac{k_2^2}{M}x^2y^2, \tag{61}$$

because it was not possible to determine the dominant terms.

For the fourth case $\{k_1' = k_1', k_2' = -1/3k_1, k_2 = -3k_1', k_1 = k_1, k_3 = 0, k_4 = 0\}$, we have the following potential (quartic):

$$Pot_{12} = 5\frac{k_1^2}{M}x^4 + 5\frac{k_1^2}{M}y^4 + 10\frac{k_1^2}{M}x^2y^2, \tag{62}$$

with $k_1' = k_1$. The resonances in this case are:

$$0, -1, 3, 4 \tag{63}$$

for the two branches.

For the first branch the coefficients of the dominant terms are:

$$x_0 = \frac{1}{10}\frac{\sqrt{(-10M^2 - 100k_1^2y_0^2)}}{k_1}, y_0 = y_0; \tag{64}$$

and, for the second branch, they are:

$$x_0 = -\frac{1}{10}\frac{\sqrt{(-10M^2 - 100k_1^2y_0^2)}}{k_1}, y_0 = y_0. \tag{65}$$

In the first branch, the arbitrary coefficients are:

$$y_0, \; x_3 \quad \text{and} \quad y_4 \tag{66}$$

and the arbitrary coefficients of the second branch are:

$$y_0, \quad x_3 \;\; \text{and} \;\; y_4. \tag{67}$$

For the fifth case $\{k_2' = 0, k_2 = k_2, k_3 = k_3, k_1' = 0, k_4 = -1/2k_3, k_1 = 0\}$, we have the following potential:

$$Pot_{13} = \frac{1}{2}\frac{k_2^2}{M}y^4 + 2\frac{k_2^2}{M}x^2y^2 + 2\frac{k_3^2}{M}x^2 + \frac{1}{2}\frac{k_3^2}{M}y^2. \tag{68}$$

This potential does not pass in the Painlevé test because the following compatibility condition appears:

$$-3ik_3^2 + 18ik_2^2y_1^2 = 0, \tag{69}$$

that is only satisfied if $k_2 = k_3 = 0$, and this eliminates our potential. Therefore this case does not pass the Painlevé test.

As this potential is of the quartic type, it is easy to verify that the result of this Painlevé analysis is in agreement with the conditions of integrability for this potential type.

5 The integrability of the bosonic sector with parity considerations for the complete model.

As verified in the previous section, by imposing parity to the bosonic sector, the task of finding integrable cases became less arbitrary, in that the choice of the coefficients in the terms of the potential was guided by the argument of parity invariance. In spite of that, it was still necessary to fix by hand the values of some parameters when applying Painlevé test to recover the integrable cases we have listed previously.

In this section, we shall impose the parity symmetry not only to the bosonic sector but also to the fermionic interactions, and we shall verify to which extent the constraints on the parameters are able to turn the model integrable without the need of fixing arbitrarily parameters in the Painlevé test.

5.1 Two-component formulation of the fermionic sector

Since the model is classic and non-relativistic, and defined in a two-dimensional Euclidean space, E^2, the covariance group is SO(2). We adopt the representation below for the Clifford algebra:

Therefore, we adopt:

$$\gamma^1 = \sigma_x, \tag{70}$$

$$\gamma^2 = \sigma_y, \tag{71}$$

$$\gamma_3 = -i\gamma^1\gamma^2 = \sigma_z, \tag{72}$$

such that:

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}1, \tag{73}$$

$$\left\{\gamma^i, \gamma_3\right\} = 0. \tag{74}$$

For a general spinor,

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}, \tag{75}$$

the action of SO(2) is as given below:

$$\Psi' = e^{-\frac{i}{2}\omega\sigma_z}\Psi, \tag{76}$$

where ω is the rotation angle; therefore $\Psi^\dagger\Psi$ is invariant.

Now, we try to identify x-and-y-parities in the spinorial space.

To do that, we start off from the Dirac equation:

$$i\gamma^1\partial_x\Psi + i\gamma^2\partial_y\Psi = 0, \tag{77}$$

to which we impose x-parity symmetry:

$$\begin{aligned}\Psi(t;\vec{x}) \xrightarrow{\quad P \quad} \Psi'(t';\vec{x'}) &= \\ &= R\Psi(t;\vec{x}) \\ &= R\Psi(t';-x',y'),\end{aligned} \tag{78}$$

where R represents the parity matrix in the spinor space:

$$\begin{aligned}\gamma^1 R &= -R\gamma^1 \\ \gamma^2 R &= R\gamma^2.\end{aligned} \tag{79}$$

Then, our parity matrix may be chosen as

$$R = \gamma^2 \tag{80}$$

and, thus,

$$\Psi'(t'; \overrightarrow{x'}) = \gamma^2 \Psi(t; \overrightarrow{x}). \tag{81}$$

So, all spinors, up to a phase factor, transform under parity by means of the γ_2-matrix.

Considering the other possibility, that is, the y-parity, one can readily check that parity is represented by the γ_1-matrix:

$$P \begin{cases} x \rightarrow x \\ y \rightarrow -y \end{cases} \tag{82}$$

$$\begin{aligned} \Psi &\rightarrow \gamma^1 \Psi \\ \Psi'\left(t'; \overrightarrow{x}\ '\right) &\rightarrow \gamma_1 \Psi\left(t; \overrightarrow{x}\ \right). \end{aligned} \tag{83}$$

5.2 The integrability with the parity constraints from the fermionic Sector

To include the constraints dictated by x- or y-parity symmetry for the complete (bosnic + fermionic) model, we propose to actually carry out the analysis directly in terms of the superfields (7) and (8). Rather than following the lengthy procedure of considering all the terms of the component-field action, we propose to work without quitting superspace.

The action of the x-parity on the superfields is given by

$$X \rightarrow -X \quad \text{and} \quad Y \rightarrow Y, \tag{84}$$

provided that

$$\begin{aligned} \Theta &\rightarrow \gamma_2 \Theta, \\ \Lambda &\rightarrow \gamma_2 \Lambda, \\ \Xi &\rightarrow -\gamma_2 \Xi, \\ f_1 &\rightarrow f_1, \\ f_2 &\rightarrow -f_2. \end{aligned}$$

With these parity assignments to the fermions and auxiliary fields, the superfield coordinates transform under parity exactly as above. Moreover, by virtue of the specific choice of γ_2, we have that parity acts on $d\theta$, $d\overline{\theta}$ and the covariant derivatives as below:

$$\begin{aligned} D \rightarrow -i\overline{D}, &\quad \overline{D} \rightarrow iD; \\ d\theta \rightarrow id\overline{\theta}, &\quad d\theta \rightarrow -id\overline{\theta}. \end{aligned}$$

With all the prescriptions, the volume element $dtd\theta d\overline{\theta}$ picks a minus sign. This means that the kinetic terms are naturally invariant, but parity symmetry of the potential sets

$$k_1 = k_3 = k_4 = k_2' = 0, \tag{85}$$

with k_2 and k_1' non-vanishing.

These parameters constraints are the same as the third set we found when only the bosonic sector was considered and we found only two integrable cases: Potentials 8 and 10 the we rename now as below:

$$Pot_{susy1-x} = \frac{1}{2}\frac{k_2^2}{M}x^4 + \frac{1}{2}\frac{k_2^2}{M}y^4 + \frac{k_2^2}{M}x^2y^2, \tag{86}$$

and

$$Pot_{susy2-x} = \frac{1}{2}\frac{k_2^2}{M}x^4 + \frac{1}{2}\frac{k_2^2}{M}y^4 + 3\frac{k_2^2}{M}x^2y^2. \tag{87}$$

So, from all integrable cases found when we considered only bosonic sector, only the two potentials above preserve x-parity under complete model consideration.

On the other hand, if we contemplate y-parity symmetry for the whole model, we have that

$$X \rightarrow X \quad \text{and} \quad Y \rightarrow -Y, \tag{88}$$

provided that

$$\begin{aligned}
\Theta &\rightarrow \gamma_1\Theta,\\
\Lambda &\rightarrow -\gamma_1\Lambda,\\
\Xi &\rightarrow \gamma_1\Xi,\\
f_1 &\rightarrow -f_1,\\
f_2 &\rightarrow f_2.
\end{aligned}$$

Also, $D \rightarrow -i\overline{D}$, $\overline{D} \rightarrow iD$, $d\theta \rightarrow id\overline{\theta}$ and $d\theta \rightarrow -id\overline{\theta}$.

So, as in previous case, y-parity invariance is ensured only for those superfield monomials that change sign under parity. This then impose:

$$k_2 = k_3 = k_4 = k_1' = 0, \tag{89}$$

while k_1 and k_2' are the only coefficients compatible with y-parity invariance.

These constraints on the parameters correspond to only one set of solutions that is found when only the bosonic setor is considered in connection with the y-parity, in a similar way to what happens for x-parity. There are only two integrable cases that we shall present below:

$$Pot_{susy1-y} = \frac{1}{2}\frac{k_1^2}{M}x^4 + \frac{1}{2}\frac{k_1^2}{M}y^4 + \frac{k_1^2}{M}x^2y^2, \tag{90}$$

and

$$Pot_{susy2-y} = \frac{1}{2}\frac{k_1^2}{M}x^4 + \frac{1}{2}\frac{k_1^2}{M}y^4 + 3\frac{k_1^2}{M}x^2y^2. \tag{91}$$

So, from all integrable cases found when only the bosonic sector is considered, only the two potentials above preserve y-parity if whole model is analysed.

6 Final discussions and general conclusions.

Along the previous sections, we carried out an integrability analysis of the bosonic sector of the supersymmetric model and we verified the appearance of integrable cases for both coupled and non-coupled systems.

The coupled cases turn out to be classified into two types: a quartic potential and a potential that is functionally the superposition of a quartic and a Henon-Heiles potential.

Contrary to the situation where we impose parity symmetry to the complete action (bosonic and fermionic interactions) and the generated potentials come out totally integrable, without the need of setting integrability constraints, the case in which parity symmetry is imposed only to the bosonic sector yields integrable potentials only after we take into account the constraints that appear in the course of Painlevé analysis. This means that, if these constraints are not fulfilled, we will be dealing with non-integrable potentials and therefore with the possibility of chaos.

For the cases where the potentials have a quartic form, there is no need to go through a chaos analysis for this has already been discussed in the literature we have previously referred to.

The cases for which the potentials are given by the superposition of a quartic and a Henon-Heiles form are under consideration and, in a forthcoming work, we shall report the results of a complete analysis [24]. However, in this section, we shall give an example to illustrate how this type of non-integrable potential admits order-chaos transition by using the potential of number 6 of Section(4.2):

$$Pot_6 = \frac{1}{2}\frac{k_1^2}{M}x^4 + 2\frac{k_1^2}{M}y^4 + 4\frac{k_1^2}{M}x^2y^2 + 4k_4\frac{k_1}{M}y^3 + \frac{(4k_1k_3 + 2k_4k_1)}{M}x^2y + 2\frac{k_3^2}{M}x^2 + 2\frac{k_4^2}{M}y^2. \qquad (92)$$

For this purpose, we make use of Lyapunov characteristic exponent (LCE) and phase portraits [25, 26] and [27]. The Lyapunov exponent is a usefull tool to quantify the divergence or convergence of initial nearby trajetories for a dynamical system. In a chaotic system, there is at least one positive Lyapunov exponent, defined as

$$\sigma_i = \lim_{t\to\infty} \ln \frac{d_i\,(t)}{d_i\,(0)}$$

where $d_i\,(t)$ is a deformation measure of the small hypersphere of initial conditions in the phase space along the trajectory. The asymptotic rate of expansion of the largest axis is given by the largest LCE. By phase portrait we mean a graph of the dynamical variables in phase space that is used to provide a qualitative insight of the dynamical behavior of the system under study. The accuracy of our computation was verified by checking if the Hamiltoniam was conserved during the simulation.

Fixing $k_1 = 10$, $M = k_3 = k_4 = 1$ the potential acquires the following form:

$$V := 50x^4 + 200y^4 + 400x^2y^2 + 40y^3 + 60x^2y + 2x^2 + 2y^2$$

We calculate de largest σ_i and its respectives phase portraits, and we present two cases for the same set of parameters fixed above, but with different initial contions. First with $p_1(0) = 0.1, p_2(0) = 0.1, q_1(0) = 0.1, q_2(0) = 0.0$, Energy=0.035; it presents regular behavior (see Figures 1 e 2 below).

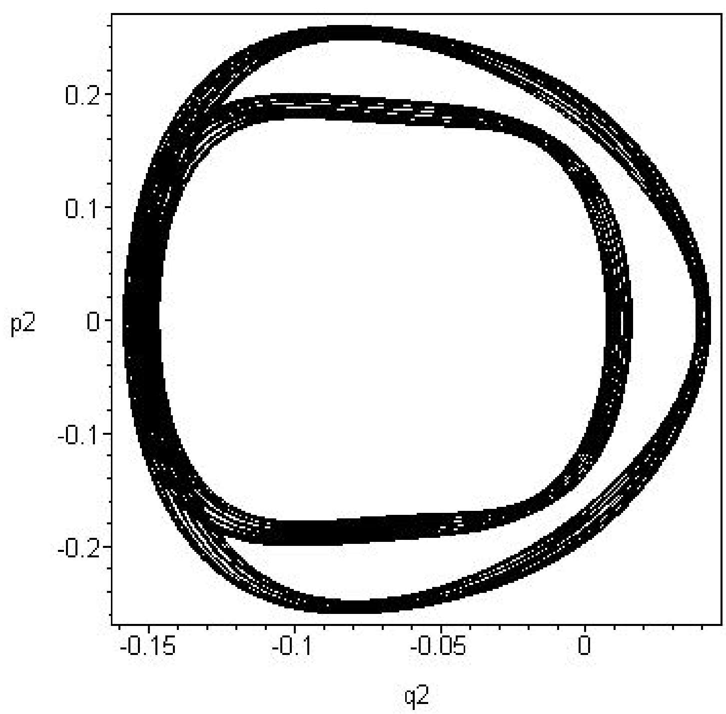

Figure 1

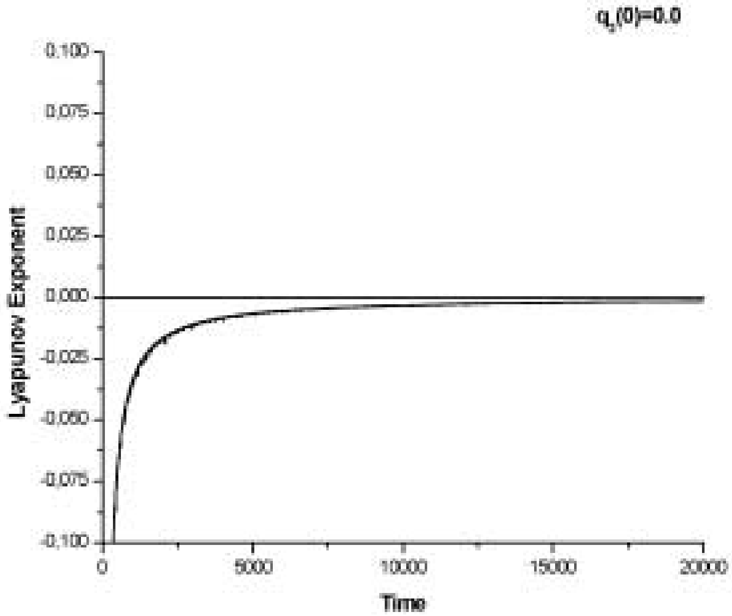

Figure 2

The second case is given by $p_1(0) = 0.1, p_2(0) = 0.1, q_1(0) = 0.1, q_2(0) = 0.18$, Energy=0.78; it presents chaotic behavior (see Figures 3 e 4 below).

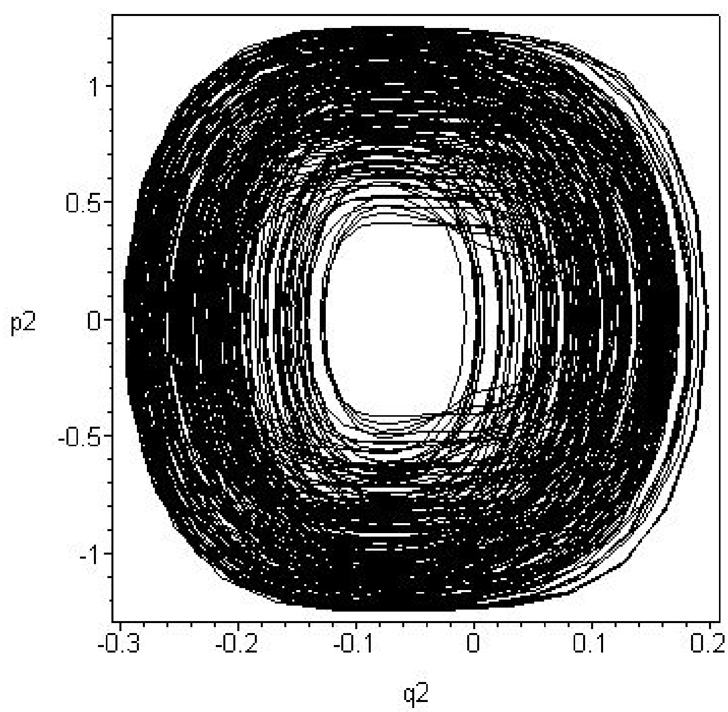

Figure 3

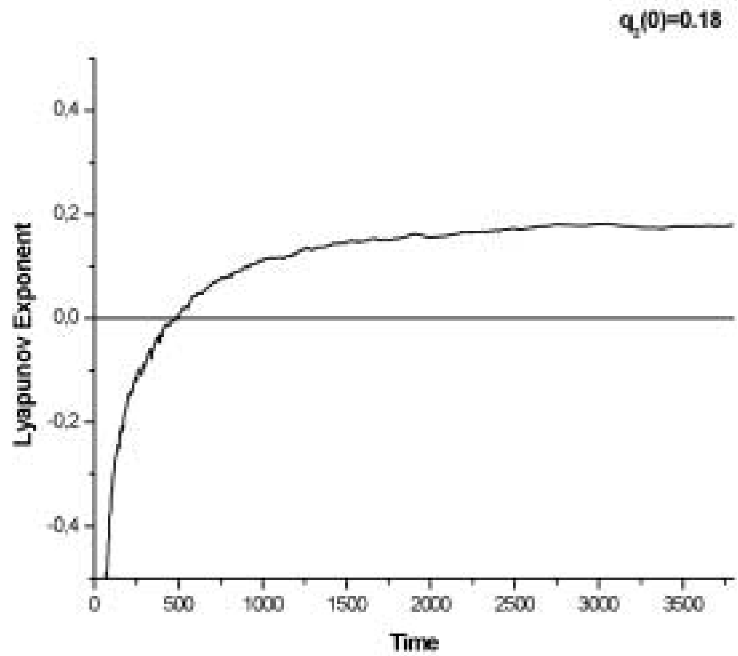

Figure 4

It is perhaps noteworthy to say that a potential of the form $\left(x^2+y^2\right)^2$ can be generated by considering the work of Refs. [29]–[31], where a classical mechanical Yang-Mills system with four degrees of freedom ξ_i is studied, with Hamiltonian given by

$$H=\frac{1}{2}\left(P_1^2+P_2^2+P_3^2+P_4^2\right)+\frac{g^2}{8}\left(\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2\right)^2. \tag{93}$$

Indeed, if we adopt hyperbolic coordinates on the $(\xi_1;\xi_2)$ and $(\xi_3;\xi_4)$ planes,

$$\begin{aligned}\xi_1&=r_1\cosh\theta_1\\ \xi_2&=r_1\sinh\theta_1\\ \xi_3&=r_2\cosh\theta_2\\ \xi_4&=r_2\sinh\theta_2,\end{aligned}$$

the Hamiltonian becomes:

$$H=\frac{1}{2}\left(P_{r_1}^2+P_{r_2}^2-\frac{1}{r_1^2}P_{\theta_1}^2-\frac{1}{r_2^2}P_{\theta_2}^2\right)+\frac{g^2}{8}\left(r_1^2+r_2^2\right)^2. \tag{94}$$

θ_1 and θ_2 are ignorable coordinates, so that their corresponding momenta are integrals of motion. If $P_{\theta_1}^2=P_{\theta_2}^2=0$, no negative contributions to the kinetic energy are present and we get an effective two-degrees-of-freedom system with the potential of the form $\left(x^2+y^2\right)^2$. Besides the discussion of integrability and classical chaos, studies of quantum chaos using this potential have received a great deal of attention in the literature[32]–[37]. This observation may be a good support in favor of the results we have got on the integrable potentials produced after parity symmetry has been imposed for the complete supersymmetric model.

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A Painlevé test

The Painlevé test[27]–[28] establishes if a system of ODEs exhibits the Painlevé property.

An ODE has the Painlevé property if its solutions in the complex plane are single-valued in the neighborhood of all its movable singularities. Given a differential system

$$L_j(u_i, u_{it}) = 0 \qquad with \qquad i,j = 1,...,n \quad , \tag{95}$$

we assume a Laurent expansion for the solution

$$u_i(t) = (t-t_0)^{\alpha_i} \sum_{k=0}^{\infty} u_{i,k}(t-t_0)^k, \tag{96}$$

with

$$u_{i,0} \neq 0 \qquad \text{and} \quad \alpha_i \in Z^- \ , \tag{97}$$

where $u_{i,k}$ are constants. The algorithm for the Painlevé test is implemented by means of the following three steps:

Step 1 (Determine the leading singularity or dominant behavior). We replace

$$u_i(t) \simeq u_{i,0}(t-t_0)^{\alpha_i} \tag{98}$$

into (95) to determine α_i and $u_{i,0}$ and we obtain an algebraic system with α_i, assuming negative integer values and t_0 arbitrary.

We require that two or more terms of each equation may balance and determine α_i and $u_{i,0}$.

If any α_i is not integer, the system is not of Painlevé type in its strong version.

If there are more than one solution for α_i or $u_{i,0}$ they define branches and the following steps of the algorithm need to be applied for each of these branches.

Step 2 (Determine the resonances).

For each α_i and $u_{i,0}$, we calculate the integers r for which $u_{i,r}$ is an arbitrary function in 95. We replace the truncated series

$$u_i(t) = u_{i,0}(t-t_0)^{\alpha_i} + u_{i,r}(t-t_0)^{\alpha_i+r}. \tag{99}$$

by (95), and we look for integer r for which $u_{i,r}$ is an arbitrary constant .

To do that, after replacing the truncated series by (95), we keep the most singular terms in $(t-t_0)$, and the coefficients of $u_{i,r}$ are set to zero. We get:

$$Qu_r = 0, \quad ur = (u_{1,r}\ u_{2,r}...u_{M,r})^T, \tag{100}$$

with Q an $M \times M$ matrix depending of r.

The resonances are the roots of $\det(Q) = 0$.

In every system with the Painlevé property, the resonance (-1) will be present and correspond to arbitrary $(t-t_0)$. The resonance with zero value may also be present, depending of the number of arbitrary values $u_{i,0}$.

Step 3 (Compatibility conditions and constants of motion).

For every resonance found in the previous step, there is a compatibility condition which must be verified in order that the system pass the Painlevé test. The compatibility conditions are verified by inserting

$$u_i(t) = (t-t_0)^{\alpha_i} \sum_{k=0}^{r_M} u_{i,k}(t-t_0)^k \tag{101}$$

into (95), where r_M is the highest positive integer resonance.

If all these compatibility conditions are satisfied so that they introduce a sufficient number of arbitrary constants, then the system is said to be of Painlevé type.

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