Numerical Studies of Relativistic Corrections to Fermion Dynamics and the Aharonov-Casher Effect

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Abstract

Dirac's equation with minimal eletromagnetic coupling yields, in the weakly relativistic regime, the Pauli's equation for two-component minimally-interacting electron endowed with magnetic moment coupling and the right value of the Landé g-factor, g=2. On the other hand, another remarkable feature associated with spinning particles is the coupling of the magnetic dipole moment to an electric field, which gives rise to the so-called Aharonov-Casher phase for the wave function of the test particle. This phase shift shows up even though there is no force acting on the particle, just as in the familiar case of the Aharonov-Bohm effect. The most accurate experimental measurements of the spin-electric field interaction are carried out with atomic systems. This work sets out to investigate how the Aharonov-Casher effect may be related to the relativistic regime: in practical terms, how it may appear by means of relativistic corrections to the Schrödinger equation, once an external eletromagnetic field is switched on. One should understand at which order of velocities the Aharonov-Casher shift arises, either by adding up higher-order gradient terms to the Schrödinger equation or by carring out the non-relativistic limit of Dirac's equation with higher-derivative terms. The latter has very interesting consequences whenever adjoined to the Dirac's equation, leading to a rich excitation spectrum and inducing interesting couplings in the low-energy regime. In view of the calculational complexity inherent to the task of finding solutions to these higher-derivatives partial differential equations, wave-function solutions and phase shifts have to be searched for with the help of numerical methods and computer-algebra softwares.

Key-words: Higher-derivatives; Exact solutions; Aharonov-Casher.

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We start up with a neutral fermion in a lower-dimensional space-time, a Majorana spinor, with the following representation for the Clifford Algebra:

$$\gamma^0 = \sigma_y$$

$$\gamma^1 = i\sigma_x, \ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1},$$

$$\eta^{\mu,\nu} = (+,-).$$

Neutral fermions, if self-conjugated, are described by Majorana spinors which, in the representation above become real:

$$\Psi^C = C\bar{\Psi}^t = -\gamma^0 C\Psi^* = \Psi_x$$

is the Majorana condition; C is the charge conjugation matrix, such that

$$C\gamma^{\mu t}C^{-1} = -\gamma^\mu$$

and

$$C^\dagger C = \mathbf{1}.$$

$C = -\gamma^0$ satisfies the conditions above and, consequently, Ψ becomes a real 2-component spinor.

Now, an open question is to study the general solution to a higher-derivative fermion equation:

$$i\gamma^\mu \partial_\mu \Box^2 = 0,$$

which is a 5-th order differential equation that we may exactly solve for a set of initial conditions left completely arbitrary. The latter specify Ψ, $\dot{\Psi}$, $\ddot{\Psi}$, $\dddot{\Psi}$ and $\ddddot{\Psi}$ at t=0, as below:

$$\Psi(0;x) = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}$$

$$\dot{\Psi}(0;x) = \begin{pmatrix} C(x) \\ D(x) \end{pmatrix}$$

$$\ddot{\Psi}(0;x) = \begin{pmatrix} F(x) \\ G(x) \end{pmatrix}$$

$$\dddot{\Psi}(0;x) = \begin{pmatrix} H(x) \\ J(x) \end{pmatrix}$$

$$\ddddot{\Psi}(0;x) = \begin{pmatrix} R(x) \\ S(x) \end{pmatrix}$$

We propose a sort of extended D'Alembert method to reduce the set of partial differential equation to 10 ordinary differential equations, which yields the following solution:

$$\begin{aligned}
\Psi_1(t;x) &= \tilde{\Psi}_1(\xi;\eta) = \frac{7}{8}A(\xi) + \frac{1}{8}A(\eta) + \frac{1}{16}\left\{2\eta C(\xi) - \xi\left[C(\xi) + C(\eta)\right]\right\} + \frac{5}{16}F(\xi)(\xi-\eta)^2 \\
&-\frac{1}{32}H(\xi)\xi(\xi-\eta)^2 + \frac{1}{32}\left\{12\eta A'(\xi) - 11\xi A'(\xi) - \xi A'(\eta)\right\} - \frac{1}{16}C'(\xi)\eta(\xi-\eta) \\
&+\frac{1}{32}A''(\xi)(\xi-\eta)(\xi-2\eta) + \frac{1}{32}C''(\xi)\xi(\xi-\eta)^2 + \frac{1}{64}A'''(\xi)\xi(\xi-\eta)^2 + \frac{3}{8}\int_\xi^\eta dy C(y) \\
&+\frac{3}{8}\left\{\int_\xi^\eta dy \int_a^y dz F(z) + (\xi-\eta)\int_a^\xi dy F(y)\right\} + \frac{1}{32}\xi\left\{\int_\xi^\eta dy \int_a^y dz \int_a^z du R(u)\right.
\end{aligned}$$

$$+(\xi-\eta)\int_a^\xi dy\int_a^y dzR(z)-\frac{1}{2}(\xi-\eta)^2\int_a^\xi dyR(y)\Big\}-\frac{1}{32}\Big\{\int_\xi^\eta dy\int_a^y dz\int_a^z duuR(u)$$
$$+(\xi-\eta)\int_a^\xi dy\int_a^y dzzR(z)-\frac{1}{2}(\xi-\eta)^2\int_a^\xi dyyR(y)\Big\}+\frac{1}{16}\Big\{2\int_\xi^\eta dy\int_a^y dz\int_a^z duH(u)$$
$$+\xi\int_a^\xi dy\int_a^y dzH(z)+\xi\int_a^\eta dy\int_a^y dzH(z)-2\eta\int_a^\xi dy\int_a^y dzH(z)+\eta(\xi-\eta)\int_a^\xi dyH(y)\Big\}$$
$$-\frac{1}{16}\Big\{\int_\xi^\eta dy\int_a^y dz\int_a^z duuH'(u)+(\xi-\eta)\int_a^\xi dy\int_a^y dzzH'(z)-\frac{1}{2}(\xi-\eta)^2\int_a^\xi dyyH'(y)\Big\}$$
$$+\frac{1}{32}\Big\{\int_\xi^\eta dy\int_a^y dz\int_a^z duuA''''(u)+(\xi-\eta)\int_a^\xi dy\int_a^y dzzA''''(z)-\frac{1}{2}(\xi-\eta)^2\int_a^\xi dyyA''''(y)\Big\}$$
$$+\frac{1}{16}\Big\{\int_\xi^\eta dy\int_a^y dz\int_a^z duuC'''(u)+(\xi-\eta)\int_a^\xi dy\int_a^y dzC'''(z)-\frac{1}{2}(\xi-\eta)^2\int_a^\xi dyyC'''(y)\Big\}\quad(1)$$

and

$$\Psi_2(t;x) = \tilde{\Psi}_2(\eta;\xi)=\frac{7}{8}B(\eta)+\frac{1}{8}B(\xi)+\frac{1}{16}\{-2\xi D(\eta)+\eta\,[D(\eta)+D(\xi)]\}+\frac{5}{16}G(\eta)(\eta-\xi)^2$$
$$+\frac{1}{32}J(\eta)\eta(\eta-\xi)^2+\frac{1}{32}\{12\xi B'(\eta)-11\eta B'(\eta)-\eta B'(\xi)\}-\frac{1}{16}D'(\eta)\xi(\eta-\xi)$$
$$+\frac{1}{32}B''(\eta)(\xi-\eta)(2\xi-\eta)-\frac{1}{32}D''(\eta)\eta(\eta-\xi)^2+\frac{1}{64}B'''(\eta)\eta(\eta-\xi)^2+\frac{3}{8}\int_\eta^\xi dyD(y)$$
$$+\frac{3}{8}\Big\{\int_\eta^\xi dy\int_a^y dzG(z)+(\eta-\xi)\int_a^\eta dyG(y)\Big\}+\frac{1}{32}\eta\Big\{\int_\eta^\xi dy\int_a^y dz\int_a^z duS(u)$$
$$+(\eta-\xi)\int_a^\eta dy\int_a^y dzS(z)-\frac{1}{2}(\eta-\xi)^2\int_a^\eta dyS(y)\Big\}-\frac{1}{32}\Big\{\int_\eta^\xi dy\int_a^y dz\int_a^z duuS(u)$$
$$+(\eta-\xi)\int_a^\eta dy\int_a^y dzzS(z)-\frac{1}{2}(\eta-\xi)^2\int_a^\eta dyyS(y)\Big\}-\frac{1}{16}\Big\{2\int_\eta^\xi dy\int_a^y dz\int_a^z duJ(u)$$
$$+\eta\int_a^\eta dy\int_a^y dzJ(z)+\eta\int_a^\xi dy\int_a^y dzH(z)-2\xi\int_a^\eta dy\int_a^y dzJ(z)+\xi(\eta-\xi)\int_a^\eta dyJ(y)\Big\}$$
$$+\frac{1}{16}\Big\{\int_\eta^\xi dy\int_a^y dz\int_a^z duuJ'(u)+(\eta-\xi)\int_a^\eta dy\int_a^y dzzJ'(z)-\frac{1}{2}(\eta-\xi)^2\int_a^\eta dyyJ'(y)\Big\}$$
$$+\frac{1}{32}\Big\{\int_\eta^\xi dy\int_a^y dz\int_a^z duuB''''(u)+(\eta-\xi)\int_a^\eta dy\int_a^y dzzB''''(z)-\frac{1}{2}(\eta-\xi)^2\int_a^\eta dyyB''''(y)\Big\}$$
$$-\frac{1}{16}\Big\{\int_\eta^\xi dy\int_a^y dz\int_a^z duuD'''(u)+(\eta-\xi)\int_a^\eta dy\int_a^y dzD'''(z)-\frac{1}{2}(\eta-\xi)^2\int_a^\eta dyyD'''(y)\Big\}\quad(2)$$

where

$$\xi\equiv x-t \qquad \textit{and} \qquad \eta\equiv x+t.$$

This exact solution may teach us a great deal about the propagation of signals with dynamics dictated by higher-order differential equations. For the particular set of initial conditions given below,

$$A(x)=e^{-x^2}$$

$$C(x)=F(x)=R(x)=0$$

$$H(x) = \frac{1}{1+x^2}$$

we present the plots of the propagation of the Ψ-excitation.

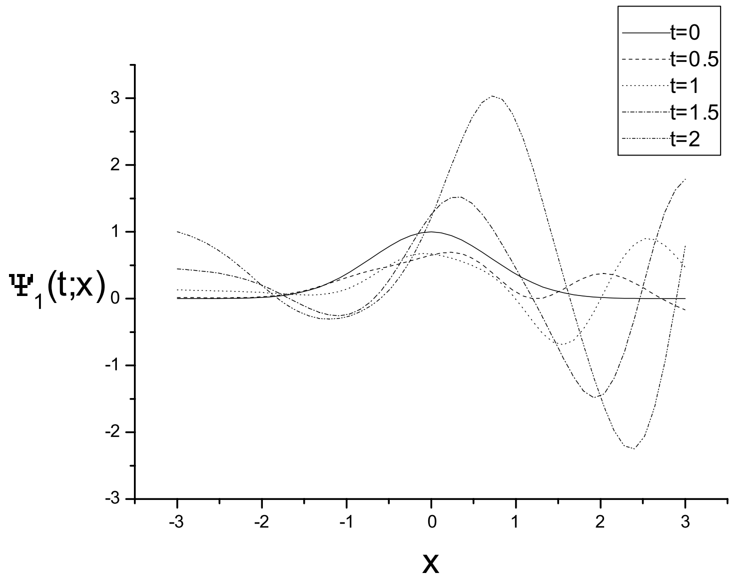

Figure 1: Time evolution for Ψ_1- excitation

Our idea here is to relate higher-order fermion dynamics with an interesting effect that describes the quantum-mechanical effect of an electrostatic field on a neutral particle of spin-$\frac{1}{2}$: the so-called Aharonov-Casher effect[1][2].

On classical grounds, a neutral particle with magnetic moment does not experiment the Coulomb force generated by an external electrostatic field, $\vec{E}$. However, quantum-mechanical systems may reveal topological effects that appear under the form of a phase on the wave function of a particle moving through a non-simply connected force-free region. This is the Aharonov-Casher effect for neutral spinning particles.

What we wish to claim here, in connection with higher-derivative fermionic wave, is that the Aharonov-Casher, contrary to the Aharonov-Bohm effect[3], has to be reassessed and a numerical study of the wave function may play a central role in rederiving the form of the phase induced by the electrostatic field.

For the Aharonov-Bohm effect, no matter the order of the derivatives in the wave equations, the minimal coupling prescription, $\partial_\mu \longmapsto \partial_\mu + ieA_\mu$, leads to an interference effect due to the phase

$$exp(iS) = exp\left(i\oint \vec{A}\cdot d\vec{x}\right).$$

Nevertheless, for the Aharonov-Casher effect, for which electric charge does not play any role, higher derivatives of the form $i\gamma^\mu\partial_\mu\Box$ or $i\gamma^\mu\partial_\mu\Box^2$ do not match with the Pauli interaction term, $\mu\sigma^{\mu\nu}F_{\mu\nu}$, to yield the ordinary non-relativistic phase of the form

$$exp(iS) = exp\left[\frac{i}{\hbar c^2}\oint\left(\vec{\mu}\times\vec{E}\right)\cdot d\vec{x}\right]$$

With our general solutions to the higher-derivative equations, we can carry out numerical studies to understand how to modify the Aharonov-Casher phase in consequence of higher derivatives. Numerical analysis indicate that the Aharonov-Casher effect is actually very significant in the low-relativistic limit. In the ultra-relativistic limit, the higher-derivative contributions dominate and we may loose the phase effect. This opens up the discussion if we should correct the Coulomb force at extreme velocities in Special Relativity. A non-zero force may reinforce the loss of the phase effect at the quantum-mechanical level.

References

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