ISSN 0029-3865

CNPq

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS

Notas de Física

CBPF-NF-011/94

January 1994

The Lagrangian of q-Poincaré Gravity

Leonardo Castellani

Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150, 22290-180, Rio de Janeiro/RJ, Brasil.

Permanent address:

Istituto Nazionale di Fisica Nucleare, Sezione di Torino

and

Dipartimento di Fisica Teorica, Via P. Giuria 1, 10125 Torino, Italy

DFTT-01/94 – January 1994

e-mail addresses:decnet=31890::castellani;

internet=castellani@to.infn.it

Abstract

The gauging of the q-Poincaré algebra of ref. [1] yields a non-commutative generalization of the Einstein-Cartan lagrangian.

We prove its invariance under local q-Lorentz rotations and, up to a total derivative, under q-diffeomorphisms. The variations of the fields are given by their q-Lie derivative, in analogy with the $q = 1$ case. The algebra of q-Lie derivatives is shown to close with field dependent structure functions.

The equations of motion are found, generalizing the Einstein equations and the zero-torsion condition.

Key-words: Quantum groups; Gravitation; Non-commutative; Differential calculus.

We describe in this Letter a geometric procedure to gauge the quantum Poincaré algebra found in ref. [1]. The lagrangian we obtain is a generalization of the Einstein-Cartan lagrangian, and has the same kind of symmetries (now q-deformed symmetries) as its classical counterpart: it is invariant under local Lorentz rotations and q-diffeomorphisms.

As one could expect, the differential calculus on the q-deformed Poincaré group is the correct framework for the program of finding a q-generalization of Einstein gravity. It was not obvious that this program could be carried to the end: in fact it can be done. We refer to [1] for most of the technicalities regarding the inhomogeneous quantum groups $ISO_q(N)$ and their differential calculus. Here we concentrate directly on the $ISO_q(3,1)$ quantum Lie algebra, and discuss its gauging. The method we follow is a natural q-extension of the geometric procedure described in [2] for classical gauge and (super)gravity theories [1].

The starting point is the q-algebra $ISO_q(3,1)$ of ref. [1]:

$$[\chi_{ab},\chi_{cd}] = C_{bc}\chi_{ad} + C_{ad}\chi_{bc} - C_{bd}\chi_{ac} - C_{ac}\chi_{bd} \tag{1}$$

$$\begin{aligned}
&[\chi_{12},\chi_a]_{q^{-1}} = q^{-\frac{1}{2}}C_{2a}\chi_1 - q^{-\frac{1}{2}}C_{1a}\chi_2\\
&[\chi_{13},\chi_a]_{q^{-1}} = q^{-\frac{1}{2}}C_{3a}\chi_1 - q^{-\frac{1}{2}}C_{1a}\chi_3\\
&[\chi_{14},\chi_a] = C_{4a}\chi_1 - C_{1a}\chi_4\\
&[\chi_{23},\chi_a] = C_{3a}\chi_2 - C_{2a}\chi_3\\
&[\chi_{24},\chi_a]_q = q^{\frac{1}{2}}C_{4a}\chi_2 - q^{\frac{1}{2}}C_{2a}\chi_4\\
&[\chi_{34},\chi_a]_q = q^{\frac{1}{2}}C_{4a}\chi_3 - q^{\frac{1}{2}}C_{3a}\chi_4
\end{aligned} \tag{2}$$

$$\begin{aligned}
&[\chi_1,\chi_2]_{q^{-1}} = 0, \qquad [\chi_1,\chi_3]_{q^{-1}} = 0\\
&[\chi_1,\chi_4]_{q^{-2}} = 0, \qquad [\chi_2,\chi_3] = 0\\
&[\chi_2,\chi_4]_{q^{-1}} = 0, \qquad [\chi_3,\chi_4]_{q^{-1}} = 0
\end{aligned} \tag{3}$$

where $[A,B]_s \equiv AB - sBA$. The subalgebra spanned by the Lorentz generators χ_{ab} $(= -\chi_{ba})$ is *classical*; the deformation parameter q appears only in the commutation relations (2) and (3), involving the momenta χ_a. The metric C_{ab} is given by

$$C_{ab} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

with Lorentz signature $(+,+,+,-)$. Only in the classical limit $q \to 1$ can one redefine the generators so as to diagonalize (4). The fact that the metric is diagonal in the indices 2,3 (and not completely antidiagonal as for the q-groups defined in [4])

[1] the so-called "group manifold approach" was initiated in ref.s [3].

is due to the existence of a particular real form on $SO_q(4;\mathbf{C})$. This real form, first discussed in ref. [5] for the uniparametric q-groups $SO_q(2n;\mathbf{C})$, was extended to the multiparametric case and to $ISO_q(2n,\mathbf{C})$ in [1], and allows to redefine antihermitian (linear combinations of the) generators, bringing the antidiagonal metric of ref. [4] in the hybrid form (4).

The algebra (1)-(3) was obtained in ref. [1] via a consistent projection from the q-Lie algebra of a particular multiparametric deformation of $SO(6)$, for which the R matrix takes a very simple form: it is diagonal and satisfies $\hat{R}^2 = 1$, with $\hat{R} \equiv PR$ $(\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd})$.

The q-Lie algebra (1)-(3) has the form

$$\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l = \mathbf{C}_{ij}{}^{k}\chi_k \tag{5}$$

where i,j...are adjoint indices running on the 10 values corresponding to the indices (a,ab) of the generators of $ISO_q(3,1)$. The non-vanishing components of the braiding matrix Λ and the structure constants $\mathbf{C}$, implicitly defined by (1)-(3), are given below (no sum on repeated indices):

$$\Lambda^{ab\ cd}{}_{ef\ gh} = \delta^a_g\delta^b_h\delta^c_e\delta^d_f \tag{6}$$

$$\Lambda^{a\ bc}{}_{de\ f} = (\alpha_{de})^2\ \delta^a_f\delta^b_d\delta^c_e \tag{7}$$

$$\Lambda^{bc\ a}{}_{f\ de} = (\alpha_{de})^{-2}\ \delta^a_f\delta^b_d\delta^c_e \tag{8}$$

$$\Lambda^{ab}{}_{cd} = \beta_{cd}\ \delta^a_d\delta^b_c \tag{9}$$

$$\mathbf{C}_{ab\ cd}{}^{ef} = \frac{1}{4}[C_{ad}\delta^e_b\delta^f_c + C_{bc}\delta^e_a\delta^f_d - C_{ac}\delta^e_b\delta^f_d - C_{bd}\delta^e_a\delta^f_c] - (e \leftrightarrow f) \tag{10}$$

$$\mathbf{C}_{ab\ c}{}^{d} = \frac{1}{2}\alpha_{ab}\ (C_{bc}\delta^d_a - C_{ac}\delta^d_b) \tag{11}$$

$$\mathbf{C}_{c\ ab}{}^{d} = -\frac{1}{2}\alpha^{-1}_{ab}\ (C_{bc}\delta^d_a - C_{ac}\delta^d_b) \tag{12}$$

with

$$\alpha_{12} = \alpha_{13} = q^{-\frac{1}{2}},\ \ \alpha_{24} = \alpha_{34} = q^{\frac{1}{2}},\ \ \alpha_{14} = \alpha_{23} = 1 \tag{13}$$

$$\beta_{12} = \beta_{13} = \beta_{24} = \beta_{34} = q^{-1},\ \ \beta_{14} = q^{-2},\ \beta_{23} = 1,\ \ \beta_{ab} = \beta^{-1}_{ba} \tag{14}$$

Note that the Λ tensor has unit square, i.e.

$$\Lambda^{ij}{}_{kl}\Lambda^{kl}{}_{mn} = \delta^i_m\delta^j_n \tag{15}$$

so that the algebra in (1)-(3) is a *minimal* deformation of $ISO(3,1)$. Deformations of Lie algebras whose braiding matrix has unit square were considered some time ago by Gurevich [6].

The Λ and $\mathbf{C}$ components in (6)-(12) satisfy the following conditions [2]

$$\mathbf{C}_{ri}{}^{n}\mathbf{C}_{nj}{}^{s} - \Lambda^{kl}{}_{ij}\mathbf{C}_{rk}{}^{n}\mathbf{C}_{nl}{}^{s} = \mathbf{C}_{ij}{}^{k}\mathbf{C}_{rk}{}^{s} \quad (q\text{-Jacobi identities}) \tag{16}$$

$$\Lambda^{nm}{}_{ij}\Lambda^{ik}{}_{rp}\Lambda^{js}{}_{kq} = \Lambda^{nk}{}_{ri}\Lambda^{ms}{}_{kj}\Lambda^{ij}{}_{pq} \quad \text{(Yang–Baxter)} \tag{17}$$

$$\mathbf{C}_{mn}{}^{i}\Lambda^{ml}{}_{rj}\Lambda^{ns}{}_{lk} + \Lambda^{il}{}_{rj}\mathbf{C}_{lk}{}^{s} = \Lambda^{pq}{}_{jk}\Lambda^{is}{}_{lq}\mathbf{C}_{rp}{}^{l} + \mathbf{C}_{jk}{}^{m}\Lambda^{is}{}_{rm} \tag{18}$$

$$\mathbf{C}_{rk}{}^{m}\Lambda^{ns}{}_{ml} = \Lambda^{ij}{}_{kl}\Lambda^{nm}{}_{ri}\mathbf{C}_{mj}{}^{s} \tag{19}$$

These are the "bicovariance conditions", see ref.s [7, 8, 9], necessary for the existence of a bicovariant differential calculus (see also the discussion in Appendix B of [1]). Whenever we have a set of matrices $\Lambda^{ij}{}_{kl}$, $\mathbf{C}_{ij}{}^{k}$ satisfying (16)-(19) we can construct a differential calculus on the quantum group $Fun_q(M_i{}^j)$, generated by the elements (adjoint representation of the q-groups) $M_i{}^j$ satisfying the "ΛMM" relations:

$$M_i{}^j M_r{}^q \Lambda^{ir}{}_{pk} = \Lambda^{jq}{}_{ri} M_p{}^r M_k{}^i \tag{20}$$

Consistency of these relations is ensured by the QYB equations (17). One can define in the usual way a coproduct $\Delta(M_i{}^j) = M_i{}^k \otimes M_k{}^j$ and a counit $\varepsilon(M_i{}^j) = \delta_i^j$. When $\Lambda^2 = 1$ one can also define a coinverse $\kappa(M_i{}^j)$ with $\kappa^2 = 1$ (This is done by enlarging the algebra $Fun_q(M_i{}^j)$, see Appendix B of [1]).

The generators χ_i of the q-Lie algebra (5) are functionals on $Fun_q(M_i{}^j)$:

$$\chi_j(M_i{}^k) = \mathbf{C}_{ij}{}^{k} \tag{21}$$

We recall that products of functionals are defined via the coproduct Δ, i.e. $\chi_i\chi_j \equiv (\chi_i \otimes \chi_j)\Delta$, whereas functionals act on products as $\chi_i(ab) = \Delta'(\chi_i)(a \otimes b)$, $a, b \in Fun_q(M_i{}^j)$ (see below the definition of Δ').

Next we introduce new functionals $f^i{}_l$ via their action on the basis $M_k{}^j$:

$$f^i{}_l(M_k{}^j) = \Lambda^{ij}{}_{kl} \tag{22}$$

The co-structures of χ and f are given by:

$$\Delta'(\chi_i) = \chi_j \otimes f^j{}_i + I' \otimes \chi_i \tag{23}$$

$$\varepsilon'(\chi_i) = 0 \tag{24}$$

$$\kappa'(\chi_i) = -\chi_j \kappa'(f^j{}_i) \tag{25}$$

$$\Delta'(f^i{}_j) = f^i{}_k \otimes f^k{}_j \tag{26}$$

$$\varepsilon'(f^i{}_j) = \delta^i_j \tag{27}$$

$$\kappa'(f^i{}_j) = f^i{}_j \circ \kappa \tag{28}$$

The algebra generated by the χ and f is a Hopf algebra, and defines a bicovariant differential calculus on the q-group generated by the $M_i{}^j$ elements. For example,

[2] this can be checked directly without too much effort. In [1] a general proof is given for all $ISO_q(N)$ obtained by the projective method.

one can introduce left-invariant one-forms ω^i as duals to the "tangent vectors" χ_i, an exterior product

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl}\omega^k \otimes \omega^l, \tag{29}$$

an exterior derivative on $Fun_q(M_i{}^j)$ as

$$da = (id \otimes \chi_i)\Delta(a)\omega^i, \quad a \in Fun_q(M_i{}^j) \tag{30}$$

and so on. The commutations between one-forms and elements $a \in Fun_q(M_i{}^j)$ are given by:

$$\omega^i a = (id \otimes f^i{}_j)\Delta(a) \tag{31}$$

The exterior derivative can be extended to the (left-invariant) one-forms via the deformed Cartan-Maurer equations [7, 9]

$$d\omega^i + C_{jk}{}^i \omega^j \wedge \omega^k = 0 \tag{32}$$

The C structure constants appearing in the Cartan-Maurer equations are related to the $\mathbf{C}$ constants of the q-Lie algebra as [9]:

$$\mathbf{C}_{jk}{}^i = C_{jk}{}^i - \Lambda^{rs}{}_{jk} C_{rs}{}^i \tag{33}$$

In the particular case $\Lambda^2 = I$ it is not difficult to see that $C = \frac{1}{2}\mathbf{C}$.

The procedure we have advocated in ref.s [10] for the "gauging" of quantum groups essentially retraces the steps of the group-geometric method for the gauging of usual Lie groups, described for instance in ref.s [2].

We consider one-forms ω^i which are not left-invariant any more, so that the Cartan-Maurer equations are replaced by:

$$R^i = d\omega^i + C_{jk}{}^i \omega^j \wedge \omega^k \tag{34}$$

where the curvatures R^i are now non-vanishing, and satisfy the q-Bianchi identities:

$$dR^i - C_{jk}{}^i R^j \wedge \omega^k + C_{jk}{}^i \omega^j \wedge R^k = 0 \tag{35}$$

due to the Jacobi identities on the structure constants $C_{ij}{}^k$ [9]. As in the classical case we can write the q-Bianchi identities as $\nabla R^i = 0$ (these define the covariant derivative ∇).

Consider now the definition (34) of the curvature R^i, and apply it to the q-Poincaré algebra of (1)-(3): the one-forms dual to χ_{ab}, χ_a are respectively denoted by ω^{ab}, V^a and the corresponding curvatures read (we omit wedge symbols):

$$R^{ab} = d\omega^{ab} + C_{cd}\omega^{ac}\omega^{db} \tag{36}$$

$$R^a = dV^a + \alpha_{af} C_{fb}\omega^{af} V^b \tag{37}$$

where $V_a \equiv C_{ab}V^b$, α_{af} and C_{ab} are given in (13) and (4), and we used $C_{ij}{}^k = \frac{1}{2}\mathbf{C}_{ij}{}^k$. We have also rescaled ω^{ab} by a factor $\frac{1}{2}$ to obtain standard normalizations.

R^{ab} is the q-Lorentz curvature, coinciding with the classical one (as a function of ω^{ab}), and R^a is the q-deformed torsion.

The Bianchi identities, deduced from (35), are:

$$dR^{ab} - C_{fe}R^{af}\omega^{eb} + C_{fe}\omega^{af}R^{eb} = 0 \tag{38}$$

$$dR^a + \alpha_{af}C_{fb}R^{af}V^b - \alpha_{af}C_{fb}\omega^{af}R^b = 0 \tag{39}$$

From the definition (29) of the exterior product we see that for $\Lambda^2 = I$ the one-forms ω^i q-commute as:

$$\omega^i\omega^j = -\Lambda^{ij}{}_{kl}\omega^k\omega^l \tag{40}$$

Inserting the Λ tensor corresponding to (6)-(9) we find:

$$\begin{aligned} V^a\omega^{12} &= -q^{-1}\omega^{12}V^a \\ V^a\omega^{13} &= -q^{-1}\omega^{13}V^a \\ V^a\omega^{14} &= -\omega^{14}V^a \\ V^a\omega^{23} &= -\omega^{23}V^a \\ V^a\omega^{24} &= -q\omega^{24}V^a \\ V^a\omega^{34} &= -q\omega^{34}V^a \end{aligned} \tag{41}$$

$$\begin{aligned} V^2V^1 &= -q^{-1}V^1V^2 \\ V^3V^1 &= -q^{-1}V^1V^3 \\ V^4V^1 &= -q^{-2}V^1V^4 \\ V^3V^2 &= -V^2V^3 \\ V^4V^2 &= -q^{-1}V^2V^4 \\ V^4V^3 &= -q^{-1}V^3V^4 \end{aligned} \tag{42}$$

and usual anticommutations between the ω^{ab} (components of the Lorentz spin connection). The exterior product of two identical one-forms vanishes (this is not true in general when $\Lambda^2 \neq I$). As a consequence, the exterior product of five vielbeins is zero.

We are now ready to write the lagrangian for the q-gravity theory based on $ISO_q(3,1)$. The lagrangian looks identical to the classical one, i.e.:

$$\mathcal{L} = R^{ab}V^cV^d\varepsilon_{abcd} \tag{43}$$

The Lorentz curvature R^{ab}, although defined as in the classical case, has non-trivial commutations with the q-vielbein:

$$\begin{aligned} V^aR^{12} &= q^{-1}R^{12}V^a \\ V^aR^{13} &= q^{-1}R^{13}V^a \\ V^aR^{14} &= R^{14}V^a \\ V^aR^{23} &= R^{23}V^a \\ V^aR^{24} &= qR^{24}V^a \\ V^aR^{34} &= qR^{34}V^a \end{aligned} \tag{44}$$

deducible from the definition (36). As in ref. [10, 9], we make the assumption that the commutations of $d\omega^i$ with the one-forms ω^l are the same as those of $C_{jk}{}^i\omega^j\omega^k$ with ω^l, i.e. the same as those valid for $R^i = 0$. For the definition of ε_{abcd} in (43) see below.

We discuss now the notion of q-diffeomorphisms. It is known that there is a consistent q-generalization of the Lie derivative (see ref.s [9, 11, 12, 1]) which can be expressed as in the classical case as:

$$\ell_V = i_V d + d i_V \tag{45}$$

where i_V is the q-contraction operator defined in ref.s [9, 11], with the following properties:

$$\begin{array}{ll} i) & i_V(a) = 0, \quad a \in A, V \text{ generic tangent vector} \\ ii) & i_{t_i}\omega^j = \delta^j_i I \\ iii) & i_{t_i}(\theta \wedge \omega^k) = i_{t_r}(\theta)\omega^l \Lambda^{rk}{}_{li} + (-1)^p\, \theta\, \delta^k_i, \quad \theta \text{ generic } p\text{-form} \\ iv) & i_V(a\theta + \theta') = a i_V(\theta) + i_V\theta', \quad \theta, \theta' \text{generic forms} \\ v) & i_{\lambda V} = \lambda i_V, \quad \lambda \in \mathbf{C} \\ vi) & i_{\varepsilon V}(\theta) = i_V(\theta)\varepsilon, \quad \varepsilon \in A \end{array} \tag{46}$$

As a consequence, the q-Lie derivative satisfies:

$$\begin{array}{ll} i) & \ell_V a = i_V(da) \equiv V(a) \\ ii) & \ell_V d = d\ell_V \\ iii) & \ell_V(\lambda\theta + \theta') = \lambda\ell_V(\theta) + \ell_V(\theta') \\ iv) & \ell_{\varepsilon V}(\theta) = (\ell_V\theta)\varepsilon - (-1)^p i_V(\theta) d\varepsilon, \quad \theta \text{ generic p-form} \\ v) & \ell_{t_i}(\theta \wedge \omega^k) = (\ell_{t_r}\theta) \wedge \omega^l \Lambda^{rk}{}_{li} + \theta \wedge \ell_{t_i}\omega^k \end{array} \tag{47}$$

In analogy with the classical case, we define the q-diffeomorphism variation of the fundamental field ω^i as

$$\delta\omega^k \equiv \ell_{\varepsilon^i t_i}\omega^k \tag{48}$$

where according to iv) in (47):

$$\ell_{\varepsilon^i t_i}\omega^k = (i_{t_i} d\omega^k + d i_{t_i}\omega^k)\varepsilon^i + d\varepsilon^k = (i_{t_i} d\omega^k)\varepsilon^i + d\varepsilon^k \tag{49}$$

As in the classical case, there is a suggestive way to write this variation:

$$\ell_{\varepsilon^i t_i}\omega^k = i_{\varepsilon^i t_i} R^k + \nabla\varepsilon^k \tag{50}$$

where

$$\begin{array}{c} \nabla\varepsilon^k \equiv d\varepsilon^k - C_{rs}{}^k i_{t_i}(\omega^r \wedge \omega^s)\varepsilon^i = \\ d\varepsilon^k - C_{rs}{}^k \varepsilon^r\omega^s + C_{rs}{}^k \omega^r\varepsilon^s \end{array} \tag{51}$$

Proof: use the Bianchi identities (35) and iii) in (46).

Notice that if we postulate:

$$\Lambda^{rk}{}_{li}\omega^l\varepsilon^i = \varepsilon^r\omega^k$$
$$\Lambda^{rk}{}_{li}\omega^l \wedge d\varepsilon^i = -d\varepsilon^r \wedge \omega^k \tag{52}$$

we find

$$\delta(\omega^j \wedge \omega^k) = \delta\omega^j \wedge \omega^k + \omega^j \wedge \delta\omega^k \tag{53}$$

i.e. a rule that any "sensible" variation law should satisfy. To prove (53) use iv) and v) of (47). The q-commutations (52) were already proposed in [10] in the context of q-gauge theories. A consequence of (52) are the following commutations between the variation parameter and $d\omega^i$:

$$\Lambda^{rk}{}_{li}d\omega^l\varepsilon^i = \varepsilon^r d\omega^k \tag{54}$$

As discussed in ref.s [10], it is consistent to postulate that R^i has the same commutations with ε as $d\omega^i$:

$$\Lambda^{rk}{}_{li}R^l\varepsilon^i = \varepsilon^r R^k \tag{55}$$

We have now all the tools we need to investigate the invariances of the q-gravity lagrangian (43). The result will be analogous to the classical one: after imposing the horizontality conditions

$$i_{t_{ab}}R^{cd} = i_{t_{ab}}R^c = 0 \tag{56}$$

along the Lorentz directions one finds that, *provided* the ε tensor in (43) is appropriately defined, the lagrangian is invariant under q-diffeomorphisms and local Lorentz rotations. Note that, as in the $q = 1$ case, the horizontality conditions (56) can be obtained as field equations (see later).

We first consider Lorentz rotations. Under these, the curvature R^{ab} and the vielbein V^c transform as:

$$\delta R^{ab} \equiv \ell_{\varepsilon^{gh}t_{gh}}R^{ab} = \mathbf{C}_{ef}{}_{gh}{}^{ab}R^{ef}\varepsilon^{gh} - \mathbf{C}_{gh}{}_{ef}{}^{ab}\varepsilon^{gh}R^{ef} \tag{57}$$

$$\delta V^c \equiv \ell_{\varepsilon^{gh}t_{gh}}V^c = -\mathbf{C}_{ef}{}_{g}{}^{c}\varepsilon^{ef}V^g + \mathbf{C}_{g}{}_{ef}{}^{c}V^g\varepsilon^{ef} \tag{58}$$

To obtain these variations, use the definition (45), iv) of (47), the Bianchi identity (38) and the horizontality conditions (56).

Now we have the lemma:

$$\delta\mathcal{L} = [(\delta R^{ab})V^cV^d + R^{ab}(\delta V^c)V^d + R^{ab}V^c(\delta V^d)]\varepsilon_{abcd} \tag{59}$$

Proof: use v) of (47) and the first of (52).

Inserting the variations (57) and (58) inside (59) we find, after ordering the terms as ϵRVV with (52) and (55):

$$\delta\mathcal{L} = 2(\mathbf{C}_{ef\ gh}{}^{ab}\varepsilon_{abcd} - \mathbf{C}_{gh\ c}{}^{s}\varepsilon_{efsd} - \mathbf{C}_{rs\ d}{}^{p}\Lambda^{q\ rs}{}_{gh\ c}\varepsilon_{efqp})\varepsilon^{gh}R^{ef}V^cV^d \tag{60}$$

Using the explicit form of the Λ and $\mathbf{C}$ tensors into (6)-(12) and imposing $\delta\mathcal{L} = 0$, yields a set of equations for the ε_{abcd} tensor. These can in fact be solved and yield:

$$\begin{array}{llll}
\varepsilon_{1234} = 1, & \varepsilon_{1243} = -q, & \varepsilon_{1324} = -1, & \varepsilon_{1342} = q, \\
\varepsilon_{1423} = q^{\frac{3}{2}}, & \varepsilon_{1432} = -q^{\frac{3}{2}}, & \varepsilon_{2134} = -1, & \varepsilon_{2143} = q, \\
\varepsilon_{3124} = 1, & \varepsilon_{3142} = -q, & \varepsilon_{4123} = -q^{\frac{3}{2}}, & \varepsilon_{4132} = q^{\frac{3}{2}}, \\
\varepsilon_{2314} = q^{\frac{1}{2}}, & \varepsilon_{2341} = -q^{\frac{5}{2}}, & \varepsilon_{2413} = -q^2, & \varepsilon_{2431} = q^3, \\
\varepsilon_{3214} = -q^{\frac{1}{2}}, & \varepsilon_{3241} = q^{\frac{5}{2}}, & \varepsilon_{4213} = q^2, & \varepsilon_{4231} = -q^3, \\
\varepsilon_{3412} = q^2, & \varepsilon_{3421} = -q^3, & \varepsilon_{4312} = -q^2, & \varepsilon_{4321} = q^3
\end{array} \tag{61}$$

Consider next the variation of L under q-diffeomorphisms, i.e.:

$$\delta\mathcal{L} = \ell_{\varepsilon^g t_g}\mathcal{L} = (\ell_{t_g}\mathcal{L})\varepsilon^g - (i_{t_g}\mathcal{L})d\varepsilon^g = \tag{62}$$

$$= d[i_{t_g}(R^{ab}V^cV^d\varepsilon_{abcd})\varepsilon^g] + i_{t_g}[d(R^{ab}V^cV^d\varepsilon_{abcd})]\varepsilon^g \tag{63}$$

Then the variation $\delta\mathcal{L}$ is a total derivative if

$$d(R^{ab}V^cV^d\varepsilon_{abcd}) = 0 \tag{64}$$

After using the expression for dR^{ab} given by the Bianchi identity (38) and the torsion definition (37) to find dV^a (note that $R^{ab}R^cV^d\varepsilon_{abcd} = 0$ because of horizontality of R^{ab}, R^c and the vanishing of the product of five vielbeins), eq. (64) yields a set of conditions on ε_{abcd}. These conditions in fact *coincide* with those found to ensure local Lorentz invariance of the q-lagrangian. This is not a miracle: indeed we could have computed the Lorentz variation of $\mathcal{L}$ in the same way as in (63); the total derivative term would have vanished because $i_{t_{gh}}(R^{ab}V^cV^d) = 0$ (horizontality of R^{ab}), and we would have found again eq. (64) as a condition for $\delta\mathcal{L} = 0$.

Thus the lagrangian (43) with the ε_{abcd} tensor as given in (61) is invariant (up to a total derivative) also under q-diffeomorphisms.

We discuss now the algebra of q-Lie derivatives. We have the theorem, analogous to the classical one:

$$\ell_{t_i}\ell_{t_j} - \Lambda^{kl}{}_{ij}\ell_{t_k}\ell_{t_l} = \ell_{(\mathbf{C}_{ij}{}^{n} - \mathbf{R}^n{}_{ij})t_n} \tag{65}$$

with

$$R^i \equiv R^i{}_{jk}\omega^j \wedge \omega^k \tag{66}$$

$$\mathbf{R}^i{}_{jk} \equiv R^i_{jk} - \Lambda^{rs}{}_{jk}R^i_{rs} \tag{67}$$

As for the structure constants, we have $R^i_{jk} = \frac{1}{2}\mathbf{R}^i{}_{jk}$ when $\Lambda^2 = 1$. The proof of the composition law (65) is computational: one applies its left-hand side to ω^k, and uses the properties of the q-Lie derivative. *Hint 1*: rewrite the Lie derivative of ω^k as:

$$\ell_{t_j}\omega^k = (\mathbf{C}_{rj}{}^k - \mathbf{R}^k{}_{rj})\omega^r \tag{68}$$

Hint 2: use the following expression for $\Lambda^{ij}{}_{kl}$ (no sums on repeated indices) :

$$\Lambda^{ij}{}_{kl} = [kl]\delta^i_l\delta^j_k \tag{69}$$

and the identities

$$[kl] = \frac{1}{[lk]} \tag{70}$$

$$\mathbf{C}_{rk}{}^s[lk][lr] = \mathbf{C}_{rk}{}^s[ls] \tag{71}$$

due to $\Lambda^2 = 1$ and the bicovariance condition (19).

From the q-algebra (65) it is not difficult to find the following composition law for q-variations:

$$\ell_{\varepsilon_2^i t_i}\ell_{\varepsilon_1^j t_j} - (1 \leftrightarrow 2) = \ell_{(\mathbf{C}_{ij}{}^n - \mathbf{R}^n{}_{ij})\varepsilon_2^i\varepsilon_1^j t_n} \tag{72}$$

The order of the factors is important in the composite parameter $(\mathbf{C}_{ij}{}^n - \mathbf{R}^n{}_{ij})\varepsilon_2^i\varepsilon_1^j$. Note also that

$$(\mathbf{C}_{ij}{}^n - \mathbf{R}^n{}_{ij})\varepsilon_2^i\varepsilon_1^j = \frac{1}{2}(\mathbf{C}_{ij}{}^n - \mathbf{R}^n{}_{ij})(\varepsilon_2^i\varepsilon_1^j - \varepsilon_1^i\varepsilon_2^j) \tag{73}$$

if we postulate the commutation rule

$$\varepsilon_1^i\varepsilon_2^j = \Lambda^{ij}{}_{kl}\varepsilon_2^k\varepsilon_1^l \tag{74}$$

Indeed $(\mathbf{C}_{ij}{}^n - \mathbf{R}^n{}_{ij})\Lambda^{ij}{}_{kl} = -(\mathbf{C}_{kl}{}^n - \mathbf{R}^n{}_{kl})$ (due to $\Lambda^2 = 1$). Then the composite parameter is explicitly $(1 \leftrightarrow 2)$ antisymmetric.

Let us derive the equations of motion corresponding to the q-lagrangian (43). We assume the same variational rule as with the Lie derivative. The q-Einstein equations are obtained by varying the vielbein in $\mathcal{L}$:

$$\delta\mathcal{L} = R^{ab}(\delta V^c V^d + V^c\delta V^d)\varepsilon_{abcd} = 0 \tag{75}$$

Postulating that $\delta\omega^i$ has the same commutations as ω^i, and noticing that

$$\varepsilon_{abef} = -\Lambda^{cd}{}_{ef}\varepsilon_{abcd} \tag{76}$$

(use the explicit entries in (9) and (61), or notice that since ε_{abcd} multiplies V^cV^d in (43), it must be Λ-antisymmetric in the indices c,d), one arrives at:

$$R^{ab}V^e\varepsilon_{abef} = 0 \tag{77}$$

The q-Einstein equations are found as in the classical case: expand (77) along three vielbeins:

$$R^{ab}{}_{cd}V^cV^dV^e\varepsilon_{abef} = 0, \tag{78}$$

multiply by another vielbein V^g and use:

$$\varepsilon^{cdeg}V^1V^2V^3V^4 \equiv V^cV^dV^eV^g \tag{79}$$

(N.B.: the entries of ε^{abcd} are different from those of ε_{abcd}) so that finally we have:

$$R^{ab}{}_{cd}\varepsilon^{cdeg}\varepsilon_{abef} = 0 \tag{80}$$

The contraction of the two alternating tensors yields a q-weighted product of Kronecker deltas. We leave to the reader to find the final form of the q-Einstein equations. Expanding (77) on ωVV, $V\omega V$ and $\omega\omega V$ yields instead the horizontality condition on R^{ab}.

The torsion equation is obtained by varying (43) in the spin connection ω^{ab}. The final result is again an equation that formally looks identical to the classical one:

$$R^cV^d\varepsilon_{abcd} = 0 \tag{81}$$

As for $q = 1$, this equation implies that the torsion vanishes as a two-form: $R^c = 0$ (hence also horizontality of R^c).

Note 1: The q-volume 4-form $V^1V^2V^3V^4 = f(q)V^aV^bV^cV^d\varepsilon_{abcd}$ (with $f(q) = \varepsilon^{abcd}\varepsilon_{abcd}$) is invariant under Lorentz rotations and, up to a total derivative, under q-diffeomorphisms. The proof is similar to the one used for the lagrangian. This means that the q-symmetries allow a cosmological constant term $V^aV^bV^cV^d\varepsilon_{abcd}$.

Note 2: It would be worthwhile to give a recipe for extracting numbers out of a q-theory of the kind discussed in this Letter. We think that there are two possible ways of doing this:

i) by finding a consistent definition of path-integral on q-commuting fields, leading to C-number amplitudes. On this question, see for example [13].

ii) by representing q-commuting objects as matrices of ordinary C-numbers. For instance, $\omega^i = \xi^i\eta^i$ (no sum on i) where ξ^i are ordinary anti-commuting one-forms and $\eta^i \equiv (\eta^i)^\alpha{}_\beta$ are constant matrices. Then the q-lagrangian becomes a C number after taking its (ordinary) trace.

Acknowledgements

It is a pleasure to thank the Centro Brasileiro de Pesquisas Físicas of Rio de Janeiro, where this work was completed, for its warm hospitality.

References

[1] L. Castellani, *Differential calculus on $ISO_q(N)$, quantum Poincaré algebra and q-gravity*, Torino preprint DFTT-70/93.

[2] L. Castellani, R. D'Auria and P. Fré, *Supergravity and Superstrings: a geometric perspective* (World Scientific, Singapore, 1991); L. Castellani, Int. J. Mod. Phys. **A7** (1992) 1583.

[3] Y. Ne'eman and T. Regge, Riv. Nuovo Cim. **5** (1978) 1; Phys.Lett.**74B** (1978) 54; A. D'Adda, R. D'Auria, P. Fré, T. Regge, Riv. Nuovo Cim. **6** (1980) 3.

[4] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Algebra and Analysis, **1** (1987) 178.

[5] E. Celeghini, R. Giachetti, A. Reyman, E. Sorace, M. Tarlini, *$SO_q(n+1,n-1)$ as a real form of $SO_q(2n,\mathbf{C})$*, Lett. Math. Phys. **23** (1991) 45.

[6] D.I. Gurevich, Soviet Math. Dokl., **33** (1986) 758.

[7] S.L. Woronowicz, Publ. RIMS, Kyoto Univ., Vol. **23**, 1 (1987) 117; Commun. Math. Phys. **111** (1987) 613 and Commun. Math. Phys. **122**, (1989) 125.

[8] D. Bernard, *Quantum Lie algebras and differential calculus on quantum groups*, Progress of Theor. Phys. Suppl. **102** (1990) 49; Phys. Lett. **260B** (1991) 389.

[9] P. Aschieri and L. Castellani, Int. Jou. Mod. Phys. **A8** (1993) 1667.

[10] L. Castellani, Phys. Lett. **B292** (1992) 93; *$U_q(N)$ gauge theories*, Torino preprint DFTT 74/92.

[11] P. Aschieri, *The space of vector fields on quantum groups*, UCLA preprint UCLA/93/TEP/25.

[12] P. Schupp, P. Watts and B. Zumino, Lett. Math. Phys. **25** (1992) 139; *Cartan calculus for Hopf algebras and quantum groups*, NSF-ITP-93-75, LBL-34215 and UCB-PTH-93/20; P. Schupp, *Quantum groups, non-commutative differential geometry and applications*, Ph.D. Thesis, LBL-34942 and UCB-PTH-93/35.

[13] L. Baulieu and E.G. Floratos, Phys. Lett. **B258** (1991) 171.