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GENERALIZED BOND PERCOLATION AND STATISTICAL MECHANICS

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ABSTRACT

A generalization of traditional bond percolation is performed, in the sens that bonds have now the possibility of partially transmitting the information (a fact which leads to the concept of "fidelity" of the bond), and also in the sens that, besides the normal tendency to equiprobability, the bonds are allowed to substantially change the information. Furthermore the fidelity is allowed to become an aleatory variable, and the operational rules concerning the associated distribution laws are determined. Thermally quenched random bonds and the whole body of Statistical Mechanics become particular cases of this formalism, which is in general adapted to the treatment of all problems whose main characteristic is to preserve a part of the information through a long path or array (critical phenomena, regime changements, thermal random models, etc). Operationally it provides a quick method for the calculation of the equivalent probability of complex clusters within the traditional bond percolation problem.

KEY WORDS: Bond percolation; Statistical Mechanics; Quenched bond; Random models.

RÉSUMÉ

Le problème traditionnel de la percolation de liaisons est géneralisé dans le sens que les liaisons ont maintenant la possibilité de transmettre seulement une partie de l'information (ce qui amène au concept de "fidélité" de la liaison), et aussi dans le sens que, mise à part la tendance normale à l'équiprobabilité, les liaisons peuvent changer essentiellement l'information. De plus la fidélité est traitée comme une variable aléatoire, et les règles opérationnelles liées aux distributions de probabilités associées sont determinées. Les liaisons aléatoires thermiquement "étoufées" et tout le corps de la Mécanique Statistique deviennent des cas particuliers de ce formalisme, qui est d'une façon générale adapté pour le traitement des problèmes dont la principale caractéristique est de préserver une partie de l'information au long d'un grand cheminement (phénomènes critiques, changements de régime, modèles thermiques aléatoires, etc). Du point de vue opérationnel il fournit une méthode rapide pour calculer la probabilité equivalente d'amas complexes dans le cadre du problème traditionnel de percolation de liaisons.

MOTS CLÉS: Percolation de liaisons; Mécanique Statistique; Liaisons "étoufées"; Modèles aléatoires.

I - INTRODUCTION

The percolation problem has been extensively studied in last years (for a review see the papers of Shante and Kirkpatrick[1] and of Essam[2]) because of its numerous applications in several areas of Physics (dilute magnetism, spin-glasses, conductivity of alloys, macromolecules, gel-sol transition, etc) or even other branches of knowledge. Different approaches exist[3-14], including the relatively recent renormalization-group[15-19] treatments, for either the site or the bond percolation.

Our purpose in the present work is to analyze and generalyze the second one, in the sens that the bonds are traditionally restricted to be either completely active or completely blocked, and we intend here to leave to them the possibility of being partially active similarly to the thermally quenched bonds; this fact will lead to the concept of "fidelity" of the bond. On the other hand, the traditional bond, when (completely) blocked, "forgets" the information (leading to equiprabability at the output); in other words, no possibility is left for an alteration of the information bigger than to forget. This restriction is also raised in the present formalism, and this leads us to another parameter called "creativity". Furthermore the fidelity is left to become an aleatory variable by introducing probability distributions, like in thermal random models[10,14,19]. The mathematical structure of these distributions is analized, and the main characteristics are exhibited in a certain amount of examples. Quick operational rules are derived for the treatment of the traditional bond percolation problem in what concerns the calculation of clusters. And finally, we reobtain the thermally quenched random bonds and the formalism of Statistical Mechanics (Boltzmann's entropy formula, microcanonical and canonical ensembles, etc) as particular cases.

In Section II, the fidelity and creativity concepts are introduced and the basic series and parallel configurations are analyzed; in Section III the probability distribution formalism is presented, and several examples are discussed; and finally, in Section IV the relation to thermally quenched random bonds and to Statistical Mechanics is exhibited.

II - FIDELITY AND CREATIVITY

II.1 - Formulation

Let us consider a simple bond between two sites of an array which might be regular or not. This bond may be seen as an agent which transmits information with a certain <u>fidelity</u> a (which is defined in the interval $[0,1]$ by relations we shall state later). We intend to test the fidelity of a given bond by verifying how well it transmits information concerning a single binary aleatory variable (which will conventionally take the values 1 and 2). In other words the input will be the probabilities $p(1)$ and $p(2)$ (satisfying $p(1)+p(2)=1$) and the output will be the probabilities $p'(1)$ and $p'(2)$ (naturally also satisfying $p'(1)+p'(2)=1$). Let us put this in more operational terms, assuming a linear transformation between the input and the output, this is to say

$$\begin{aligned} p'(1) &= f(a)\, p(1) + g(a)\, p(2) \\ p'(2) &= [1-f(a)]\, p(1) + [1-g(a)]\, p(2) \end{aligned} \qquad (1)$$

where $0 \leq f(a), g(a) \leq 1 \ \forall a \in [0,1]$. It is straightforward to verify that this is the most general linear relation which transforms a binary probability law into another.

Let us now impose that input and output coincide when the fidelity equals one, in other words

$$f(1)= 1 \qquad \text{and} \qquad g(1)= 0 \tag{2}$$

In the other extremity ($a= 0$) there is no need to make a rigid imposition, and this leads us to a classification of the bonds, namely we shall say that our bond is an ideally dissipative one if

$$f(0)= g(0)= 1/2 \tag{3}$$

an ideally creative one if

$$f(0)= 1- g(0)= 0 \tag{3'}$$

and an intermediate one if

$$f(0) = j(\alpha) \qquad \text{and} \qquad g(0) = h(\alpha) \ ,$$

where the creativity coefficient α is defined in the interval $[0,1]$, and leads to the ideally dissipative bond in the limit $\alpha = 0$ by imposing $j(0) = h(0)= 1/2$; and to the ideally creative bond in the limit $\alpha = 1$ by imposing $j(1) = 0$ and $h(1) = 1$. The simplest functions which satisfy these facts are

$$\begin{aligned} f(0) &= (1-\alpha)/2 \\ g(0) &= (1+\alpha)/2 \end{aligned} \tag{3''}$$

Within restrictions (2) and (3'') we still conserve a great freedom in what concerns the choice of $f(a)$ and $g(a)$; we shall make the simplest one, namely

$$\begin{aligned} f(a) &= \frac{1-\alpha}{2} + \frac{1+\alpha}{2} a \\ g(a) &= \frac{1+\alpha}{2} (1-a) = 1 - f(a) \end{aligned} \tag{4}$$

These expressions substituted in transformations (1) lead to what we shall consider as the operational definitions of fidelity a and creativity α .Let us however point out that if we are restricted to perform only the basic "experience" which consists in "presenting" to our bond an input and "measuring" the output,we shall not be able to determine simultaneously a and α,as definition (4) into (1) leaves a degree of freedom.But if we are permited to control let us say the fidelity (as it is usually the case in thermal systems by controlling the temperature), then we shall be able to "measure" the creativity.To make this point more clear let us assume we are allowed to vanish the fidelity:in this case if we present to the bond the input $p(1)=1$,the measure of the output $p'(1)$ will directly give α by the relation $\alpha = 1-2p'(1)$.Let us finally remark that the particular case $a=\frac{\alpha}{1+\alpha}$ leads to equiprobability in all cases.

II.2 - Series configuration

Let us now assume two ideally dissipative bonds (with fidelities a_1 and a_2 respectively) in series,and let us determine (assuming the transformation law (4) with $\alpha=0$) the fidelity a_s of the whole array (see Fig. 1).It is sufficient to write down the probabilities for the binary value 1:

$$p''(1)=\frac{1+a_2}{2}p'(1)+\frac{1-a_2}{2}p'(2) = \frac{1+a_1a_2}{2}p(1)+\frac{1-a_1a_2}{2}p(2)$$

so,if we consider the array also as an ideally dissipative bond,it comes

$$a_s = a_1a_2 \qquad (5)$$

relation that we intend to exploit later.Obviously,this relation generalizes, for n bonds in series, into

$$a_s = \prod_{i=1}^{n} a_i \qquad (5')$$

If we perform the above operations for two ideally creative bonds in series, it comes

$$a_s = 1-a_1-a_2+2a_1a_2 \qquad (6)$$

Among the particular cases of (6), exists an amusing one: $a_1 = a_2 = 0$ leads to $a_s \equiv 1$ (two perfect lies restore the truth!).

If we assume finally that the two bonds in series have creativities α_1 and α_2 respectively, and we interpret the whole array as a bond with creativity α_s, it comes

$$(1+\alpha_s)a_s-\alpha_s=(\alpha_1-a_1)(\alpha_2-a_2)-\alpha_1\alpha_2(a_1+a_2)$$
$$+(\alpha_1+\alpha_2)a_1a_2+\alpha_1\alpha_2a_1a_2 \qquad (7)$$

In the particular case $\alpha_1=\alpha_2=\alpha_s\equiv\alpha$ this expression becomes

$$a_s = a_1a_2+\alpha(1-a_1)(1-a_2) \qquad (7')$$

Remark also that expression (7) is invariant under permutation of the bonds 1 and 2, which means that the series array has no priveleged sens for the transmission of information. Let us finally call the attention onto the fact that relation (7) leaves to us a degree of freedom (for example α_s).

II.3 - Parallel configuration

In this Section we shall be concerned only with ideally dissipative bonds which will be assumed in parallel configuration. We expect for the fidelity a_p of the array as a whole to have the following basic properties:

1) a_p to be bigger than the biggest of our set a_i $(i=1,2,\ldots,m)$;
2) $\lim_{m\to\infty} a_p = 1$ as long as we have an infinite number of parallel bonds with non vanishing fidelities;
3) a bond whose fidelity vanishes is, in what concerns a_p, like if it was not there;
4) a_p equals unity whenever is there at least one bond whose fidelity equals one (corollary of property (1));
5) a_p is invariant under any permutation of the parallel bonds.

All these properties are satisfied by the simple following relation

$$1-a_p = \prod_{i=1}^{m} (1-a_i) \tag{8}$$

which, for $m=2$, becomes

$$a_p = a_1 + a_2 - a_1 a_2 \tag{8'}$$

In the more general case where the bonds may have a certain amount of creativity ($\alpha_i \neq 0$), the parallel messages may be frankly contradictory, a fact which will lead to the appearance of the phenomenon of "frustration". It is out of the aim of this paper to analyze this eventuality, however let us point out that we have voluntarily used the word "frustration" introduced by Toulouse[20,21] for spin-glasses, because we believe that the core of the concept is the same.

III - ALEATORY FIDELITY

III.1 - Generalities

Let us now work with bonds of known and equal nature (one fixed value α for the creativity of all of them) but whose fidelity a is an aleatory variable, and let us note $P_i(a)$ the associated probability distributions ($P_i(a) \geq 0$ $\forall a \in [0,1]$ and

$\int_0^1 da\ P_i(a) = 1$).Situations where the bonds could be of different natures (or even where the nature itself could be an aleatory variable) may be treated similarly to what we shall present next.

III.2 - Series configuration

We assume here two bonds (with probability distributions $P_1(a)$ and $P_2(a)$ respectively,and common creativity α) in series configuration.We intend to construct next the probability distribution $P_s(a)$ of the array as a whole,which is assumed to have the same creativity α .If bonds 1 and 2 have fidelities a' and a'' respectively,then the fidelity of the set will be given by (7'),and its contribution to $P_s(a)$ will be essentially given by $P_1(a')P_2(a'')$.However many other couples of values (a',a'') will contribute onto the same point,so we shall finally have that

$$P_s(a) = \int da'\ N(a')P_1(a')P_2\left(\frac{a-\alpha(1-a')}{a'-\alpha(1-a')}\right) \qquad (10)$$

where $N(a')$ is a function introduced for norm purposes.The particular case* $P_1(a) = \delta(a-a_1)$ and $P_2(a) = \delta(a-a_2)$ must lead,through product (10) and by using (7'),to

$$P_s(a) = \delta\,(a-a_1a_2-\alpha(1-a_1)(1-a_2)).$$

This imposition straightforwardly leads to

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* In order to have $\int_0^1 \delta(a-a_0)\ da = 1 \quad \forall a_0 \in [0,1]$,we adopt the following definition for the δ-function: $\delta(a-a_0) = \delta_{Dirac}(a-a_0)$ if $0 < a_0 < 1$,and $\delta(a-a_0) = 2\,\delta_{Dirac}(a-a_0)$ if $a_0 = 0,1$.

$$N(a) = \left(\left| (1+\alpha)a - \alpha \right| \right)^{-1} \qquad \text{(see Fig. 2)}$$

and (10) may be rewritten as follows

$$P_s(a) = \int \frac{da'}{\left|(1+\alpha)a' - \alpha\right|} P_1(a') P_2\left(\frac{a-\alpha(1-a')}{a'-\alpha(1-a')}\right) \qquad (10')$$

where the integration domain is given (see Fig. 3) by

$a' \in [0,a]$ and $a' \in \left[\frac{\alpha - a}{\alpha}, 1\right]$ if $0 \leq a < \frac{\alpha}{1+\alpha}$,

$a' \in [0,1]$ if $a = \frac{\alpha}{1+\alpha}$,

$a' \in \left[0, \frac{\alpha - a}{\alpha}\right]$ and $a' \in [a,1]$ if $\frac{\alpha}{1+\alpha} < a \leq \alpha$,

$a' \in [a,1]$ if $\alpha < a \leq 1$.

Remark that the product (10') takes, for ideally dissipative bonds ($\alpha = 0$) the simple form

$$P_s(a) = \int_a^1 \frac{da'}{a'} P_1(a')\, P_2(a/a') \equiv P_1 \circledS P_2 \qquad (10'')$$

which, in a certain way, is to multiplication what the convolution product is to sum. We easily verify that $P_s(a)$ is normed:

$$\int_0^1 da\, P_s(a) = \int_0^1 da \int_a^1 \frac{da'}{a'} P_1(a')P_2(a/a') = \int_0^1 da' \int_0^{a'} \frac{da}{a'} P_1(a')P_2(a/a')$$

$$= \int_0^1 da'\, P_1(a') \int_0^1 dx\, P_2(x) = 1$$

Furthermore product (10'') (called series product or $\circledS$ -product from now on) is commutative, associative and admits a neutral element but not an inverse, in other words it has the mathematical structure of a monoid. This fact can be proved either by direct verification of each property, or by remarking[22] that the variable changement $a=e^{-x}$ isomorphically transforms the series product into a convolution product. The neutral element of the series product is $I_s(a)=\delta(a-1)$, which clearly corresponds to the intuitive idea that the information be not deteriorate. Within the same ideas it is clear to understand why the series product admits no inverse (once the information has been partially destroyed, there is no way to recover it back). This last point makes an essential difference with the ideally creative bonds, where it is sometimes possible to completely restore the original information (see in Section II.2 the example of the two lies cancelling each other).

III.3 - Particular cases

In order to achieve a more intuitive insight of the series product, we shall list in this Section several examples. Let us introduce the notation $P^{Ⓢ n}$ to mean the n-factor product $P Ⓢ P Ⓢ \dots Ⓢ P$.

1st example: $P(a)=\delta(a-a_o)$ leads to $P^{Ⓢ n}=\delta(a-a_o^n)$. We see that there are only two fixed points, namely $a_o=1$ (perfect transmission of the information) and $a_o=0$ (complete loss of the information, i.e. immediate equiprobabilization of the possible events). We also see that, for $a_o<1$, $\lim_{n\to\infty} P^{Ⓢ n}=\delta(a)$, a property we shall verify in other examples.

2nd example: Let us present here the traditional bond percolation (probabilities p of being active and (1-p) of being blocked) : $P(a)=(1-p)\,\delta(a)+p\,\delta(a-1)$. We obtain $P^{Ⓢ n}=(1-p^n)\,\delta(a)+p^n\,\delta(a-1)$, which clearly means that, for percolating through n bonds in series, all of them must be active. We also verify that, for $p<1$, $\lim_{n\to\infty} P^{Ⓢ n}=\delta(a)$.

3rd example: If we consider $P(a) = (1+r)a^r$ $(r \geq 0)$, we obtain

$$P^{\circledS n} = (1+r)^n \frac{a^r}{(n-1)!} \ln^{n-1} \frac{1}{a}$$

Remark that, for increasing n, there is a more and more pronounced transfer of probability density from the neighborhood of a=1 to the neighborhood of a=0 (continuous and irreversible loss of information, similarly to what happens with the entropy of an isolated thermodynamical system during the "travel" towards equilibrium). Let us prove that $\lim_{n\to\infty} P^{\circledS n} = \delta(a)$ by verifying that, for any function f(a) analytic in a = 0, we have that

$$\lim_{n\to\infty} \int_0^1 da\, P^{\circledS n}(a)\, f(a) = f(0) \qquad (11)$$

By considering that $f(a) = f(0) + f'(0)\, a + \dots$, statement (11) will be proved if

$$\lim_{n\to\infty} \int_0^1 da\, P^{\circledS n}(a)\, a = 0 \quad ,$$

and this is so because

$$\langle a \rangle_n \equiv \int_0^1 da\, P^{\circledS n}(a)\, a = \left(\frac{1+r}{2+r}\right)^n$$

Incidentally we verify that $\langle a \rangle_n = (\langle a \rangle_1)^n$, a result which is typical of series arrays.

4th example: Let us consider

$$P(a) = (\lambda - \mu)^{-1} \quad \text{if } \mu \leq a \leq \lambda \; ,$$
$$= 0 \quad \text{otherwise,}$$

with $0 \leq \mu < \lambda \leq 1$. We obtain (see Fig. 4(a))

$$P^{\circledS 2} = (\lambda-\mu)^{-2} \ln \frac{a}{\mu^2} \quad \text{if} \quad \mu^2 \leq a \leq \lambda\mu$$
$$= (\lambda-\mu)^{-2} \ln \frac{\lambda^2}{a} \quad \text{if} \quad \lambda\mu \leq a \leq \lambda^2 .$$

Once again we verify the shift towards $a = 0$.

<u>5^{th} example</u>: If we consider the Lorentzian distribution

$$P(a) = \frac{1}{\operatorname{arctg}\frac{1}{\lambda}} \frac{\lambda}{a^2+\lambda^2} \qquad (\lambda > 0)$$

we obtain

$$P^{Ⓢ 2} = \frac{\lambda^2}{(a^2-\lambda^4)\operatorname{arctg}^2\frac{1}{\lambda}} \ln\frac{a(1+\lambda^2)}{a^2+\lambda^2} \qquad \text{(see Fig. 4(b))}$$

<u>6^{th} example</u>: Let us finally consider a mixtured situation:

$P_1 = -\ln a$,

$P_2 = 1/\lambda$ if $0 \leqslant a \leqslant \lambda$ and vanishes otherwise $(0 < \lambda \leqslant 1)$,

$P_3 = 2a$.

We straightforwardly obtain

$$P_1 Ⓢ P_2 Ⓢ P_3 = \frac{1}{\lambda}\left[\left(\ln\frac{\lambda}{a} - 1\right)^2 + 1 - \frac{2a}{\lambda}\right] \qquad \text{if } a \leqslant \lambda ,$$

$$= 0 \text{ otherwise} .$$

This distribution is "worse" (dense near $a = 0$) than the "worst" among the three distributions we have considered.

Let us close this list of examples by stating (though we have not attempted a general proof) that if we have an infinite set of distributions $\{P_i\}$ with the unique restriction that they be different from $\delta(a-1)$,it will be

$$\lim_{n\to\infty} \left\{ P_1 Ⓢ P_2 Ⓢ \ldots Ⓢ P_n \right\} = \delta(a) \qquad (12)$$

III.4 - <u>Parallel configuration</u>

Let us now consider our two ideally dissipative bonds in a parallel configuration,and let us note $P_p(a)$ the resulting probability distribution.The same type of arguments used in the series configuration lead, by using relation (8') instead of (7'), to

$$P_p(a) = \int_0^a \frac{da'}{1-a'} P_1(a') P_2\left(\frac{a-a'}{1-a'}\right) \equiv P_1 \circledcirc\!\!\!\!p\; P_2 \qquad (13)$$

Let us introduce the complementary distribution $\overline{P}(a)$ of a given one $P(a)$ by the following definition:

$$\overline{P}(a) = P(1-a) \qquad \forall\, a \in [0,1] \qquad (14)$$

A corollary we shall use later is that $\overline{P} = P$ whenever $P(a)$ is symmetric with respect to the axis $a = 1/2$. Furthermore it is straightforward to prove, by using (13) and (14), the following general statement

$$P_3 = P_1 \,ⓟ\, P_2 \iff \overline{P}_3 = \overline{P}_1 \,ⓢ\, \overline{P}_2 \qquad (15)$$

It follows immediately that the parallel product (or ⓟ-product) has the same mathematical structure as the series product (normed, associative, commutative, admiting a neutral element $I_p(a) = \delta(a)$, but not an inverse). Clearly the statement (12) becomes

$$\lim_{n\to\infty} \left\{ P_1 \,ⓟ\, P_2 \,ⓟ \cdots ⓟ\, P_n \right\} = \delta(a-1) \qquad (16)$$

with the unique restriction that the P_i be different from $\delta(a)$. Another interesting property, valid whenever $P = \overline{P}$, concerns the mean fidelities $\langle a \rangle_{n,s}$ and $\langle a \rangle_{n,p}$ of n-bond series and n-bond parallel arrays respectively:

$$\begin{aligned} \langle a \rangle_{n,s} + \langle a \rangle_{n,p} &= \int_0^1 da\; a\, P^{ⓢ n}(a) + \int_0^1 da\; a\, P^{ⓟ n}(a) \\ &= \int_0^1 da\; a\, P^{ⓢ n}(a) + \int_0^1 da\; a\, \overline{P^{ⓢ n}}(a) \\ &= \int_0^1 da\; a\, P^{ⓢ n}(a) + \int_0^1 dx(1-x) P^{ⓢ n}(x) = 1 \end{aligned} \qquad (17)$$

where we have introduced the nomenclature $P^{\textcircled{p}\, n}$ for the n-factor parallel product $P \textcircled{p} P \textcircled{p} \ldots \textcircled{p} P$.

Let us now present an example, namely n bonds (with $P(a) = 1$ for each one) in parallel configuration. We have, by using the results of the 3rd example of Section III.3 with $r = 0$, that

$$P^{\textcircled{p}\, n} = \overline{P^{\textcircled{s}\, n}} = \frac{1}{(n-1)!} \ln^{n-1} \frac{1}{1-a}$$

and, by using also the property (17), that

$$\langle a \rangle_n = 1 - \frac{1}{2^n}$$

Another instructive situation that might be considered is the traditional bond percolation problem (probability p for each bond being active): the classical result (see, for example, references (11) and (13))

$$p_p = p^2 + 2p(1-p)$$

appears, in our language, in the compact form $P_p = P^{\textcircled{p}\, 2}$, with $P(a) = (1-p)\,\delta(a) + p\,\delta(a-1)$.

III.5 - Complex configurations

The concepts presented in the last Sections enable for calculation of the equivalent distribution $P_e(a)$ of any array of bonds whose topology con be solved in terms of series and parallel configurations of subsets of bonds. A few examples will exhibit the basic characteristics better we could explain in general terms.

1st example: Let us assume the configuration of Fig. 5(a) and let us discuss it for "vertical" percolation. Clearly we have

$$P_e = P_1 \textcircled{s} (P_2 \textcircled{p} (P_3 \textcircled{s} P_4))$$

which, in the particular case

$$P_1=P_2=P_3=P_4=P_5=(1-p)\,\delta(a)+p\,\delta(a-1) \quad ,$$

becomes $P_e=(1-p_e)\,\delta(a)+p_e\,\delta(a-1)$

with $p_e=p^4+3p^3(1-p)+p^2(1-p)^2$.

In this way we have compactly obtained the same result to which lead the usual bond percolation arguments (see, for example, references (11) and (13)). It might be worth while to point out that, in order to get quickly the solution of this kind of problem, it is convenient to have in mind the following properties:

$$\delta(a-1) \circledS \delta(a-1)= \delta(a-1) \; ⓟ \; \delta(a-1)= \delta(a) \; ⓟ \; \delta(a-1)= \delta(a-1)$$
$$\delta(a) \circledS \delta(a) = \delta(a-1) \circledS \delta(a) = \delta(a) \; ⓟ \; \delta(a) = \delta(a)$$

2nd example: If we assume the configuration of Fig. 5(b) with the same probability distribution P(a) for each bond, we obtain $P_e = (P^{\circledS n})^{ⓟ m}$. This expression, in the simple case $P(a) = (1-p)\,\delta(a)+ p\,\delta(a-1)$, becomes

$$P_e = (1-p_e)\,\delta(a)+p_e\,\delta(a-1)$$
$$\text{with} \quad p_e = 1-(1-p^n)^m \qquad (18)$$

The mean fidelity is given by $\langle a\rangle_{nm} = p_e$, and satisfies, for $0<p<1$ and $n,m \gg 1$, the assymptotic behaviour

$$\langle a\rangle_{nm} \sim 1-e^{-mp^n}$$

and tends to the limit $1-e^{-K}\in[0,1]$, where $K\equiv\lim_{n,m\to\infty} mp^n$ and tends to 0 $(+\infty)$ if n diverges faster (slower) than ln m.

Let us have a look at the entropy variation through this array. If we assume that the input is p(1)=1, we obtain, by using relations (1) and (4) (with $\alpha=0$), the following output

$$p'(1) \equiv 1-p'(2) = \frac{1+a}{2} \qquad (19)$$

hence the entropy variation is given by

$$\Delta S = -k_B \left\{ \frac{1+a}{2} \ln \frac{1+a}{2} + \frac{1-a}{2} \ln \frac{1-a}{2} \right\}$$

where k_B is the Boltzmann constant. Let us now evaluate the mean entropy variation associated to the distribution (18):

$$\langle \Delta S \rangle_{nm} = k_B \ln 2\,(1-p_e) = k_B \ln 2\,(1-\langle a \rangle_{nm})$$

Within this last relation it becomes evident that the eventual tendency, for diverging m and n, of the mean fidelity towards a non vanishing limit, is directly related to a certain amount of information which has been "saved" through its macroscopic propagation. For example, in all critical phenomena, it is intuitive that the saved information is exactly the one which allowds for the existence of a non vanishing long-range order parameter. Similar remarks may be done for regime changements in macroscopic physical systems.

The configuration of Fig. 5(c) leads to $P_e=(p^{\textcircled{p}\, m})^{\textcircled{s}\, n}$. We shall not discuss in detail this case, which certainly drives to results analogous to those obtained for the configuration of Fig. 5(b).

IV - RELATION TO STATISTICAL MECHANICS

Let us now make the joint between our formalism and Statistical Mechanics.

IV.1 - A simple example

If we have a single 2-states system (let be zero the energy of the fundamental state, and ε the energy of the unique excited state) at thermal equilibrium at temperature T, the occupancy probabilities are given by

$$p(0) = 1/1 \quad e^{-\varepsilon/k_BT}$$
$$p(\varepsilon) = e^{-\varepsilon/k_BT}/1 \quad e^{-\varepsilon/k_BT}$$

If we use now relation (19) we obtain

$$a = \mathrm{th}\,\frac{\varepsilon}{2k_BT} \qquad (20)$$

which leads to the intuitive limits $a=1$ for $T=0$, and $a=0$ for $T\to\infty$. So we may say that this simple Statistical Mechanics example corresponds to the following particular case of our formalism: $P(a)=\delta(a-a_o)$, with a_o given by (20). This is exactly the concept of the so called quenched bond[14,19].

Let us now assume we do not exactly know the value of ε but only a probability distribution $g(\varepsilon)$ (in other words, we have a thermal random model[10,14,19]). This case will correspond, in our formalism, to a distribution law P(a) given by

$$P(a) = g(\varepsilon)\,\frac{d\varepsilon}{da} = \frac{2k_BT}{1-a^2}\,g(2k_BT\ \mathrm{argth}\ a)$$

where we have used relation (20). This last situation is the one which appears in dilute magnetism, spin-glasses, some insulator-conductor transitions, etc.

IV.2 - Generalization

We intend to present here a possible generalization of the concept of fidelity for ideally dissipative bonds, in what concerns the input. More precisely, the input is no more restricted to be a binary aleatory variable, but it refers to a domain of possibilities containing W elements. So the input and output are respectively given by

$$\vec{p} \equiv \begin{pmatrix} p(1) \\ p(2) \\ \vdots \\ p(W) \end{pmatrix} \quad \text{and} \quad \vec{p}\,' \equiv \begin{pmatrix} p'(1) \\ p'(2) \\ \vdots \\ p'(W) \end{pmatrix}$$

Our basic assumption will be that they are related like follows:

$$\vec{p}\,' = (A)\,\vec{p}$$

where

$$(A) \equiv \frac{1}{W} \begin{pmatrix} 1+a_1 & 1-a_2 & 1-a_3 & \cdots & 1-a_W \\ 1-a_W & 1+a_1 & 1-a_2 & \cdots & 1-a_{W-1} \\ 1-a_{W-1} & 1-a_W & 1+a_1 & \cdots & 1-a_{W-2} \\ \vdots & \vdots & \vdots & \ddots & \\ 1-a_2 & 1-a_3 & 1-a_4 & \cdots & 1+a_1 \end{pmatrix}$$

with $a_1 = \sum_{i=2}^{W} a_i$ ($a_i \in [0,1]$ for $i=2,3,\ldots,W$ hence $a_1 \in [0, W-1]$).

If the input is $\vec{p} = (1,0,0,\ldots,0)$, then the output will be $\vec{p}\,' = \left(\frac{1+a_1}{W}, \frac{1-a_2}{W}, \ldots, \frac{1-a_W}{W} \right)$, and the entropy variation will be

$$\Delta S = -k_B \left\{ \frac{1+a_1}{W} \ln \frac{1+a_1}{W} + \sum_{i=2}^{W} \frac{1-a_i}{W} \ln \frac{1-a_i}{W} \right\}$$

We remark that the particular case $a_i = 0$ ($i=1,2,\ldots,W$) leads, by taking into account that the input entropy vanishes, to the celebrated Boltzmann's formula

$$S = -k_B \ln W$$

which traditionally drives to the solution of thermal physics of isolated systems (microcanonical ensemble), and from them to the whole body of Statistical Mechanics (canonical and grand canonical ensembles, etc). Nevertheless let us detail a little bit the canonical situation. Our system is assumed to accede to W different states with energies ε_j $(j=1,2,\ldots,W)$ respectively (some of them might be degenerate). Then, at thermal equilibrium at temperature T, the occupancy probabilities are given by

$$p(j) = e^{-\varepsilon_j/k_BT}/Z$$

$$Z \equiv e^{-\varepsilon_j/k_BT}$$

and identifying them with $\vec{p}\,' = \left(\frac{1+a_1}{W}, \frac{1-a_2}{W}, \ldots, \frac{1-a_W}{W}\right)$ we obtain

$$a_1 = \frac{W\,e^{-\varepsilon_1/k_BT}}{Z} - 1 \qquad (18.a)$$

and

$$a_j = 1 - \frac{W\,e^{-\varepsilon_j/k_BT}}{Z} \qquad (j=2,3,\ldots,W) \qquad (18.b)$$

Relations (18) are the "bridge" between the present formalism and the canonical thermal Gibbs distribution. In other words, if we define a vector fidelity $\vec{a} \equiv (a_2, a_3, \ldots, a_W)$ defined in $\mathbb{R}^{W-1}$, the canonical ensemble is represented by $P(\vec{a}) = \delta(\vec{a}-\vec{a}_0)$ where $\vec{a}_0$ is given by (18.b). As it should be, the microcanonical ensemble is reobtained from the canonical one, in the limit $T \to \infty$ hence $\vec{a}_0 \to 0$.

V - CONCLUSION

Let us conclude by saying that the present formalism generalizes and unifies the bond percolation problem (quenched or not), Statistical Mechanics and in general all problems whose main characteristic is to preserve a part of the information through a (long) path or array (critical phenomena, regime changements, theory of decisions, theory of communications, etc). Operationally it simplifies, in comparison with the traditional methods[11,13], the task of establishing the equivalent probability for complex clusters in the usual bond percolation problem. Furthermore it seems that this formalism could find practical applications in the thermal treatment of random models[10,14,19] (dilute magnetism, spin-glasses, some insulator-conductor transitions, gel-sol transition, etc). Let us finally add that the concept of frustration introduced by Toulouse[20,21] for spin-glasses seems to emerge naturally in parallel configurations, though we have not attempted a detailed analysis of this aspect.

I am happy to aknowledge here very encouraging discussions with E.M.F. Curado.

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CAPTION FOR FIGURES

Fig. 1 - The input (p), intermediate (p') and output (p'') probabilities in a series array.

Fig. 2 - The norm function N(a) associated to the series product of two bonds with creativity α .

Fig. 3 - Determination of the integration domain for the series product of two bonds with creativity α ;

$$a'' = \frac{a-\alpha(1-a')}{a'-\alpha(1-a')} \text{ . (a) } 0 < a < \frac{\alpha}{1+\alpha} \text{; (b) } \frac{\alpha}{1+\alpha} < a < 1$$

(note that the bisectrix $a'' = a'$ is a mirror, and that the branch near the origin leaves the validity domain of a' if $a > \alpha$).

Fig. 4 - Distribution associated to two bonds in series. (a) P(a) is a square "barrier"; (b) P(a) is Lorentzian-shaped.

Fig. 5 - Three examples of complex arrays.

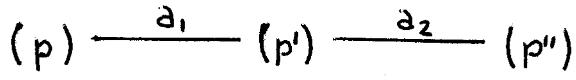


Fig. 1

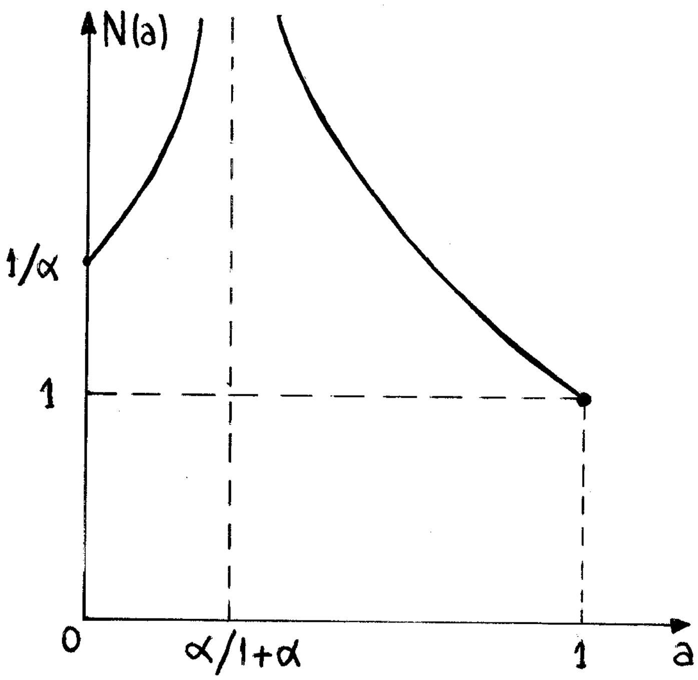


Fig. 2

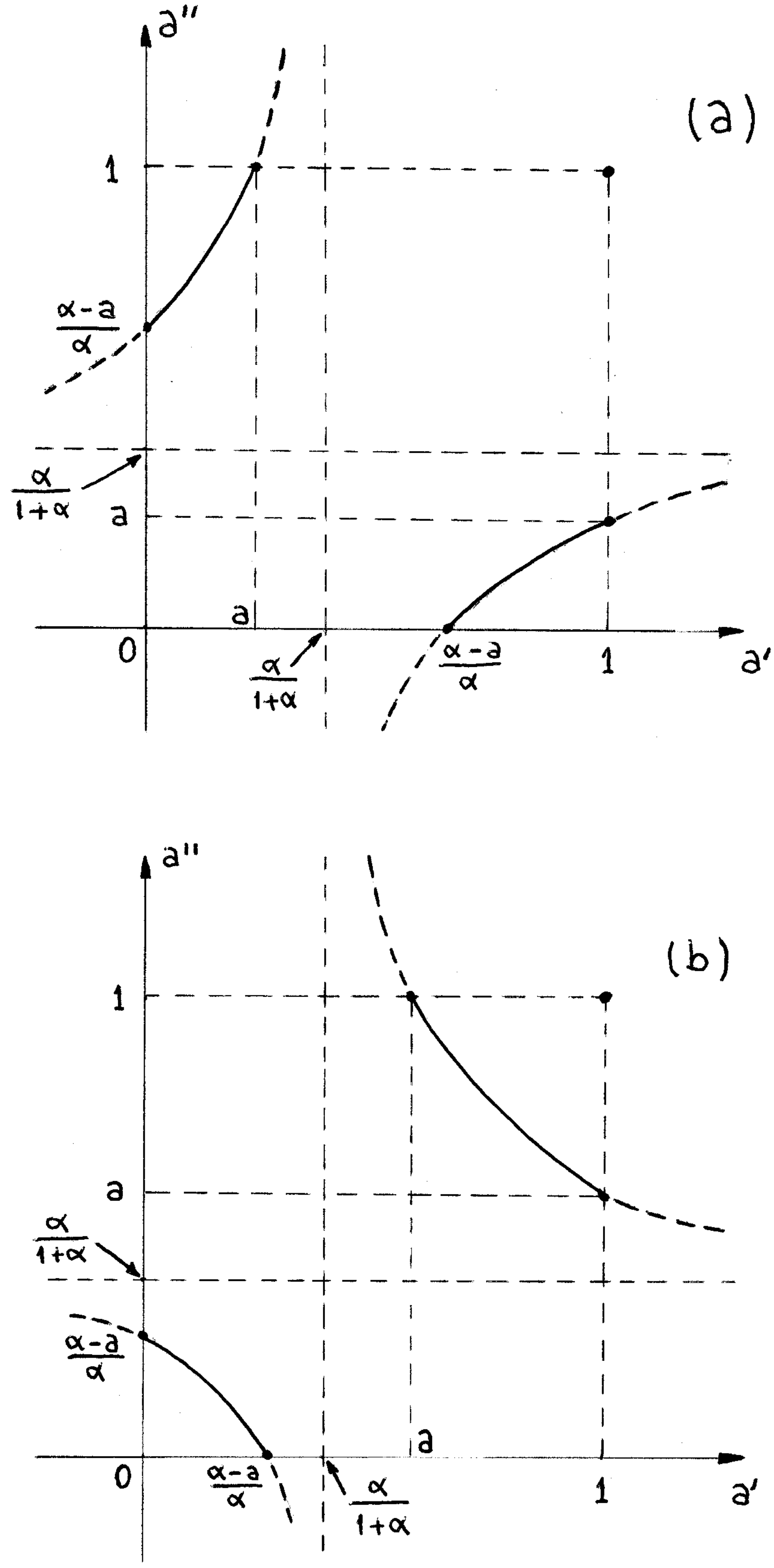
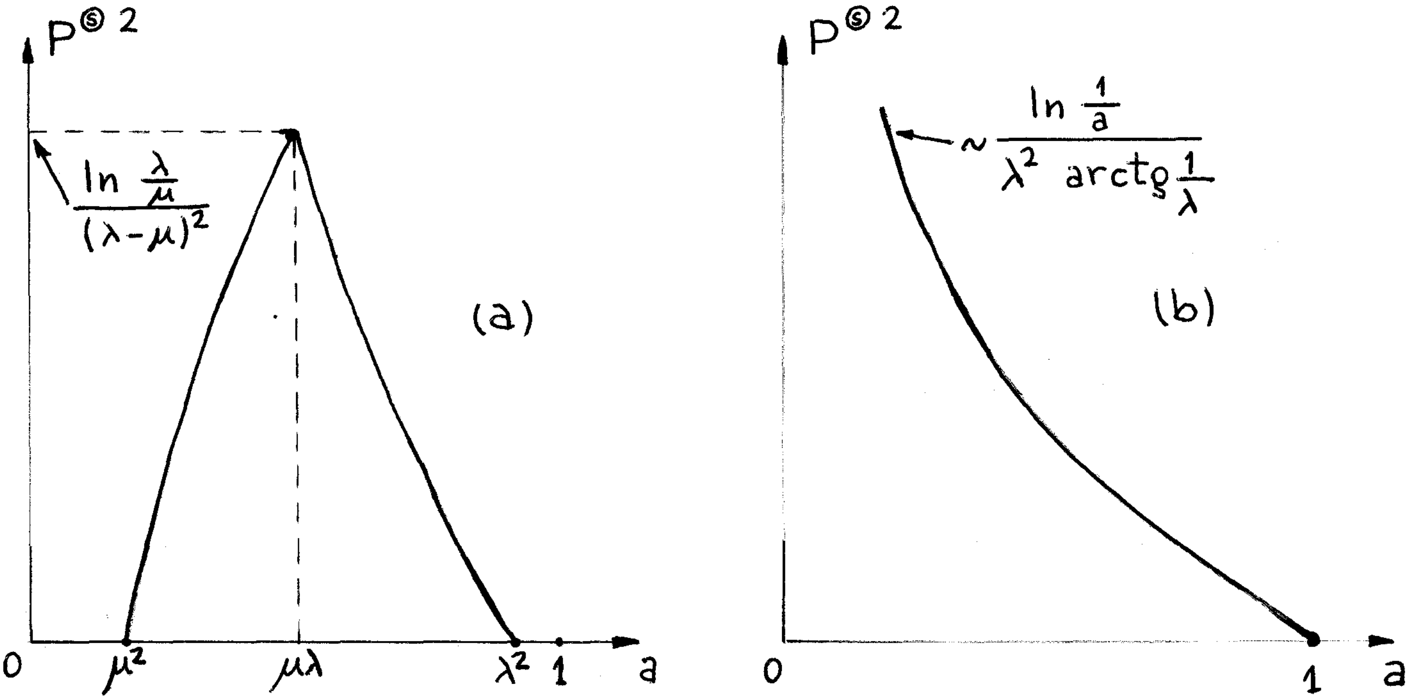
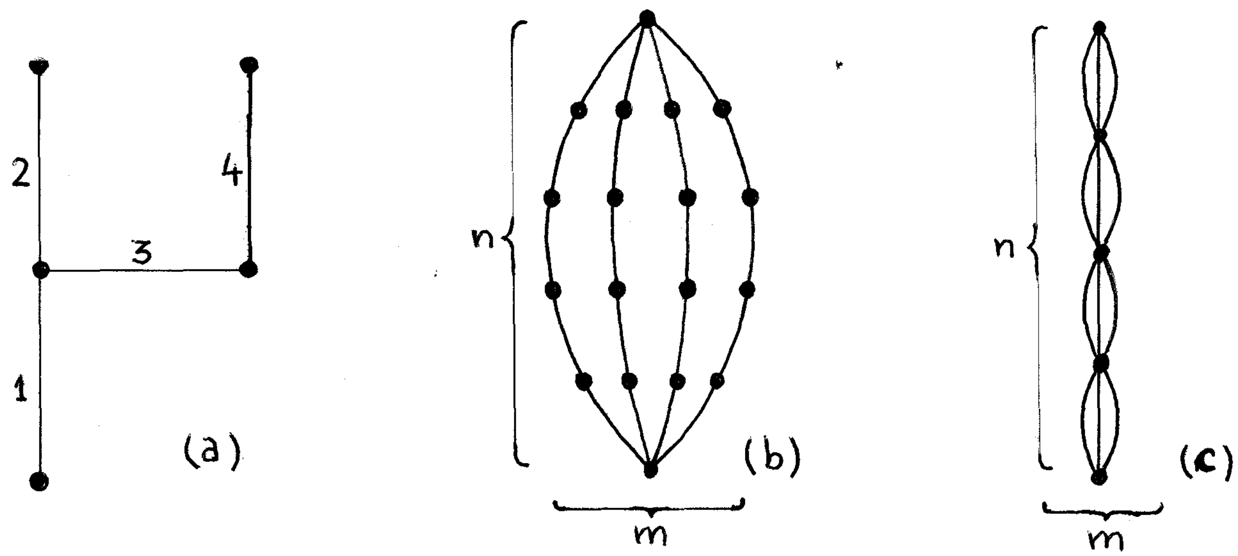


Fig. 3



$P^{\circledS 2}$
$\frac{\ln \frac{\lambda}{\mu}}{(\lambda-\mu)^2}$
(a)
0
μ^2
$\mu\lambda$
λ^2
1
a
$P^{\circledS 2}$
$\sim \frac{\ln \frac{1}{a}}{\lambda^2 \operatorname{arctg} \frac{1}{\lambda}}$
(b)
0
1
a

Fig. 4



2
4
3
1
(a)
n
m
(b)
n
m
(c)

Fig. 5