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ABSTRACT

A source free disordered distribution of electromagnetic radiation is considered in Einstein's theory, and a time independent exact solution with cylindrical symmetry is obtained. The gravitation and pressure effects of the radiation alone are sufficient to give the distribution an equilibrium. A finite maximum concentration is found on the axis of symmetry, and decreases monotonically to zero outwards. Timelike and null geodesics are discussed.

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1. INTRODUCTION

One of the most fascinating physical systems described by general relativity is that of an electromagnetic radiation evolving only under its own gravitation and pressure effects. There are some circumstances under which the attractive effects associated to the energy density of the radiation can balance the respulsive effects due to the corresponding gradient of pressure.

The pioneer work in this line is that of Klein [1]: he obtained an approximate solution of Einstein's equations for a distribution of diffused radiation with spherical symmetry. His distribution in equilibrium shows a maximum condensation at the center and dilutes monotonically to a zero value at infinity; however, his solution at infinity does not coincide with the vacuum solution of Schwarzschild. Very recently the present authors [2] obtained an exact solution of a physically analogous systems with plane symmetry: their solution also presents a larger condensation in the innermost regions and dilutes outwards, tending asymptotically to the plane vacuum solution of Levi-Civita [3].

The main purpose of the present study is to obtain the general characteristics of a physically analogous system, but with a different symmetry, and to compare the new results with those corresponding to the previous symmetries. We then consider a distribution of disordered electromagnetic radiation with cylindrical symmetry, in equilibrium. We are calling cylindrical symmetry the property of invariance of the system under rotations about a given axis, and under translations parallel to the same axis. The most widely known cylindrical coordinate systems (Weyl's canonical and Einstein-Rosen) cannot be used for the description of our isotropic distribution, as explained in Sec. 3; however, we succeded in obtaining an exact solution in a reference system where $g_{zz}^2 = r^{-4}\ g_{\phi\phi}^2 = -\ g_{rr}/g_{00}$.

2. GENERAL EQUATIONS

We start with the line element

$$ds^2 = e^{2\alpha}(dx^0)^2 - e^{2\beta}\ dr^2 - (dz^2 + r^2\ d\phi^2)e^{\beta-\alpha} \quad , \qquad (1)$$

where α and β are functions of r alone; the corresponding non-zero Christoffel symbols of the second kind are

$$\left\{{0 \atop 0\ 1}\right\} = \alpha_1 \quad , \left\{{1 \atop 0\ 0}\right\} = \alpha_1\ e^{2\alpha-2\beta} \quad , \left\{{1 \atop 1\ 1}\right\} = \beta_1 \quad ,$$

$$\left\{{1 \atop 2\ 2}\right\} = \tfrac{1}{2}(\alpha_1-\beta_1)e^{-\alpha-\beta} \quad , \quad \left\{{1 \atop 3\ 3}\right\} = \tfrac{1}{2}(\alpha_1-\beta_1-2/r)e^{-\alpha-\beta} \quad , \qquad (2)$$

$$\left\{{2 \atop 1\ 2}\right\} = \tfrac{1}{2}(\beta_1-\alpha_1) \quad , \quad \left\{{3 \atop 1\ 3}\right\} = \tfrac{1}{2}(\beta_1-\alpha_1+2/r) \quad ,$$

where a subscript 1 means d/dr. And the surviving components of the Ricci tensor are

$$R_0^0 = -(\alpha_{11} + \alpha_1/r)\, e^{-2\beta} \quad ,$$

$$R_1^1 = (\beta_{11} - \beta_1^2/2 - \alpha_1\beta_1 + 3\alpha_1^2/2 - \alpha_1/r)\, e^{-2\beta} \quad , \tag{3}$$

$$R_2^2 = R_3^3 = \tfrac{1}{2}(\alpha_{11} - \beta_{11} + \alpha_1/r - \beta_1/r)\, e^{-2\beta} \quad .$$

The energy momentum tensor of a perfect fluid is

$$T_\nu^\mu = (\rho c^2 + p)\, u^\mu u_\nu - p\, \delta_\nu^\mu \quad , \tag{4}$$

where ρc^2, p and u^μ are the rest energy density, the pressure and the macroscopic velocity field of the fluid; this velocity satisfies $u^\lambda u_\lambda = 1$. For our disordered radiation we follow Tolman's [4a] prescriptions and make $\rho c^2 = 3p$; we then obtain in static condition

$$T_\nu^\mu = p(r)\ \mathrm{diag}\left[\ 3,-1,-1,-1\ \right] \quad . \tag{5}$$

The Einstein equations for a traceless T_ν^μ are

$$R_\nu^\mu = -\kappa\, T_\nu^\mu \quad , \qquad \kappa = 8\pi G/c^4 \quad ; \tag{6}$$

in our system these equations reduce to

$$\alpha_{11} + \alpha_1/r = 3\kappa p\, e^{2\beta} \quad , \tag{7}$$

$$\beta_{11} - \beta_1^2/2 - \alpha_1\beta_1 + 3\alpha_1^2/2 - \alpha_1/r = -\kappa p\, e^{2\beta} \quad , \tag{8}$$

$$\alpha_{11} - \beta_{11} + \alpha_1/r - \beta_1/r = 2\kappa p\, e^{2\beta} \quad . \tag{9}$$

The two functions α and p are related by the Bianchi identity

$$p_1 + 4p\,\alpha_1 = 0 \quad . \tag{10}$$

3. SOLUTIONS OF EQUATIONS

Eliminating p from (7) and (9) we obtain $\alpha-3\beta=a+b\log r$, with a and b constants of integration. We are free to impose the boundary conditions $g_{00} = -g_{rr} = 1$ on the axis $r = 0$; from (1) one finds then that $\alpha(0) = \beta(0) = 0$. With the constants $a=b=0$ we then have

$$\alpha = 3\beta . \tag{11}$$

Again eliminating p from (7) and (8) and using (11) we obtain $\beta_{11} + 5\beta_1^2 - \beta_1/r = 0$, whose solution is

$$5\beta = \log(m + nr^2) , \tag{12}$$

with m and n constants of integration. The imposition $\beta(0) = 0$ implies $m = 1$.

We finally obtain for the presure the expression

$$5\kappa p = 4n\ (1 + nr^2)^{-12/5} ; \tag{13}$$

this expression satisfies the Bianchi identity (10).

If we now call the constant $p_0 = 4n/5\kappa$, our solution may be written as

$$ds^2 = F^3(dx^0)^2 - F\,dr^2 - F^{-1}\,(dz^2 + r^2 d\phi^2) , \tag{14}$$

$$p = p_0\ F^{-6} , \tag{15}$$

$$F(r) = (1 + r^2/R^2)^{2/5} , \qquad R^{-2} = 5\kappa p_0/4 . \tag{16}$$

On the axis of symmetry $(r = 0)$ this line element (14) reduces to

$$ds^2(0) = (dx^0)^2 - dr^2 - dz^2 - r^2\, d\phi^2 \quad , \tag{17}$$

and in regions close to this axis ($\kappa p_0 r^2 << 1$) we have

$$g_{00} \simeq 1 + \frac{3}{2}\,\kappa p_0 r^2 \quad , \quad p = p_0 (1 - 3\kappa p_0 r^2) \quad . \tag{18}$$

The significance of these results will be discussed later.

We can evaluate the energy content of the distribution from the axis of symmetry to the radius r, per unit length measured on that axis: we take Tolman's [4b] general result

$$d^3E = (-g)^{1/2} \left[2T_0^0 - T\right] dx^1\, dx^2\, dx^3 \quad , \quad g = \det g_{\mu\nu} \tag{19}$$

and use (14) and (15); we obtain

$$d^3E = 6 p_0 r\, F^{-5}\, dr\, dz\, d\phi \quad . \tag{20}$$

If we now integrate (20) in ϕ and in r (from 0 to r) we get for the linear density of energy $c^2\lambda(r)$ the expression

$$c^2\lambda(r) = dE(r)/dz = 6\pi p_0 r^2\, (1 + r^2/R^2)^{-1} \quad . \tag{21}$$

One finds from (21) that the total energy content per unit length measured on the z axis is finite and only depends on natural constants:

$$c^2\lambda(\infty) = 3c^4/5G \quad . \tag{22}$$

One also finds from (21) that half of the total energy of the distribution is contained inside the radius R given by (16); we may then consider this value R as a measure of the radius of our distribution.

In regions far from the axis of symmetry we have from

(14) and (16) the asymptotic line element

$$ds^2(\infty) = (r/R)^{12/5}\,(dx^0)^2 - (r/R)^{4/5}\,dr^2 - (r/R)^{-4/5}\,(dz^2+r^2 d\phi^2)\ ; \qquad (23)$$

this line element is an exact solution of the Einstein's equations for the vacuum ($R^\mu_\nu = 0$) and corresponds to the exterior gravitational field produced by an infinitely long cylinder of matter, with uniform linear density. The value λ of this density can be obtained in the weak field approximation: in this approximation one has $g_{00} = \exp(2\Phi/c^2)$, where $\Phi = 2G\lambda \log r + \text{const}$ is the Newtonian potential. Comparing these relations with g_{00} from (23) one finds $\lambda = 3c^2/5G$, a value that coincides with (22).

With appropriate coordinate transformations we can express our asymptotic result (23) both in Weyl's canonical coordinates ($g_{00}\, g_{\phi\phi} = - r^2$, $g_{rr} = g_{zz}$) and in Einstein-Rosen coordinates ($g_{00} = - g_{rr}$, $g_{zz}\, g_{\phi\phi} = r^2$); however, none of these two sets of coordinates is suitable for describing our entire system. The reason is that Weyl's canonical coordinates can only be used when $T^0_0 = - T^\phi_\phi$; and Einstein-Rosen coordinates require $T^z_z = - T^\phi_\phi$; none of these equalities is satisfied by the energy momentum (5). Our coordinates, however, are suitable for systems with $T^z_z = T^\phi_\phi$, a relation obeyed by our system. Only in the asymptotic regions, where all components of T^μ_ν vanish, we can use any of these coordinates.

4. TIMELIKE GEODESICS

Some quite simple results for timelike geodesics can be obtained from the general line element (1). We start with the geodesic equation

$$du^{\mu}/ds + \begin{Bmatrix} \mu \\ \nu \;\; \rho \end{Bmatrix} u^{\nu}u^{\rho} = 0 \quad , \quad u^{\mu} = dx^{\mu}/ds \tag{24}$$

and use the Christoffel symbols (2); we obtain for $\mu = 0,2,3$

$$du^0/ds + 2\alpha_1 \, u^0u^1 = 0 \quad , \tag{25}$$

$$du^2/ds + (\beta_1-\alpha_1)u^1u^2 = 0 \quad , \tag{26}$$

$$du^3/ds + (\beta_1-\alpha_1+2/r)u^1u^3 = 0 \quad , \tag{27}$$

while u^1 is directly obtained from u^0, u^2 and u^3 through $u^{\lambda}u_{\lambda} = 1$. The first integral of these equations is then

$$\begin{aligned} &u^0 = D^2 \, e^{-2\alpha} \quad , \quad u^2 = B \, e^{\alpha-\beta} \quad , \quad u^3 = (C/r^2) \, e^{\alpha-\beta} \quad , \\ &(u^1)^2 = (D^4e^{-2\alpha} - 1)e^{-2\beta} - (B^2 + C^2/r^2)e^{\alpha-3\beta} \quad , \end{aligned} \tag{28}$$

with B, C and D^2 constants of integration related to the three components of a given "initial" velocity of the test particle.

These results are valid for all physical systems with symmetries compatible with the line element (1). One finds from (28) and (1) that while the covariant component u_1 usually varies along the motion of a given test particle all other components are kept unchanged along that motion,

$$u_0 = D^2 \quad , \quad u_2 = - B \quad , \quad u_3 = - C \quad . \tag{29}$$

In view of the difficulty in obtaining the integrals of (28) with the line element (14) we only consider the motions of test particles with velocities small in comparison with that of light, and in regions near the axis of symmetry $r = 0$. In other words, we take the dimensionless quantities B^2, D^2-1, C^2/r^2, $\kappa p_0 r^2$ all very small. The singularity of C^2/r^2 on $r=0$ will be seen to be only apparent, since only particles with $C=0$ can cross that axis. We then obtain from (28) with the line element (14) the equations

$$dx^0/ds \simeq D^2(1-3\kappa p_0 r^2/2) \simeq 1 \quad , \quad dz/ds \simeq B \quad , \quad d\phi/ds \simeq C/r^2 \quad ,$$
$$(dr/ds)^2 \simeq D^4 - 1 - B^2 - C^2/r^2 - 3\kappa p_0 r^2/2 \quad . \tag{30}$$

These equations can now be integrated; we call $x^0 = ct$ and obtain

$$r^{-2} = M^{-2} \cos^2(\phi-\phi_0) + m^{-2} \sin^2(\phi-\phi_0) \quad , \tag{31}$$

$$M \tan(\phi-\phi_0) = m \tan(\omega t - \phi_0') \quad , \tag{32}$$

$$z - z_0 = c\,B\,t \quad , \tag{33}$$

where ϕ_0, ϕ_0' and z_0 are new constants of integration, and

$$\omega^2 = 3\kappa c^2 p_0/2 \tag{34}$$

is a constant not depending on the nonrelativistic velocity of the particle. The two constants M and m are related to the velocity parameters through

$$Mm = c\,C/\omega \quad , \quad M^2 + m^2 = c^2(D^4 - 1 - B^2)/\omega^2 \quad . \tag{35}$$

From (31) to (33) one finds that the most general motion of a "never relativistic" test particle is an helix drawn on a cylinder with axis on $r = 0$, having an elliptical section with semiaxes M and m; each test particle has a constant longitudinal velocity ($dz/dt = cB$), and all particles complete a revolution around the z axis in the same interval of time $2\pi/\omega$. Particles that cross the axis $r = 0$ correspond to cylinders with semiminor axis $m = 0$; then from (35) one finds that for such particles $C = 0$, as stated before.

5. NULL GEODESICS

Null geodesics can formally be obtained from (24), but now $u^\lambda u_\lambda = 0$. A first integral is then

$$dz/dx^0 = B\, e^{3\alpha-\beta} \quad , \quad d\phi/dx^0 = C\, r^{-2}\, e^{3\alpha-\beta} \quad ,$$
$$(dr/dx^0)^2 = \left[e^\beta - (B^2 + C^2/r^2)e^{3\alpha}\right] e^{2\alpha-3\beta} \quad ; \qquad (36)$$

the two constants B and C are related to a given "initial" direction of the null geodesic. These results do not depend on the particular form of the functions α and β.

In the case of our line element (14) one has from (36)

$$(dr/dx^0)^2 = F^2 - (B^2 - C^2/r^2)F^6 \quad , \quad dz/dx^0 = BF^4 \quad , \quad d\phi/dx^0 = CF^4/r^2 ; \qquad (37)$$

again we have not been able to obtain the general exact solution of these equations. However, a few interesting results can be presented.

Given an arbitrary radius $r_c \leq (3\kappa p_0/4)^{-1/2}$ we can have helical null geodesics drawn on a circular cylinder with that radius, with direction parameters

$$5C^2=8R^{-2}r_c^4F_c^{-13/2} \quad , \quad B^2=(1-3\kappa p_0 r_c^2/4)F_c^{-13/2} \quad , \quad F_c=F(r_c). \qquad (38)$$

A plane circular null geodesic (B=0) is obtained when

$$r_c = (3\kappa p_0/4)^{-1/2} \quad . \qquad (39)$$

For small values of the constant radius r_c we have lightlike helical motions almost parallel ($B^2 \simeq 1$) to the axis of symmetry; the angular velocity of this motion can be obtained from (37) and (38):

$$\Omega^2 = (d\phi/dt)^2 = 2\kappa c^2 p_0 \quad . \qquad (40)$$

Null geodesics along the axis $r = 0$ have $C = 0$ and $B^2 = 1$.

Similarly to the timelike geodesics, all null geodesics close to the axis of symmetry ($1-B^2$, C^2/r^2, $\kappa p_0 r^2$ all very small) present elliptic helical characteristics; the semiaxes M and m are now given by

$$Mm = cC/\Omega \quad , \quad M^2 + m^2 = c^2(1-B^2)/\Omega^2 \qquad (41)$$

where Ω is the frequency (40).

We finally consider lightlike motions on planes containing the axis of symmetry: we put C=0 in (37) and obtain

$$(dr/dz)^2 = B^{-2}\,F^{-6} - F^{-2} \quad . \qquad (42)$$

If one defines $\tan \nu(r) = dr/dz$ one finds from (42) that the angle $\nu = \nu(0)$ of incidence on the axis of symmetry is simply given by

$$\cos \nu = |B| \quad ; \tag{43}$$

one also finds from (42) that the maximum distance ($dr/dz = 0$) reached by a light ray is

$$r_{max} = R(|B|^{-5/4} - 1)^{1/2} \quad . \tag{44}$$

Combining then (43) and (44) one finds that null geodesics almost parallel to z-axis reach only

$$r_{max} \simeq (5/8)^{1/2} \, \nu R \quad , \qquad \nu \simeq 0 \quad , \tag{45}$$

and null geodesics almost normal to z axis reach

$$r_{max} \simeq R/(\pi/2 - \nu)^{5/8} \quad , \qquad \nu \simeq \pi/2 \quad ; \tag{46}$$

for an angle of incidence equal to 45^{o} one obtains

$$r_{max} \simeq 4R/5 \quad , \qquad \nu = \pi/4 \quad , \tag{47}$$

a distance of the order of the radius R.

6. DISCUSSIONS

We obtained the exact solution given by general relativity corresponding to a distribution of disordered radiation in equilibrium under its own gravitation, and with cylindrical symmetry. According to Tolman's [4a] prescriptions we used an isotropic and traceless energy momentum tensor to represent the distribution. Our solution then may also correspond to a disordered distribution of neutrinos, and to disordered distributions of ultrarelativistic particles [1].

Our line element (14) is well behaved except in the asymptotic regions $r \to \infty$. In these regions the solution (23) is equivalent to the Levi-Civita's vacuum cylindrical one [5],

$$ds^2 = V_0^2\, \rho^{2h}\, dt^2 - (dr'^2 + dz'^2)\rho^{2h(h-1)} - r'^2 \rho^{-2h}\, d\phi^2 ,$$

$$\rho = r'/r_0 \quad ; \quad r_0,\ V_0,\ h = \text{constants},$$

where his coordinates (Weyl canonical) are related to ours through

$$x^0 = a^{-3}\, V_0 t \quad , \quad r^9 = a^{15}\, R^4\, r'^5 \quad ; \quad z = az' ,$$

$$r_0 = a^{3/2}\, R \quad , \quad a = (9/5)^{1/2} ,$$

and where the special value $h = 2/3$ is assigned; the quantity $4h$ essentially corresponds to our $\lambda(\infty)$ given in (22), the difference in numerical value being due to the change of coordinates.

We found that both the plane solution [2] and the present cylindrical solution tend to vacuum solutions at infinity, while the Klein's spherical solution [1] does not. The origin of this difference is that in the spherical configuration the concentration of the distribution "decreases but not very rapidly" with increasing radius; one finds that according to

Tolman's definition [4a] the energy content between two concentric spherical shells of large radii r and $r + \Delta r$ is proportional to $r^{1/2}\Delta r$, a quantity that increases with r. In the plane- and in the cylindrically-symmetric systems, however, the concentration decreases much more rapidly so as to allow a far placed observer to attribute the gravitational field to a slab of finite thickness or to a cylinder of finite radius.

Similarly to the spherical and plane distributions our solution contains only one parameter, the central pressure p_0; and the spectrum of frequency of our isotropic radiation is arbitrary.

Since the trace T of the energy momentum of the distribution vanishes, the scalar curvature R^{μ}_{μ} also vanishes. However, the "square" of the Ricci tensor does not,

$$R^{\mu\nu}R_{\mu\nu} = (192/25)R^{-4}\,(1+r^2/R^2)^{-24/5} \quad ;$$

also the Kretschmann scalar is non-zero,

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = (24/25)^2R^{-4}(1+r^2/R^2)^{-24/5}(25+10r^2/R^2+7r^4/R^4).$$

Both these scalars are finite at $r = 0$ and decrease monotonically to zero at infinity.

In studying the timelike geodesics we found that the nonrelativistic test particles perform harmonic motions around the axis of symmetry, together with a uniform displacement parallel to that axis. This behaviour of particles is characte ristic of gravitational fields produced by homogeneous distributions of energy; indeed, one finds from (18) that the concentration of the radiation is fairly uniform in the central zone.

The null geodesics deserve a special consideration in our study, since these are the geodesics followed by the very constituents of the physical system. We found (39) a closed null geodesic, a circle with radius nearly 1.3R, in a region where the concentration is still considerable. We also found (40) that the frequency Ω of revolution of almost longitudinal central null geodesics is larger than the frequency ω (34) of nonrelativistic test particles, $\Omega^2 = 4\omega^2/3$; an interpretation of this factor 4/3 has already been tried [2].

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