Notas de Física - Volume XXII - Nº 11

STATIONARY VACUUM FIELD WITH CYLINDRICAL SYMMETRY*

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(Received 6th July 1974)

ABSTRACT

It is demonstrated that stationary vacuum field solution with cylindrical symmetry is always reducible to static cylindrically symmetric solution.

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* To be submitted for publication to Comm. Math. Phys.

I. INTRODUCTION

In an ealier work, Lewis[1] examined the stationary cylindrically symmetric vacuum field and obtained a special class of solutions of Einstein equations; these solutions are linear combinations of static cylindrically symmetric solutions with constant coefficients. In a recent work Som et al.[2] extended his work to a stationary axially symmetric case, and presented a class of solutions which are linear combinations of static axially symmetric Curzon[3] fields. In both cases, they showed that stationary metrics are reducible to fundamental quadratic forms with suitable coordinate transformation; so one can interprete these solutions as descriptions of static fields by an observer who is rotating with constant angular speed.

In the present work we have studied the question whether the diagonalizability of stationary metric by coordinate transformation is essentially linked with the specialization of the solutions. Of course with axial symmetry one has a special class of solutions given by Kerr[4] which is not diagonalizable by coordinate transformation. But in the case of cylindrical symmetry our observation is that the diagonalizability is a general feature of stationary vacuum fields.

II. FIELD EQUATIONS

We start with the general stationary cylindrically symmetric line element

$$ds^2 = f\, dx^{o2} - e^{2\psi}(dr^2 + dz^2) - \ell d\phi^2 - 2m dx^o\, d\phi \quad , \qquad (2.1)$$

where f, ℓ, m and ψ are functions of r alone; we shall number the coordinates (x^o, r, z, ϕ) as (0, 1, 2, 3) respectively.

Einstein field equations in empty space are given by

$$R^{\mu}_{\nu} = 0 \quad ; \tag{2.2}$$

for the line element (2.1) the field equations (2.2) are

$$D\Psi_{11} - \Psi_1 - (f_1\,\ell_1 + m_1^2)/2D = 0 \ , \tag{2.3}$$

$$D\Psi_{11} + \Psi_1 = 0 \quad , \tag{2.4}$$

$$[(\ell f_1 + mm_1)/D]_1 = 0 \quad , \tag{2.5}$$

$$[(f\ell_1 + mm_1)/D]_1 = 0 \quad \text{and} \tag{2.6}$$

$$[(m\ell_1 - \ell m_1)/D]_1 = 0 \quad , \tag{2.7}$$

where

$$D^2 = f\ell + m^2 \tag{2.8}$$

and subscript 1 denotes differentiation with respect to r.

Summing equations (2.5) and (2.6) one gets immediately on integrations

$$D = ar + b \tag{2.9}$$

where a and b are constants; by a simple transformation of coordinate we can now introduce the Weyl-type canonical coordinates where

$$D^2 = f\ell + m^2 = r^2 \quad . \tag{2.10}$$

Subtracting equation (2.6) from (2.5) one gets

$$[(\ell f_1 - f\ell_1)/D]_1 = 0 \quad ; \tag{2.11}$$

we take equations (2.7), (2.10) and (2.11) as the basic equations to determine f, ℓ and m.

To obtain a general class of solutions we introduce here similar functions as done by Datta and Raychaudhuri[5],

$$u = f/\ell \tag{2.12}$$

and

$$v = m/\ell \; ; \tag{2.13}$$

equation (2.10) gives

$$\ell^2 = r^2 / (u + v^2) \quad , \tag{2.14}$$

and equations (2.11) and (2.7) reduce to

$$[ru_1/(u + v^2)]_1 = 0 \tag{2.15}$$

and

$$[rv_1/(u + v^2)]_1 = 0 \quad . \tag{2.16}$$

The solution of these equations gives a linear relation between u and v,

$$v = \delta u + \gamma \quad , \tag{2.17}$$

where δ and γ are constants of integration so f, ℓ and m are linearly related. With the help of equation (2.17) one then immediately obtains from any of the two equations (2.15) and (2.16)

$$2\delta^2 u = (1 + 4\delta\gamma)^{1/2} (1 + y^2)(1 - y^2)^{-1} - (1+2\delta\gamma) \tag{2.18}$$

with

$$y = (r/\zeta)^{\varepsilon} \tag{2.19}$$

where ε and ζ are constants of integration. From (2.12)-(2.14) and (2.17)-(2.19) we now get

$$f = (r/4\delta\eta) \; [(1+\eta)^2 y^{-1} - (1-\eta)^2 y] \quad , \tag{2.20}$$

$$\ell = (r\delta/\eta) \; [y - y^{-1}] \tag{2.21}$$

and

$$m = (r/2\eta) \; [(1+\eta) y^{-1} - (1-\eta) y] \quad , \tag{2.22}$$

where we have used for brevity

$$\eta = - (1 + 4\delta\gamma)^{1/2} \quad .$$

Now from (2.4) and (2.10) we get

$$e^{2\Psi} = (r/\beta)^{2\alpha} \quad , \tag{2.23}$$

where β and α are constants of integrations; substituting (2.23) in (2.3) we have

$$f_1 \, \ell_1 + m_1^2 = - 4\alpha \quad , \tag{2.24}$$

then from (2.20)-(2.22) and (2.24) we get

$$\alpha = (\varepsilon^2 - 1)/4 \ . \tag{2.25}$$

For particular values of constants of integrations our solutions reduce to Lewis solution.

III. DIAGONALIZATION

We shall now consider the diagonalization of the line element (2.1) for this general class of solutions. Let us consider the following transformation of coordinates:

$$x^o = (\delta/\zeta\eta)^{1/2} \left[x^{o'} (\zeta/\beta)^{-(1+\varepsilon)/2} + \zeta\phi' \right] \quad , \tag{3.1}$$

$$r = r' \ (\zeta/\beta)^{(1+\varepsilon)/2} \quad , \tag{3.2}$$

$$z = z' \ (\zeta/\beta)^{(1+\varepsilon)/2} \tag{3.3}$$

and

$$\phi = (4\delta\zeta\eta)^{-1/2} \left[x^{o'}(1-\eta)(\zeta/\beta)^{-(1+\varepsilon)/2} + (1+\eta)\zeta\phi'\right] \quad ; \tag{3.4}$$

by this coordinate transformation one obtains the line element (2.1) reduced to the form

$$ds^2 = (r'/\lambda)^{2C}\, dx^{o'^2} - (r'/\lambda)^{-2C+2C^2}\,(dr'^2 + dz'^2) - r'^2(r'/\lambda)^{-2C} d\phi'^2 \ , \tag{3.5}$$

where

$$1 - \varepsilon = 2C \tag{3.6}$$

and

$$\lambda = \zeta(\zeta/\beta)^{-2+C+1/C} \ . \tag{3.7}$$

This line element is already obtained by Marder[6]; the solutions correspond to static radial exterior field of an infinite cylinder. So one can infer that stationary vacuum field with cylindrical symmetry is equivalent to a cilindrically symmetric static field.

ACKNOWLEDGEMENTS

One of the authors (M.M.S.) acknowledges financial support from Conselho Nacional de Pesquisas (Brazil).

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