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ENTIRE FUNCTIONS OF EXPONENTIAL TYPE BOUNDED ON THE REAL AXIS AND FOURIER TRANSFORMS OF DISTRIBUTIONS WITH BOUNDED SUPPORTS *

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Let E and F be real Banach spaces; $\mathcal{P}_N(^mE;F)$ the Banach space of nuclear m-homogeneous polynomials from E to F, the nuclear norm being denoted by $\| \cdot \|_N$; E' the dual space to E; E_C a normed complexification of E and $(E_C)'$ its dual; and $(E')_C$ a normed complexification of E', where $(E')_C$ and $(E_C)'$ are isometric under the natural isomorphism between them.

We shall denote by $\varepsilon_N(E;F)$ the vector space of all infinitely differ-

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entiable mappings $f:E \to F$ such that

$$\hat{d}^m f(E) \subset \mathcal{P}_N(^mE;F)$$

for m = 0, 1, 2, ..., and each mapping

$$\hat{d}^m f: E \to \mathcal{P}_N(^mE; F)$$

is differentiable of first order when $\mathcal{P}_N(^mE; F)$ is endowed with its nuclear norm. An element of $\varepsilon_N(E; F)$ is referred to as an infinitely nuclearly differentiable mapping from E to F.

We shall also introduce the vector subspace $\varepsilon_{Nb}(E; F)$ of all $f:E \to F$ in $\varepsilon_N(E; F)$ such that every

$$\hat{d}^m f:E \to \mathcal{P}_N(^mE; F)$$

is bounded on bounded subsets. An element of $\varepsilon_{Nb}(E; F)$ is said to be an infinitely nuclearly differentiable mapping of bounded type from E to F. Then $\varepsilon_{Nb}(E; F)$ is a Fréchet space with respect to the topology defined by the following seminorms on it

$$f \in \varepsilon_{Nb}(E; F) \to \sup \{ \|\hat{d}^m f(x)\|_N : \|x\| \leq n \} \in R_+$$

for m, n = 0, 1, 2 ...

For every $\phi \in E'$ and m = 0, 1, 2, ... (the case in which both $\phi = 0$ and m = 0 being excluded), consider the functions ϕ^m on E, it being understood that $\phi^0 = 1$. We shall denote by $\varepsilon_{Nbc}(E; F)$ the closure in $\varepsilon_{Nb}(E; F)$ of its vector subspace obtained as the set of finite sums of all mappings of the form $\phi^m b: E \to F$, where $\phi \in E'$, m = 0, 1, 2, ... and $b \in F$. An element of $\varepsilon_{Nbc}(E; F)$ is said to be an infinitely nuclearly differentiable mapping of bounded-compact type E to F. Thus $\varepsilon_{Nbc}(E; F)$ is a Fréchet space with respect to the topology induced on it by that of $\varepsilon_{Nb}(E; F)$.

The vector subspace of $\varepsilon_{Nbc}(E; F)$ obtained as the set of finite sums of all mappings of the form

$$e^{\phi}\, b: E \to F$$

where $\phi \in E'$ and $b \in F$, is dense in $\varepsilon_{Nbc}(E; F)$. If F is a complex Banach space, not only the vector subspace of $\varepsilon_{Nbc}(E; F)$ obtained as the set of finite sums of all mappings of the form

$$e^{\phi + i\psi}\, b: E \to F$$

where $\phi \in E'$, $\psi \in E'$ and $b \in F$, is dense in $\varepsilon_{Nbc}(E; F)$, but also the same assertion is true if we focus our attention only on the mappings of the form

$$e^{i\psi}\, b: E \to F$$

where $\psi \in E'$ and $b \in F$.

If T is a continuous complex linear form on $\varepsilon_{Nbc}(E; C)$, its Fourier transform $\mathfrak{F}T$ is the complex valued function on E' defined by

$$(\mathfrak{F}T)(\psi) = T(e^{i\psi})$$

for $\psi \in E'$. The mapping $T \to \mathfrak{F}T$ thus defined is linear and injective.

Each such $\mathfrak{F}T$ may be extended in a necessarily unique way to an entire function on $(E')_C$ by setting

$$(\mathfrak{F}T)(\phi + i\psi) = T(e^{\phi + i\psi})$$

for $\phi \in E'$ and $\psi \in E'$. Then $\mathfrak{F}T$ is entire of exponential type on $(E')_C$ and slowly increasing on E' in the sense that there an integer $\alpha \geq 0$ and a real number $C \geq 0$ such that

$$|(\mathcal{F}T)(\psi)| \leq C\cdot(1 + \|\psi\|)^{\alpha}$$

for $\psi \in E'$. More precisely, there are an integer $\alpha \geq 0$ and real numbers $C \geq 0$ and $c \geq 0$ such that

$$|(\mathcal{F}T)(\zeta)| \leq C\cdot(1 + \|\zeta\|)^{\alpha} \cdot e^{c\cdot\|I\zeta\|}$$

for $\zeta \in (E')_C$, where $I\zeta$ is the imaginary part of ζ. Notice that this more stringent condition on the entire function $\mathcal{F}T$ on $(E')_C$ not only implies trivially, but also is implied via the Phragmen-Lindelöf theory [2, §47] by the fact that T is entire of exponential type on $(E')_C$ and slowly increasing on E'.

If we consider the Fréchet space $\mathcal{H}_{Nb}(E_C; C)$ of nuclearly entire complex valued functions of bounded type on E_C, then a complex valued function on $(E_C)'$ is the Borel (or equivalently, for that matter, the Fourier) transform of a continuous complex linear form on $\mathcal{H}_{Nb}(E_C; C)$ if and only if it is entire of exponential type on $(E_C)'$ (see [1]). In view of this fact, one might naively expect to prove that an entire complex valued function of exponential type on $(E')_C$ which is slowly increasing on E' must be the Fourier transform of a continuous complex linear form on $\varepsilon_{Nbc}(E; C)$, a true statement if E is finite dimensional (Paley-Wiener-Schwartz). This, however, will be discarded for every infinite dimensional E, even if slow increasingness on E' is guaranteed by boundedness on E'.

THEOREM - <u>If</u> E <u>is infinite dimensional, there is an entire complex valued function of exponential type on</u> $(E')_C$ <u>bounded on</u> E' <u>which is not the Fourier transform of any continuous complex linear form on</u> $\varepsilon_{Nbc}(E;C)$.

The proof will be based on the following known result.

LEMMA - <u>For every integer</u> $n \geq 1$ <u>there is a distribution</u> T_n <u>on</u> R^n

with bounded support, such that its Fourier transform $\mathcal{F}T_n$ is bounded on R^n and the order of T_n tends to infinity as $n \to +\infty$.

PROOF OF THE THEOREM - Set $E_0 = E$ and construct inductively a vector subspace D_n of E of dimension n and a closed vector subspace E_n of E such that $E_{n-1} = D_n \oplus E_n$ in the topological vector space sense for every $n = 1,2,3 \ldots$ Notice that

$$E = D_1 \oplus \ldots \oplus D_n \oplus E_n$$

in the topological vector space sense; let π_n denote the continuous projection of E onto D_n corresponding to such a decomposition. By virtue of the above lemma, let T_n be a distribution on D_n with support contained in the closed ball in D_n centered at 0 and of radius 1, having a Fourier transform $\mathcal{F}T_n$ bounded in modulus by 1 on the dual space $(D_n)'$, and such that

$$|T_n(F)| \leq \sum_{0 \leq m \leq \nu_n} \sup \left\{ \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_N : x \in D_n, \|x\| \leq 1 \right\}$$

for every $f \in \varepsilon(D_n; C)$, where ν_n is supposed to be the order of T_n and to tend to infinity as $n \to +\infty$. We may also assume that $T_n(1) = 0$ for every n, since it suffices to replace T_n by

$$\frac{T_n - T_n(1)\delta}{2}$$

If $f \in \varepsilon_{Nb}(E; C)$, then

$$|T_n(f|D_n) \leq \sum_{0 \leq m < +\infty} \sup \left\{ \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_N : x \in E, \|x\| \leq 1 \right\}$$

and therefore

$$\sum_{1 \leq n < +\infty} \frac{1}{2^n} |T_n(f|D_n)| \leq$$

$$\sum_{0 \leq m < +\infty} \sup\left\{\left\|\frac{1}{m!} \hat{d}^m f(x)\right\|_N : x \in E, \ \|x\| \leq 1\right\}.$$

We define the continuous complex linear T on $\mathcal{H}_{Nb}(E_C; C)$ by setting

$$T(f) = \sum_{1 \leq n < +\infty} \frac{1}{2^n} |T_n(f| D_n)|$$

for $f \in \mathcal{H}_{Nb}(E_C; C)$ and by noticing that then $f|E \in \varepsilon_{Nb}(E; C)$ and

$$\sum_{0 \leq m < +\infty} \sup\left\{\left\|\frac{1}{m!} \hat{d}^m f(x)\right\|_N : x \in E_C, \|x\| \leq 1\right\} < +\infty.$$

The Fourier transform $\mathcal{F}T$ is an entire complex valued function of exponential type on $(E_C)'$ and thus can be looked upon as an entire complex valued function of exponential type on $(E')_C$ in the natural way. For every $\psi \in E'$ we have

$$(\mathcal{F}T)(\psi) = \sum_{1 \leq n < +\infty} \frac{1}{2^n} T_n(e^{i\psi}|D_n)$$

$$= \sum_{1 \leq n < +\infty} \frac{1}{2^n} (\mathcal{F}T_n)(\psi|D_n) ;$$

hence $\mathcal{F}T$ is bounded in modulus by 1 on E'. We claim that $\mathcal{F}T$ is not the Fourier transform of any continuous complex linear form U on $\varepsilon_{Nbc}(E; C)$. In fact, assume that it were. Then we would have $(\mathcal{F}T)(\psi) = (\mathcal{F}U)(\psi)$, that is

$$\sum_{1 \leq n < +\infty} \frac{1}{2^n} T_n(e^{i\psi|Dn}) = U(e^{i\psi})$$

for every $\psi \in E'$. Continuity of U means that there are real numbers $C \geq 0$ and $r \geq 0$ and some integer $\nu \geq 0$ such that

$$|U(f)| \leq C \cdot \sum_{0 \leq m \leq \nu} \sup \left\{ \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_N : x \in E, \ \|x\| \leq r \right\}$$

for every $f \in \varepsilon_{Nbc}(E; C)$. Define a distribution U_n on D_n by setting $U_n(f) = U(f \circ \pi_n)$ for every $f \in \varepsilon(D_n; C)$, and by noticing that $f \circ \pi_n \in \varepsilon_{Nbc}(E; C)$ and

$$|U_n(f)| \leq C \sum_{0 \leq m \leq \nu} \|\pi_n\|^m \sup\left\{ \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_N : x \in D_n, \ \|x\| \leq r \right\}.$$

Hence U_n is of bounded support and of order at most equal to ν. If now $\psi \in (D_n)'$, then $\psi \circ \pi_n \in E'$ and $\psi \circ \pi_n$ vanishes on every D_k with $k \neq n$. Therefore

$$\sum_{1 \leq k < +\infty, \ k \neq n} \frac{1}{2^n} T_k(1) + \frac{1}{2^n} T_n(e^{i\psi}) = U_n(e^{i\psi}) ,$$

that is $\mathfrak{F}T_n = 2^n \cdot \mathfrak{F}U_n$ and so $T_n = 2^n \cdot U_n$. This implies that $\nu_n \leq \nu$ for every n, a contradiction. QED

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ABSTRACT

On the complexification of the dual space E' of a real Banach space E of infinite dimension, there is an entire complex valued function of exponential type, bounded on E', hence slowly increasing on E', which is not the Fourier transform of any distribution with bounded support on E.

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