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# ON THE ROLE OF THE MASS IN THE THEORY OF FREE FERMIONS *

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ABSTRACT

This paper investigates the possibility of regarding the mass as an operator in Dirac's equation for free fermions.

The mass transformation induced by general coordinate transformations which leave invariant Dirac's equation is obtained and includes mass-reversal. Invariance under the Poincaré group imposes strong restrictions on the mass-transformation but seems to allow mass-reversal.

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## I. GENERAL TRANSFORMATION OF FREE SPINORS WITH VARIABLE MASS

Let us represent a Dirac spinor field which describes a free particle with mass m by $\psi(x; m)$, where x is a point in space-time. The corresponding Dirac equation will be written:

$$\left\{ i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m \right\} \psi(x, m) = 0 \tag{1}$$

in the well-know convention; ħ and c are units of action and velocity respectively and the scalar product of two vectors is:

$$a^{\mu} b_{\mu} = a^{\mu} b^{\nu} g_{\mu\nu}$$

$$g_{\mu\nu} = 0, \text{ for } \mu \neq \nu ; \quad g_{00} = - g_{11} = -g_{22} = -g_{33} = 1$$

The equation (1) is obtained by an observer in a reference-frame R.

Let us assume that a passive transformation is carried out and let R' denote the new frame. We make the assumption that the free-field under consideration is now described by the spinor $\psi'(x', m')$ where, in general, the geometrical coordinates x' are connected to the former ones, x, by the relations:

$$x'^{\mu} = f^{\mu}(x) \tag{2a}$$

which can be inverted:

$$x^{\alpha} = g^{\alpha}(x') \tag{2b}$$

The new point in the assumption is that we allow that the mass, as measured in the new frame, be different from m. We impose that Dirac's equation (1) be invariant:

$$\left\{ i\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}} - m' \right\} \psi'(x', m') = 0 \quad . \tag{3}$$

The transformations (2) induce a correlation between the fields

$\psi(x; m)$ and $\psi'(x', m')$:

$$\psi'(x',m') = S(x',m',x,m)\,\psi(x,m) \tag{4}$$

The conditions for the invariance of Dirac's equation are obtained in the familiar way, the operator S being non-singular.

These conditions are the following:

$$S^{-1}\gamma^{\mu}\frac{\partial x^{\nu}}{\partial x'^{\mu}}S = \pm\gamma^{\nu}$$

$$S^{-1}\,m'S - i\,S^{-1}\gamma^{\mu}\frac{\partial x^{\nu}}{\partial x'^{\mu}}\;\frac{\partial S}{\partial x^{\nu}} = \pm\, m\,. \tag{5}$$

The operator S, besides being a matrix which acts on the spinor index of the fields $\psi$, is assumed to depend on the points x and x' and on the possible masses m and m'.

In the particular case of a Poincaré transformation (a, L):

$$X' = a + Lx$$

we have:

$$\frac{\partial X^{\nu}}{\partial X'^{\mu}} = (L^{-1})^{\nu}_{\mu} \tag{6}$$

and the conditions (5) are reduced to the well-known relations:

$$\gamma^{\mu}(L^{-1})^{\nu}_{\mu} = S\,\gamma^{\nu}S^{-1}$$
$$\frac{\partial S}{\partial X^{\nu}} = \frac{\partial S}{\partial X'^{\nu}} = 0\,. \qquad m = m'\,. \tag{7}$$

## II. MASS GENERATED BY GAUGE FUNCTIONS

It is well known that the field $\psi$ may undergo phase transformations of the type:

$$\psi'(X) = e^{i\alpha}\psi(x) \tag{8}$$

where $\alpha$ is a real constant.

When, however, the phase transformation is of the first kind:

$$\psi'(x) = e^{ie\varphi(x)}\psi(x) \tag{9}$$

that is, when the phase is a function $\varphi(x)$ of the coordinates x, it is usually stated that the invariance of Dirac's equation is broken for free fields and restored if the field $\psi$ interacts with an electromagnetic field $A^{\mu}(x)$ according to the equation:

$$\left\{\gamma^{\mu}\left(i\frac{\partial}{\partial x^{\mu}} - e\,A_{\mu}\right) - m\right\}\psi(x) = 0 \tag{10}$$

in which case the field $A^{\mu}$ undergoes a gauge transformation of the second kind [1] induced by equation (9):

$$A'_{\mu}(x) = A_{\mu}(x) - \frac{\partial\varphi(x)}{\partial x^{\mu}}\,. \tag{11}$$

This type of gauge transformation rests on the assumption that the field $\psi$ describes charged particles with constant mass.

We now wish to consider the case of free spinor fields and assume the invariance of Dirac's equation under the transformations (9). In this case, the mass cannot be constant. Indeed if one sets ($\Lambda(x)$ being a real function of x):

$$x' = x$$

$$S(x, m, m') = e^{i\Lambda(x)} \tag{12}$$

then conditions (5) reduce to:

$$m = m' + \gamma^{\alpha} \frac{\partial \Lambda(x)}{\partial x^{\alpha}} \qquad (13)$$

The gauge function $\Lambda(x)$ appears as a kind of internal field which generates a mass which, according to relation (13), is a matrix depending on x:

$$m_{\alpha\beta} = m'_{\alpha\beta} + (\gamma^{\mu})_{\alpha\beta} \frac{\partial \Lambda}{\partial x^{\mu}} \qquad (14)$$

if in some reference frame S' the mass is a constant matrix.

Under this view, the gauge transformation (12) is a mass-transformation.

Let us now assume that the field $\Psi(x, m)$ describes particles with charge e which is revealed through interaction with an electromagnetic field $A^{\mu}(x)$. In this case we let the field $\psi$ undergo the phase transformation:

$$\psi'(x, m') = e^{i[\Lambda(x) + e\varphi(x)]} \psi(x, m) \qquad (15)$$

and assume the invariance of the equation:

$$\left[\gamma^{\mu}\left(i \frac{\partial}{\partial x^{\mu}} - e A'_{\mu}(x)\right) - m'\right] \psi'(x, m') = 0 . \qquad (16)$$

Note that the charge e multiplies $\varphi(x)$ only, in equation (15). The assumed invariance will be satisfied by equations (11) and (14).

## III. IMPROPER-MASS-TRANSFORMATIONS

The components of the spinor $\psi'(x', m')$ transform under equation (4) according to the relation:

$$\psi'_\alpha(x', m') = S_{\alpha\beta}(x', m'; x, m)\, \psi_\beta(x, m) \; . \qquad (17)$$

An interesting example of such a matrix is one which has the $\gamma^5$- matrix as a factor:

$$S(x', m'; x, m) = \gamma^5\, F(x', m'; x, m) \qquad (18)$$

In this case, the invariance condition imposed on equation (3) leads to the relations.

$$F^{-1}\gamma^5\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu}\, \gamma^5\, F = \pm\, \gamma^\nu$$

$$F^{-1}\gamma^5\, m'\, \gamma^5\, F - iF^{-1}\gamma^5\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu}\, \gamma^5 \frac{\partial F}{\partial x^\nu} = \pm\, m \qquad (19)$$

which are simply a result of the substitution (18) into equations (5).

Clearly, one must be careful and regard m and m' as functions of x and of the matrices $\gamma^\mu$.

A particular case is the mass-reversal transformation [2] where F' is the identity in equation (18). Then the equation (3), which is now:

$$\left\{ i\gamma^\mu \frac{\partial}{\partial x^\mu} - m' \right\} \psi'(x, m') = 0$$

becomes:

$$\left\{ i\gamma^\mu \frac{\partial}{\partial x^\mu} - m' \right\} \gamma^5 \psi(x, m) = 0$$

or:

$$\left\{ - i\gamma^\mu \frac{\partial}{\partial x^\mu} - \gamma^5\, m'\, \gamma^5 \right\} \psi(x, m) = 0 \; .$$

Therefore one must have:

$$\gamma^5 m' \gamma^5 = - m \quad . \tag{20}$$

The $\gamma^5$ - transformation therefore belongs to the type of transformations (19) where the minus sign is to be taken in the right-hand side.

In general, transformations (5) may be written:

$$S^{-1}\gamma^\mu \frac{\partial}{\partial x'^\mu} S = e^{i\theta}\gamma^\nu \tag{21}$$

$$S^{-1} m' S - i S^{-1}\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial S}{\partial x^\nu} = e^{i\theta} m$$

where θ is a constant. The constant phase factor $e^{i\theta}$ may be absorbed into the field $\psi$. We shall call proper-mass transformations those for θ = 0 and improper-mass transformations those for θ = π. Mass-reversal transformation is the simplest improper-mass transformation, (20).

## IV. THE RESTRICTIONS IMPOSED BY THE POINCARÉ GROUP

The hermiticity conditions which equations (5) must obey, under the choice:

$$(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0 \tag{22}$$

are:

$$\gamma^0 S^+ \gamma^0 = S^{-1}$$

$$\gamma^\mu \left( \frac{\partial S}{\partial x'^\mu} S^{-1} \right) = \left( \frac{\partial S}{\partial x'^\mu} S^{-1} \right) \gamma^\mu \tag{23}$$

provided that the hermitian conjugate of the mass be of the form:

$$m^{+} = \gamma^{0} \, m \, \gamma^{0} \tag{23a}$$

The first of the equations (23) is well known. The second equation (23) is satisfied by the ansatz:

$$S(x, x') = f(x, x') \mathcal{G} \tag{24}$$

where $f(x, x')$ is an ordinary function of modulus one:

$$f^{*}f = 1 \tag{25}$$

and $\mathcal{G}$ is a matrix in the spinor space.

The usual treatment of Dirac's equation is a consequence of the theory of the representations of the Poincaré group.

This theory requires that the operator $P^{\mu} P_{\mu}$ be a multiple of the identity, where $P^{\mu}$ are the generators of space-time translations:

$$P^{\mu} P_{\mu} = m_{o}^{2} \, I \tag{26}$$

$m_o$ is a number and I is the identity.

In this case, the term which contains the space-time derivatives of the transformation operator S in equation (5) must vanish. Equation (5) reduces to:

$$\begin{aligned} S^{-1}\gamma^{\mu}(L^{-1})^{\nu}{}_{\mu} S &= \pm \gamma^{\nu} \\ S^{-1} \, m' \, S &= \pm m \end{aligned} \tag{27}$$

provided that it is assumed that:

$$m'^{2} = m^{2} = m_{o}^{2} \, I \, .$$

It is clear that the general formulae (5) for the mass-operator would violate condition (26).

If equation (26) holds the gauge transformations (12) cannot give rise to equation (13). The $\Lambda$-function would have to fulfill the condition:

$$g^{\alpha\beta} \frac{\partial \Lambda}{\partial x^{\alpha}} \frac{\partial \Lambda}{\partial x^{\beta}} = 0 \qquad (28)$$

and the mass operator would have to be such that:

$$m' \gamma^{\alpha} + \gamma^{\alpha} m' = 0 \qquad (29)$$

i.e. the mass would have to be proportional to $\gamma^5$, which would violated the hermiticity requirement (23a). Thus invariance of Dirac's equation both under the Poincaré and the gauge group (12) requires a vector field transforming like equation (11).

We see that mass-reversal (equation (20)) is a special case of formulae (27), where

$$S = \gamma^5$$

$$(L^{-1})^{\nu}_{\mu} \longrightarrow S^{\nu}_{\mu}$$

and the minus sign holds in the right hand side of equs. (27).

Mass reversal would be satisfied by an ansatz of the form:

$$m = m_o \, a_{\mu} \gamma^{\mu}$$

Where $a_{\mu}$ is a time-like unit vector (for instance, normal to a space-like surface at every point) and $m_o$ is a real number:

$$a_{\mu} a^{\mu} = 1$$

so that:

$$m^2 = m_o^2$$

and:

$$m^{+} = \gamma^{o} \, m \, \gamma^{o}$$

Thus an equation invariant under mass-reversal

$$\gamma^{\mu}\left(i\frac{\partial}{\partial x^{\mu}} - m_{o}\, a_{\mu}(x)\right)\psi(x) = 0$$

is similar to the equation of a mass-less charged particle in interaction with an electromagnetic field.

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