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WEIGHTED APPROXIMATION FOR FUNCTION ALGEBRAS AND QUASI-ANALYTIC MAPPINGS *

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In this note, we shall sketch a different approach to the proofs of some results in weighted approximation theory that we established elsewhere.

Such results, in their general form, were concerned with modules of continuous functions. Algebras of continuous functions constituted just a particular case, since every algebra is a module over itself.

The viewpoint that we shall adopt here consists precisely in inverting the order we preferred previously. We shall now deal firstly with the case of algebras of continuous functions, and then use it to treat the general case of modules of continuous functions.

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* This work was done when the author was at the University of Rochester, Rochester, New York.

Let E be a completely regular space. Denote R, or C, the system of all real, or complex, numbers; use K to refer to either R or C. In case it appears to be convenient, we shall include K in the notation for each function space in order to emphasize that the functions take values in K. Let $\mathcal{C}(E)$ be the algebra of all continuous K-valued functions on E. Fix a set V of weights on E, that is, of upper-continuous positive functions on E; we shall assume that:

(1) If $v_1, v_2 \in V$, there are $\lambda > 0$ and $v \in V$ such that $v_1 \leq \lambda v$ and $v_2 \leq \lambda v$.

(2) If $x \in E$, there is $v \in V$ such that $v(x) > 0$.

We then denote by $\mathcal{C}V_\infty(E)$ the vector subspace of $\mathcal{C}(E)$ of all f such that, for every $v \in V$ and every $\epsilon > 0$, the subset of E

$$\{x \in E \mid v(x) \cdot |f(x)| \geq \varepsilon\}$$

is compact. We shall use on $\mathcal{C}V_\infty(E)$ the locally convex topology defined by the family of the following seminorms:

$$f \longmapsto \|f\|_v = \sup\{v(x) \cdot |f(x)| \mid x \in E\}$$

for all $v \in V$.

We shall denote by $\mathcal{A}$ a subalgebra of $\mathcal{C}(E)$ containing the unit 1. It defines an equivalence relation $E/\mathcal{A}$ on E if we consider $x_1, x_2 \in E$ as being equivalent whenever $f(x_1) = f(x_2)$ for all $f \in \mathcal{A}$.

We shall denote by $\mathcal{W}$ a vector subspace of $\mathcal{C}V_\infty(E)$ and shall assume that $\mathcal{W}$ is a module over $\mathcal{A}$, that is $\mathcal{A}\mathcal{W} \subset \mathcal{W}$

$\mathcal{W}$ is localizable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$ when the following condition holds true: a function f belonging to $\mathcal{C}V_{\infty}(E)$ is in the closure of $\mathcal{W}$ in $\mathcal{C}V_{\infty}(E)$ if (and always only if), for any $v \in V$, any $\epsilon > 0$ and any equivalence class X modulo $E/\mathcal{Q}$, there exists some $w \in \mathcal{W}$ such that

$$v(x).|w(x) - f(x)| < \epsilon \quad \text{for any } x \in X .$$

In particular, if $\mathcal{Q}$ is contained in $\mathcal{C}V_{\infty}(E)$, we shall say that $\mathcal{Q}$ is localizable in $\mathcal{C}V_{\infty}(E)$ if $\mathcal{Q}$ is localizable under $\mathcal{Q}$ in $\mathcal{C}V_{\infty}(E)$; in this case we have $\mathcal{W} = \mathcal{Q}$ and $\mathcal{Q}$ is treated as a modulo over $\mathcal{Q}$. It is imediate that the following conditions are equivalent:

(a) $\mathcal{Q}$ is localizable in $\mathcal{C}V_{\infty}(E)$.

(b) A function f belonging to $\mathcal{C}V_{\infty}(E)$ is in the closure of $\mathcal{Q}$ in $\mathcal{C}V_{\infty}(E)$ if (and always only if) f is constant on every equivalence class modulo $E/\mathcal{Q}$.

Two basic results for the approach to be adopted in the present note are the following ones.

THEOREM 1. Assume that the set of all submodules of $\mathcal{W}$ over $\mathcal{Q}$ that are localizable under $\mathcal{Q}$ in $\mathcal{C}V_{\infty}(E)$ do generate $\mathcal{W}$ (in the sense that the vector space sum of all such submodules is dense in $\mathcal{W}$). Then $\mathcal{W}$ is localizable under $\mathcal{Q}$ in $\mathcal{C}V_{\infty}(E)$.

THEOREM 2. Assume that $\mathcal{W} = \mathcal{Q}\, w$, where $w \in \mathcal{C}V_{\infty}(E)$. Then $\mathcal{W}$ is localizable under $\mathcal{Q}$ in $\mathcal{C}V_{\infty}(E)$ if and only if $\mathcal{Q}$ is localizable in $\mathcal{C}(V.|w|)_{\infty}(E)$, where $V.|w| = \{v.|w| \mid v \in V\}$.

Let Ω_n denote the set of all fundamental weights on R^n in the classical sense of Serge Bernstein; that is, the set of all upper-semicontinuous positive functions ω on R^n such that the algebra of all K-valued polynomials on R^n is contained and dense in $\mathcal{C}\omega_{\infty}(R^n)$. Let Γ_n be the set of all upper-semicontinuous positive functions γ on R^n such that $\gamma^h \in \Omega_n$ for any $h > 0$.

THEOREM 3. Assume that K = R and that, for every $v \in V$, $\mathcal{A}$ is generated as an algebra with unit by the set of all $a \in \mathcal{A}$ for each of which there exists some $\gamma \in \Gamma_1$ satisfying the estimate

$$v(x) \leq \gamma\left[a(x)\right] \quad \text{for any } x \in E.$$

Then $\mathcal{A}$ is localizable in $\mathcal{C}V_{\infty}(E)$.

The proof of this result is based on the following straightforward extension of a classical lemma due to Kakutani and Stone.

LEMMA 1. Let K = R and $\mathcal{L}$ be a sublattice of $\mathcal{C}V_{\infty}(E)$. A function f belonging to $\mathcal{C}V_{\infty}(E)$ is in the closure of $\mathcal{L}$ in $\mathcal{C}V_{\infty}(E)$ if and only if, for every $x_1, x_2 \in E$ and every $\epsilon > 0$, there exists some $g \in \mathcal{L}$ such that

$$|g(x_i) - f(x_i)| < \varepsilon \quad \text{for } i = 1, 2.$$

Once Theorem 3 is proved, we use it together with Theorem 1 and Theorem 2 to establish Theorem 4 of [3] in the case K = R.

THEOREM 4. Assume that, for every $v \in V$, $\mathcal{A}$ is generated as an algebra with unit by the set of all $a \in \mathcal{A}$ for which

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt[m]{M_m}} = +\infty.$$

where

$$M_m = \sup \left\{ v(x).|a(x)^m| \; | x \in E \right\} \text{ for } m = 0, 1, \ldots,$$

and that $\mathcal{A}$ is self-adjoint in the complex case. Then $\mathcal{A}$ is localizable in $\mathcal{C}V_\infty(E)$.

The proof of this result is based on the following lemma.

Let $\mathcal{E}$ and $\mathcal{F}$ be two real locally convex spaces, where $\mathcal{F}$ is assumed to be separated. Let U be an open subset of $\mathcal{E}$, and ϕ an indefinitely differentiable mapping of U into $\mathcal{F}$. Denote by $d^m \phi(x)$ the m-th differential of ϕ at $x \in U$ for $m = 0, 1, \ldots$; it is a symmetric m-linear mapping of $\mathcal{E}^m$ into $\mathcal{F}$. We shall say that ϕ is quasi-analytic on U if, letting $d\,\phi(x)$ denote for each $x \in U$ the closed vector subspace of $\mathcal{F}$ generated by all $d^m\phi(x).(x_1,\ldots,x_m)$, where $m = 0, 1, \ldots$, and $x_1,\ldots,x_m \in \mathcal{E}$ are arbitrary, then the mapping $x \mapsto d\,\phi(x)$ is constant on every connected component of U; then for every $x \in U$, $d\phi(x)$ is equal to the closed vector subspace of $\mathcal{F}$ generated by $\phi(U_x)$, where U_x is the connected component of x in U. We refer to [2].

Let V denote the set of all $v^{1/m}$, for arbitrary $v \in V$ and $m = 1, 2, \ldots$. Notice that $\mathcal{C}V_\infty(E)$ is the largest subalgebra of $\mathcal{C}V_\infty(E)$; therefore, if $\mathcal{A}$ is contained in $\mathcal{C}V_\infty(E)$, then actually $\mathcal{A}$ is contained in $\mathcal{C}V_\infty(E)$.

LEMMA 2. Let K = R and $\mathcal{X}$ be a vector subspace of $\mathcal{A}$ generating $\mathcal{A}$ as an algebra with unit. Then $\mathcal{A}$ is localizable in $\mathcal{C}V_\infty(E)$ if and only if the indefinitely differentiable mapping

$$f \mapsto e^{if}$$

of $\mathfrak{X}$ into $\mathcal{C}V_\infty(E;C)$ is quasi-analytic, where $\mathfrak{X}$ is endowed with the topology induced by $\mathcal{C}V_\infty(E;R)$.

Once Theorem 4 is proved, we combine it with Theorem 1 and Theorem 2 to establish Theorem 7 of [3].

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