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by

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INTRODUCTION

The parametric representation of a dynamical system is the more simple example of a Lagrangian formalism where the initial Cauchy data cannot be arbitrariely specified, and even after the proper specification of the initial data, the evolution of the basic configuration variables remains not fixed.

The Hamiltonian version of the theory has a phase space which is not entirely determined since it contains a constraint, the so called Hamiltonian constraint. Therefore, the initial condition problem in terms of canonical pair of variables is also subjected to a subsidiary condition, the constraint equation, which limits the possible choice of the initial variables.

Furthermore, the evolution of the canonical variables out of some proper initial values remains arbitrary.

Both kinds of methods may be presented from a purely geometric approach, which is the point of view used in this paper. The Action integral is invariant under a parameter change, since the Lagrangian is a homogeneous function of the velocities Q' of the first degree, which in turn implies that the Hamiltonian is equal to zero. In our geometric language, the Action integral gives the lenght of the arc element in a one-dimentisonal space, the "parameter space", equipped with a metric which is given by the square of the Lagrangian. The invariance of the Action integral under parameter changes is therefore the statement of the invariance of lenght of the arc element under a transformation in the "parameter space", which we call a coordinate transformation. Under such transformation the Lagrangian changes as a first order covariant vector, which mathematically is equivalent to the statement that the Lagrangian is a homogeneous function of the velocities of the first degree.

The affine connection associated to the "parameter space" is a well determined quantity, and as it is clear, no curvature exists in this space.

The property that the initial condition problem has no unique solution is geometrically interpreted by the rather simple statement that the configuration and velocity variables are respectively scalars and vectors and therefore depend on the choice made on the coordinates (or parameters). A similar interpretation can also be made for the initial condition problem in terms of canonical pair of variables.

It is treated the problem of determination of invariant dynamical functions, that means, functions of the dynamical variables, and possibly also of the corresponding velocities, which posess a well prescribed value when we carry out a coordinate transformation. The method used for the determination of those invariant dynamical functions in the Hamiltonian representation is an extension for the classical dynamics of a method used by Komar in general relativity.

The classical commutator algebra for those invariant dynamical functions is given. It is shown directly the equivalence of this commutator algebra with Dirac's method of introducing a new bracket for giving account of the presence of second class variables.

The intrinsic value for those invariant dynamical functions is independent of the particular coordinate condition which is used. However, the commutator between invariant dynamical functions depends on the choice made on the coordinates. In a future paper, we will study up to what extension we can determine a modification on the definition of what we called a starred variable for obtaining a commutator algebra with a more symmetrical behavior with respect to the choice of coordinates.

1. THE CLASSICAL DYNAMICS UNDER PARAMETRIC FORM

We start from the usual Hamilton's variational principle which states that the equations of motion are the extremals of the variational problem,

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0$$

for variations δq which vanish at the boundaries of integration. The same physical content may also be presented by means of first order differential equations if we expand the dimensions of the configuration space by introducing the momenta variables as independent variables and use in place of the Lagrangian, the Hamiltonian.

$$\delta \int_{t_1}^{t_2} (p_a \dot{q}^a - H(p, q, t))\, dt = 0$$

$$\delta p = \delta q = 0 \text{ at the boundaries,}$$

which gives as result the canonical equations of motion,

$$\dot{p}_a = -\frac{\partial H}{\partial q^a}, \quad \dot{q}^a = \frac{\partial H}{\partial p_a}$$

In the present treatment we will be interested in a formulation which contains besides the usual invariance properties of classical dynamics, another invariance property, now with respect to a function group of transformations which depends on one arbitrary function. As is known, such formulation is obtained when we introduce another parameter to take place of time, and let this variable in the same ground as any other q-type variable.

$$\delta\int_{\theta_1}^{\theta_2} Lt'\,d\theta = \int_{\theta_1}^{\theta_2} \mathcal{L}\,d\theta = 0$$

$$\mathcal{L} = L\left(q, \frac{q'}{t'}, t\right) t' \tag{1}$$

If the dimension of the q-space is n, the extended representation in its canonical form will belong to a phase space with dimension $2n+2$.

It is a simple matter to verify that the Lagrangian in the extended representation is a homogeneous function of the "velocities" q'^a, t' (which we call for short as Q'^i) of the first order. As consequence of this, the Action Integral is invariant under an arbitrary change of the parameter θ.

$$\int \mathcal{L}\,d\theta = \int \tilde{\mathcal{L}}\,d\tilde{\theta} \tag{2}$$

where $\tilde{\theta}$ is an arbitrary function of θ. Since the momenta P_i are given by,

$$P_i = \frac{\partial \mathcal{L}}{\partial Q'^i} \tag{3}$$

it follows that,

$$P_i Q'^i = \mathcal{L} \tag{4}$$

which implies that the Hamiltonian in the extended representation vanishes.

$$\mathcal{H} = P_i \, Q'^{\,i} - \mathcal{L} = 0 \tag{5}$$

Thus, as consequence of the invariance of the integrand of the Action Integral under arbitrary transformations of the parameter θ, the Hamiltonian representation contains a constraint, the so called Hamiltonian constraint.

We may present the relation (1) also as,

$$\mathcal{L} = \left(p_i \frac{q'^{\,i}}{t'} - H \right) t' \, , \tag{6}$$

if we call by π the momenta conjugate to t, we can write Eq. (4) as

$$\mathcal{L} = p_i \, q'^{\,i} + \pi t' \tag{7}$$

A comparison between the equations (6) and (7) gives,

$$H(p, q, t) + \pi = 0 \tag{8}$$

Since $\mathcal{H}$ vanishes too, we may set it as proportional to the left hand side of equation (8).

$$\mathcal{H} = \alpha \, (H + \pi)$$

The value for the constant multiplicative factor is easily found by using the equations of motion in the canonical form

$$P'_i = - \frac{\partial \mathcal{H}}{\partial Q^i} \, , \quad Q'_i = \frac{\partial \mathcal{H}}{\partial P_i} \, ,$$

we find, $\alpha = t'$. Then, we can write the Hamiltonian constraint as,

$$\mathcal{H} = t'(H + \pi) = 0$$

where it is understood that the part which gives the constraint is really that given by equation (8).

From now on we shall adopt the following notation: we call the variables Q and P as dynamical variables, the parameter θ will be called as coordinate, the reason for this comes from the fact that presently the variable θ appears similarly as it does in any generally covariant theory, that means, playing the role of a coordinate. It is the intention of this paper to pursue further this analogy. Indeed, we can replace the previous statement which explained the invariance of the Action Integral by a purely geometric statement: Under a coordinate transformation θ changes as a first order contravariant vector,

$$d\tilde{\theta} = \frac{d\tilde{\theta}}{d\theta}\, d\theta \qquad (9)$$

The Lagrangian $\mathcal{L}$ changes as a first order covariant vector,

$$\tilde{\mathcal{L}} = \frac{d\theta}{d\tilde{\theta}}\, \mathcal{L} \qquad (10)$$

Which is mathematically equivalent to the statement that $\mathcal{L}$ is a homogeneous function of the Q'^{i} of the first degree.

Thus, we have to treat with a one-dimensional tensor calculus, with coordinate θ which is allowed to transform as,

$$\tilde{\theta} = \varphi(\theta)$$

where the function $\varphi(\theta)$ is arbitrary, except for eventual boundary conditions.

The Lagrangian is a linear function of the Q'^i, and these dynamical variables form a first order covariant vector. The dynamical variables Q^i and P_i are scalars (recall that P_i being the partial derivaties of $\mathcal{L}$ with respect to Q'^i, will be a homogeneous function of the Q'^i of the degree zero, which means to form a scalar in our geometric language).

The Lagrangian in the old representation, the L, is also a scalar, $\tilde{L} = L$.

2. GEOMETRIC INTERPRETATION OF PARAMETRIZED DYNAMICS

We look at the variational principle which leads to the equations of motion as if it were an extension of the concept of a geodesic for a one-dimensional space.

$$\delta \int \mathcal{L}\, d\theta = \delta \int ds \; .$$

The reason for this interpretation, even it it is difficult to understand what might be a geodesic in one-dimensional space, is the following: It is known that the canonical formulation of general relativity represents a theory where the four coordinates pay the role of parameters, as result the theory contains constraints. The motion of a particle in a gravitational field is described by the geodesic associated to the metric which is solution of the gravitational field equations. Presently, we treat with a theory which is much more poor in structure, but nevertheless contains some of the properties of general relativity.

Namely, instead of four parameters it has just one parameter, and as consequence its canonical formulation contains just one constraint, which as we will see is associated to the generator of infinitesimal displacements along the θ-space. We may, as we will do, interpret the Lagrangian equations of motion as a geodesic in θ-space (the term geodesic is to be understood purely in technical sense, since no intuitive geometric interpretation is possible), where the metric is given by,

$$ds^2 = g\, d\theta^2,$$
$$g = \mathcal{L}^2 . \tag{11}$$

We will see that no curvature tensor exists in the θ-space, as it is clear from the fact that this is a one-dimensional space. Thus, we have to take the metric as a given variable [1], which is equivalent to take the Lagrangian as a given variable as we do in all cases where a Lagrangian is used (it has to satisfy the necessary invariance requirements, but nothing else is required concerning its origin).

With respect to the transformation laws of $\mathcal{L}$ or g, we may use the terms covariant tensors or relative scalars with weight -1 and -2 respectively, if we take for the Jacobian of the transformation $J = \frac{d\tilde{\theta}}{d\theta}$. However, we prefer to use the terms tensors, since that at one dimension there is no way to know what is the difference between determinant and matrix element. [2] In such notation g transforms as a second order covariant tensor,

$$\tilde{g}(\tilde{\theta}) = g(\theta) \left(\frac{d\theta}{d\tilde{\theta}}\right)^2$$

We proceed to introduce an affine connection in θ-space. Let $f(\theta)$ be a covariant vector of the first order,

$$\tilde{f}(\tilde{\theta}) = f(\theta)\,\frac{d\theta}{d\tilde{\theta}}$$

The derivatives $\frac{df}{d\theta}$ will transform as,

$$\frac{d\tilde{f}(\tilde{\theta})}{d\tilde{\theta}} = \frac{d^2\theta}{d\tilde{\theta}^2}\,f + \left(\frac{d\theta}{d\tilde{\theta}}\right)^2 \frac{df}{d\theta}\,. \tag{12}$$

Due to the term $\frac{d^2\theta}{d\tilde{\theta}^2}\,f$ in the right hand side of equation (12), the quantity f' does not form a tensor. We introduce the quantity,

$$f_{;}(\theta) = f' - f\Delta\,, \tag{13}$$

and impose that it behaves as a second order covariant tensor,

$$\tilde{f}_{;}(\tilde{\theta}) = f_{;}(\theta)\left(\frac{d\theta}{d\tilde{\theta}}\right)^2 \tag{14}$$

Equations (12), (13) and (14) imply that Δ transform as,

$$\tilde{\Delta} = \frac{d\theta}{d\tilde{\theta}}\,\Delta + \frac{d^2\theta}{d\tilde{\theta}^2}\,\frac{d\tilde{\theta}}{d\theta}\,. \tag{15}$$

It is easy to verify that a Δ transforming in the way given by the relation (15) is,

$$\Delta = \frac{1}{2g}\,g' = \frac{1}{2}\,\frac{d}{d\theta}\,\ln\ g\,. \tag{16}$$

Thus, the quantity Δ given by (16) is the affine connection in θ -space, and $f_{;}$ is the covariant derivative of the covariant vector $f(\theta)$. Similarly we can define the covariant derivative of a contravariant vector $\varphi(\theta)$.

$$\varphi_{;} = \varphi' + \varphi\Delta \tag{17}$$

which transforms as,

$$\tilde{\varphi}_{;}(\tilde{\theta}) = \frac{d\tilde{\theta}}{d\theta}\,\frac{d\theta}{d\tilde{\theta}}\,\varphi_{;}(\theta)$$

The covariant derivative of the metric g vanishes,

$$g_{;} = g' - 2g\Delta = 0$$

where we have made use of the relation (16). From the formulas given up to here, it is a simple matter to generalize the operation of covariant derivative for a tensor of arbitrary order,

$$j_{;} = j' \pm nj\Delta \quad ,$$

the positive sign holds if j is a contravariant tensor of the order n, the oposite sign is used if j is a covariant tensor of this same order.

The operation of raising or lowering the order of variance of tensors is similar to the usual operation in tensor calculus (conventional tensor calculus). For instance, the covariant components associated to the contravariant vector j is,

$$k = gj$$

The following results are obvious: Every function $f(Q^i, Q'^i)$ which is a homogeneous function of the Q'^i of the degree n, form a covariant tensor of the order **n**. As consequence, all functions of the above form which are homogeneous in Q'^i with degree $-n$, will form contravariant tensors of the order n.

The Lagrange equations of motion,

$$\frac{d}{d\theta}\frac{\partial \mathcal{L}}{\partial Q'^{i}} - \frac{\partial \mathcal{L}}{\partial Q^{i}} = 0$$

in terms of g read as,

$$\left(\frac{d}{d\theta} - \Delta\right)\frac{\partial g}{\partial Q'^{i}} = \frac{\partial g}{\partial Q^{i}} \tag{18}$$

Since $\frac{\partial g}{\partial Q'^{i}}$ is a covariant vector, we can rewrite this relation in compact covariant form,

$$\left(\frac{\partial g}{\partial Q'^{i}}\right)_{;} - \frac{\partial g}{\partial Q^{i}} = 0 \tag{19}$$

Since $\frac{\partial \mathcal{L}}{\partial Q'^{i}}$ is a scalar [3], its derivative respect to θ is the same as a covariant derivative, thus, we may also write the Lagrange equation in the same form as the relation (19).

Another interesting result, which will be used later on, is the equation which gives the expression of g' in terms of the Q^{i} and Q'^{i},

$$g' = - Q'^{i}\left(\frac{\partial g}{\partial Q^{i}} - \frac{d}{d\theta}\frac{\partial g}{\partial Q'^{i}}\right) \tag{20}$$

From now on, we shall supose that $\mathcal{L}$ is explicitly independent of θ.

3. THE INITIAL CONDITION PROBLEM

We can represent the configuration variables $Q^i(\theta)$ by the following power series expansion,

$$Q^i(\theta) = Q^i(\theta) + \theta(Q'_i)_o + \frac{\theta^2}{2}(Q''^i)_o + \ldots \tag{21}$$

since the Lagrange equations are second order differential equations, we need two quantities as initial conditions, taking into account the correct number of variables, we state the Cauchy initial value problem as: Given at $\theta = \theta_o$ the $2n + 2$ variables $q^i(\theta)$, $q'^i(\theta)$, $t(\theta)$ and $t'(\theta)$, then the evolution of the system throughout the region $\theta > \theta_o$ of the $n+1$ variables $q^i(\theta)$, $t(\theta)$ is to be determined from the Lagrangian equations of motion. For convenience we will take $\theta_o = 0$. For solving the Cauchy problem we have to obtain Q'' as function of the remaining variables. We have,

$$\frac{\partial^2 \mathcal{L}}{\partial Q^k \partial Q'^i} Q'^k + \frac{\partial^2 \mathcal{L}}{\partial Q'^k \partial Q'^i} Q''^k - \frac{\partial \mathcal{L}}{\partial Q^i} = 0 \tag{22}$$

Let's consider $\dfrac{\partial^2 \mathcal{L}}{\partial Q'^k \partial Q'^i}$ as the matrix element Y_{ik} of a matrix Y

$$Y = \left(\frac{\partial^2 \mathcal{L}}{\partial Q'^k \partial Q'^i} \right)$$

If the matrix Y has an inverse, we can solve (22) for the Q'',

$$Q''^k = \overset{-1}{Y}{}^{ki} \left(\frac{\partial \mathcal{L}}{\partial Q^i} \quad \frac{\partial^2 \mathcal{L}}{\partial Q^\ell \partial Q'^i} Q'^\ell \right) \tag{23}$$

From which any higher derivative in θ may be computed. Replacing (23) and its derivatives into (21) we obtain the solution of the Cauchy problem. Thus, the existence of solutions of this problem reduces to the existence of Y^{-1}. We will see however that such matrix does not exist in the present formulation. For proving this we recall that $\mathcal{L}$ satisfies,

$$\frac{\partial \mathcal{L}}{\partial Q'^{i}} Q'^{i} = \mathcal{L}$$

differentiating in Q'^{k} we get

$$Y_{ik} Q'^{k} = 0 .$$

which means that Q' is an eigenvector of Y belonging to the eigenvalue zero. Thus, the matrix Y is singular, and we cannot predict the values of $Q^{i}(\theta)$ if we know the Cauchy data at any $\theta_o < \theta$. The geometric reason for this is similar to that happening in general relativity (as we said before, both theories are similar in structure), namely, if we know the Cauchy data at some value of θ, say θ_o, we cannot predict the value of $Q^{i}(\theta)$ at a later value of θ since we can always consider a transformation $\tilde{\theta} = f(\theta)$, such that $f(\theta) \rightarrow \theta_o$ as $\theta \rightarrow \theta_o$ and otherwise is arbitrary. The Cauchy data at $\theta = \theta_o$ remains the same, but the value of $Q^{i}(\theta)$ for $\theta > \theta_o$ remains arbitrary since the argument of this function is arbitrary. As it is clear, this arbitrariety in choice of Q^{i} as function of θ has no physical implication, since everything which posess physical significance is independent of the choice of the "coordinate" θ.

The metric g is also undetermined as function of θ, since g is a function of $Q^{i}(\theta)$ and $Q'^{i}(\theta)$. For instance, for a particle in a conservative field,

$$g(\theta) = \left\{ \frac{m}{2} \frac{q'^{2}}{t'} - V(q)t' \right\}^{2}$$

Now, it appears as an important question, the problem of the determination of quantities which possess a well defined behavior when we allow θ to vary. In the next section we will treat this problem in the framework of the Lagrangian formulation. Later on, we will treat the same problem in the Hamiltonian formulation.

4. OBSERVABLES IN THE LAGRANGIAN REPRESENTATION

We call by the term observable any quantity which has a well defined evolution when we allow the coordinate θ to vary. Geometrically the name observable is equivalent to invariant.

As it turns out clear, we cannot hope to construct such a quantity out of functions of θ. For an example, let's consider once more the metric $g(\theta)$. Suppose that at $\theta = 0$ we know the value of g. Since a coordinate transformation has no intrinsic physical significance, we can always consider a transformation

$$\tilde{\theta} = f(\theta) = \theta + \xi(\theta)$$

(where $\xi(\theta)$ is an infinitesimal of the first order). The original $g(\theta)$ as well as the g transformed according to this transformation have the same physical meaning. Since g transforms as a second order covariant tensor, we will obtain

$$\tilde{g}(\theta) = g(\theta) - 2g(\theta)\xi' - \xi g'(\theta) , \qquad (24)$$

Now, we can take the arbitrary function $\xi(\theta)$ in such way that $\xi(\theta)$ and its first derivative, $\xi'(\theta)$ vanish at $\theta = 0$,

$$\xi(0) = \xi'(0) = 0 .$$

This implies that at $\theta = 0$, the metric is a well determined quantity, $\tilde{g}(0) = g(0)$. But for all $\theta > 0$ we will have $g(\theta)$ and $\tilde{g}(\theta)$ of (24) as the description of the same physical situation. Since then $\xi(\theta)$ is fully arbitrary it follows that $g(\theta)$ cannot be a well determined quantity throughout all region $\theta > 0$.

However, it is possible, as we will see, to construct observables out of functionals of the dynamical variables. We take these functionals in the form of integrals of the type,

$$M = \int M(\theta)\, d\theta \; , \tag{25}$$

where we consider for $M(\theta)$ the general expression,

$$M(\theta) = \int f^{ij}(Q^k(\bar{\theta}),\, Q'^k(\bar{\theta}))K_{ij}(\theta,\bar{\theta})d\bar{\theta} \; . \tag{26}$$

We note that for the particular choice,

$$K_{ij} = \delta_{ij}\, \delta(\theta,\, \bar{\theta}) \; ,$$

$$f^{ii} = \mathcal{L} = (g)^{\frac{1}{2}} \; ,$$

M turns out to be equal to the Action integral. We consider the factor K_{ij} which appears in Eq. (26) as independent of the dynamical variables, so that all dependence on those variables is contained in f^{ij}. We mention that θ is completely independent of the dynamical variables as long as we do not impose any coordinate condition. In this case, θ becomes some function of the dynamical variables, for instance, we may use as coordinate condition $\theta = f(t)$, and t presently is a dynamical variable.

We proceed to determine the condition which M has to satisfy for being an observable. This condition is obtained when we equal to zero the variation in M which comes by effect of a coordinate transformation of the form given before, with a generating function $\xi(\theta)$. Under this transformation the dynamical variables change as,

$$\delta Q^k(\theta) = \tilde{Q}^k(\theta) - Q^k(\theta) = - Q'^k(\theta)\, \xi(\theta) \,,$$
$$\delta Q'^k(\theta) = \tilde{Q}'^k(\theta) - Q'^k(\theta) = \frac{d}{d\theta}\, \delta Q^k(\theta) \,.$$

Where we have considered only the first order terms in ξ. We consider the function $\xi(\theta)$ in such way that it vanishes at the boundaries of integration (which is the unique restriction imposed on this function). Using this condition we can drop all surface terms in the variation δM, and obtain as result,

$$\delta M = - \iint d\theta d\bar{\theta} \left\{ K_{ij}(\theta,\bar{\theta}) \frac{\partial f^{ij}(\bar{\theta})}{\partial Q^k(\theta)} - \frac{d}{d\theta} \left(K_{ij}(\theta,\bar{\theta}) \frac{\partial f^{ij}(\bar{\theta})}{\partial Q^{ik}(\theta)} \right) \right\} \xi(\theta) Q'^k(\theta) = 0.$$

Since $\xi(\theta)$ is arbitrary, this implies that,

$$\int d\bar{\theta} \left\{ K_{ij}(\theta,\bar{\theta}) \frac{\partial f^{ij}(\bar{\theta})}{\partial Q^k(\theta)} - \frac{d}{d\theta} \left(K_{ij}(\theta,\bar{\theta}) \frac{\partial f^{ij}(\bar{\theta})}{\partial Q'^k(\theta)} \right) \right\} Q'^k(\theta) = 0 \,. \quad (27)$$

Which is the condition satisfied by an observable M. Again we note that in the case where K_{ij} is the Kronecker delta in i, j times the Dirac delta function in θ, $\bar{\theta}$, and $f^{ii} = \mathcal{L}$, we get,

$$\left\{ \frac{\partial \mathcal{L}}{\partial Q^k} - \frac{d}{d\theta} \left(\frac{\partial \mathcal{L}}{\partial Q'_k} \right) \right\} Q'^k = 0 \,. \quad (28)$$

A direct calculation shows that this result holds. Geometrically it means that the integral of the arc element in θ-space is invariant.

The relation (22) and (28) give,

$$Q'^k\, Q'^i \frac{\partial^2 \mathcal{L}}{\partial Q^i \partial Q'^k} + Q'^k\, Y_{ik}\, Q''^i - Q'^k \frac{\partial \mathcal{L}}{\partial Q^k} = 0$$

Using that Q' is an eigenvector of Y belonging to the eigenvalue zero, we can write this as,

$$Q'^k\, Q'^i \frac{\partial^2 \mathcal{L}}{\partial Q^i \partial Q'^k} - Q'^k \frac{\partial \mathcal{L}}{\partial Q^k} = 0 \qquad (29)$$

The equation (29) is a relation of the type,

$$F(Q^i(\theta),\, Q'^i(\theta)) = 0$$

Thus, the relation (29) represents one condition on the choice of the Cauchy data on the point θ = constant. Therefore, the property of the Action integral being invariant under coordinate transformations implies that the initial data cannot be chosen arbitrariely, but is restricted by one condition. This fact is the equivalent in the Lagrangian formulation of what happens in the Hamiltonian formulation where we have one constraint limiting the behavior the canonical variables, and therefore limiting too the Cauchy data in terms of canonical pairs of variables.

5. OBSERVABLES IN THE HAMILTONIAN REPRESENTATION, AND THE CO-ORDINATE CONDITION

The Hamiltonian formulation of the parametrized dynamics contains one constraint, the Hamiltonian constraint $\mathcal{H} = 0$, which vanishes weakly according to Dirac's terminology [4].

Given any function of the canonical variables Q, P we obtain the following value for the Poisson bracket of this function (which we consider as explicitly independent of θ) with the Hamiltonian,

$$[F, \mathcal{H}] = \frac{dF}{d\theta} \tag{30}$$

We saw before that both Q and P are scalars in θ-space, then, any function of them will represent a function of scalars, and its variation under a coordinate transformation will contain only a transport term (derivative of this function with respect to θ). This property along with the Eq. (30) allow us to introduce the weakly vanishing quantity

$$G(\theta) = -\xi(\theta)\mathcal{H}(\theta) \tag{31}$$

as the generator of infinitesimal coordinate transformations, since then,

$$[F, G] = -\xi(\theta)\, F'(\theta) = \delta F(\theta)$$

is the variation in a scalar function when we carry out an infinitesimal coordinate transformation with generating function $\xi(\theta)$. We calculate now the Poisson bracket between Q'^{i} and G. Using Hamilton's equation we get,

$$[Q'^{i}, G] = \left[[Q^{i}, \mathcal{H}], G\right] . \tag{32}$$

The Jacobi identity allow us to rewrite this as,

$$\left[[Q^{i}, \mathcal{H}], G\right] = \left[[Q^{i}, G], \mathcal{H}\right] = \frac{d}{d\theta}[Q^{i}, G] ,$$

which gives,

$$\left[Q'^{i}, G\right] = \frac{d}{d\theta}(-\xi Q'^{i}) = \delta Q'^{i}(\theta) . \tag{33}$$

Thus, the Poisson bracket of G with Q' gives the correct transformation law for Q' under a coordinate transformation, as should be the case according to our geometric interpretation for G.

Using the relation (33) we may calculate any type of Poisson bracket (of course, we also use the relation (30) along with (33)). For instance, the Poisson bracket of the metric g with the generator G is determined as,

$$[g, G] = 2\mathcal{L}[\mathcal{L}, G] = 2\mathcal{L}[P_i Q'^i, G]$$
$$= 2\mathcal{L} P_i [Q'^i, G] + 2\mathcal{L} Q'^i [P_i, G]$$

Using (30) and (33),

$$[g, G] = -2\mathcal{L} P_i \frac{d}{d\theta} (\xi Q'^i) - 2\mathcal{L} Q'^i \xi P_i' ,$$

which gives as result the variation in g which results from a coordinate transformation (Eq. (24)).

$$[g, G] = -2g\xi' - \xi g' = \delta g(\theta) . \qquad (34)$$

According to these results, an invariant quantity in the Hamiltonian representation is a quantity which has a vanishing Poisson bracket with the generator G,

$$[\mathcal{F}, G] = 0 . \qquad (35)$$

A first example of such quantity is given directly by the Hamiltonian itself, which form with G a set of two first-class quantities. [5] Presently, the Poisson bracket of $\mathcal{H}$ with G does not vanish weakly, but instead it vanishes strongly due to the fact that no further weakly vanishing constraint is present in the right hand side of this Poisson bracket. However, the fact that $\mathcal{H}$ is an invariant (or observable) does not help too much since it vanishes. We want to get some other non-zero quantity with an invariant behavior under the coordinate transformation group. First of all, it is of obvious interest to find out an invariant quantity associated to $Q^i(\theta)$, and also to $P_i(\theta)$. Any other quantity with the property of being invariant, may be in principle a certain function of such invariant canonical variables. The method of determination of these invariants is the same for any kind of given quantity, so that the presentation done for Q and P may be extended for any other quantity.

The first step in this direction is a fixation of the coordinate system by means of a coordinate condition [6]. We will consider the coordinate θ as some function of the dynamical variables Q and P, with the requirement that it must be an increasing function of time,

$$\theta = f(Q^i, P_i) \; , \tag{36}$$

$$\frac{\partial \theta}{\partial t} > 0 \; .$$

The relation (36) is the coordinate condition. In the appendix we give an example of how to choose the necessary conditions for obtaining an explicit form for the function f(Q, P).

The relation (36) may also be read as,

$$F = \theta - f(Q,P) = 0 \tag{37}$$

Thus, to impose coordinate conditions is equivalent to introduce in the formalism further constraints. In our case we get one new constraint, given by (37), which we call the coordinate constraint. The set of two constraints given by the Hamilton ian and coordinate constraints form a set of second class constraints. Then, the theory of a general covariant system with a chosen coordinate condition posess a Hamiltonian formulation where second class constraints are present. Dirac has shown that in this case the Poisson bracket must be properly modified to a new bracket, the usually called Dirac bracket (see reference (2)). We will develop a different approach, due to Bergmann and Komar [7], and which is mathematically equivalent to Dirac's method [8].

The method of Bergmann-Komar was used for the case of general relativity, presently we treat with a more simple situation, so that we will be able to show the equivalence of both methods in a more simple way. The Bergmann-Komar (for short we will call this method by BK approach) uses the original Poisson bracket, but modifies the form of the dynamical functions. To each dynamical variable A is associted a new variable A* defined by

$$A^* = A + \alpha G + \beta F \tag{38}$$

along with the requirements,

$$[A^*, G] = 0 \tag{39}$$

$$[A^*, F] = 0 \,. \tag{40}$$

The relation (39) means that A* is an invariant quantity, so that we call A* as the observable associated to the dynamical variable A. The two relations (39) and (40) determine uniquely the expressions for the coefficients α and β which appear in the definition (38). Incidentally, those coefficients may depend on the dynamical variables Q and P, but they will not contribute to any Poisson bracket calculation since they always have a weakly vanishing constraint as a multiplicative factor.

From (39) and (40) we get

$$\alpha = -\frac{[A, F]}{\xi(1-f')} , \tag{41}$$

$$\beta = -\frac{A'}{1-f'} , \tag{42}$$

which gives for (38) the value

$$A^* = A + \frac{A'}{f'-1} F - \frac{[A, F]}{\xi(1-f')} G . \tag{43}$$

Thus, any explicit form for A* will depend on the form of the function f.

The Poisson bracket between A* and B*, where B is some other dynamical variable, is given by

$$[A^*, B^*] = [A,B] + \frac{A'}{1-f'} [B, F] - \frac{B'}{1-f'} [A, F] . \tag{44}$$

The Dirac bracket between A and B is defined by,

$$\{A,B\} = [A,B] - [A, C_a] [C_a, C_b]^{-1} [C_b, B] , \tag{45}$$

where a sum over a and b is to be made, and C_a along with C_b represent the set of second class constraints. The inverse of the Poisson bracket between C_a and C_b is calculated as the inverse of a matrix. In our present situation we have,

$$[C_a, C_b] = \begin{pmatrix} 0 & \xi F' \\ -\xi F' & 0 \end{pmatrix}$$

where we have taken the first matrix element $[C_1, C_1]$ as $[C_G, C_G]$. Then,

$$\{A, B\} = [A, B] - \frac{A'}{1-f'} [F, B] - \frac{B'}{1-f'} [A, F]$$

which is equal to the Poisson bracket A* and B* (see equation (44)),

$$\{A, B\} = [A^*, B^*] \ .$$

Which proves the equivalence of Dirac's method with the BK formalism. Both methods have the same commutator algebra.

From the general relation (43) we may obtain the observable associated to Q as well as to P, in each case we equal A to Q or P. We note that the Poisson bracket between two different Q* type variables (or P* type variables) does not vanish,

$$[Q_i^*, Q_j^*] = \frac{Q_i'}{1-f'} [Q_j, F] - \frac{Q_j'}{1-f'} [Q_i, F]$$

with a similar formula for the Poisson bracket P_i^* and P_j^*. Finally,

$$[Q_i^*, P_j^*] = \delta_{ij} + \frac{Q_i'}{1-f'} [P_j, F] - \frac{P_j'}{1-f'} [Q_i, F] \ .$$

These relations represent the canonical commutation relations in the starred representation, from them we can calculate any commutation relation between two dynamical functions of Q* and P*.

In the appendix which follows we give an example of how to choose a coordinate condition.

APPENDIX

The expression for the metric g may be writen in the form,

$$g = g_o + h$$

where g_o and h are a short for,

$$g_o = E^2 \, t'^2 , \qquad (A\text{-}1)$$

$$h = 4V(V-E)t'^2 , \qquad (A\text{-}2)$$

where E represents the total energy and V is the potential energy. We have used the symbol g_o to denote the part of g which does not vanish in the case of free systems. We shall consider two cases, the first one for a free system and the second for interactions. In the case of a free system, the coordinate condition will be taken in a form which implies in a simple and intuitive value for the metric.

This is obtained by imposing as coordinate condition the relation,

$$\theta = E/k \, . \, t + b , \qquad (A\text{-}3)$$

along with V = 0. In this relation, both k and b are constants. From (A-3) we get easily that in this coordinate system the metric g_o is constant,

$$g_o = k^2$$

and is equal to the total metric since h vanishes. The set of coordinate transformations which keep unchanged this constant value for g is given by the set of uniform translations in θ-space,

$$\tilde{\theta} = \theta + \xi$$

with ξ an infinitesimal constant. Such transformations may be generated by infinitesimal uniform translations in the time t, and are the unique transformations which may be allowed after our previous choice of coordinates.

In the case where interactions are present, as for instance when it exists a potential function $V = V(q)$, we may use as coordinate condition a formula which is an extension of the simple relation (A-3),

$$\theta = T/k \, . \, t + b \qquad \text{(A-4)}$$

$$T = E - V(q) \; . \qquad \text{(A-5)}$$

From these relations we get,

$$t' = \left(1 - \frac{t}{k} F_i \, q'^i\right) \frac{k}{T}$$

and the value for the components of the metric in this coordinate system are,

$$g_o = \frac{E^2 \, k^2}{T^2} \left(1 - \frac{t}{k} F_i \, q'^i\right)^2$$

$$h = 4V(V-E) \, \frac{k^2}{T^2} \left(1 - \frac{t}{k} F_i \, q'^i\right)^2$$

Both being functions of t, q^i and q'^i.

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