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by

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EVANESCENT-MODE EFFECTS IN THE DOUBLE-WEDGE PROBLEM

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SUMMARY. The diffraction of the principal mode at the open end of a semi-infinite parallel-plate waveguide terminated by an infinite plane flange (double-wedge) is studied in the interval $0.1 \leq ka \leq 1.7$, where k is the wave number and 2a is the guide width. The evanescent-mode correction to the reflection amplitude and the amplitudes of the evanescent modes are computed by three different approximation methods. The agreement among these methods in the considered interval is good. The evanescent-mode correction to the reflection coefficient is small, but the correction to the phase is quite large, attaining values of the order of 10% for $ka \sim 1.7$. The domain of applicability of the various methods is discussed.

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1. INTRODUCTION

The study of discontinuities in a waveguide is a problem of great mathematical difficulty. There are only a few known exact solutions and, even if an exact solution is obtained, its reduction to a practically useful form may be extremely difficult. Such problems are usually solved by approximation methods, the accuracy of which is hard to assess. Even those which can be solved exactly have such mathematical complexity that the physical interpretation of the results becomes very difficult. Nussenzveig [1] (hereafter referred to as (N)) studied the problem of wave propagation in a semi-infinite parallel-plate waveguide terminated by an infinite plane flange (double wedge) applying methods which permit a clear physical interpretation of the phenomena which occur in the short and long wavelength. However the intermediate-wave-length region was not considered. Only the domains $ka \ll 1$ and $ka \gg 1$ were studied, where k is the wave number and 2a the diameter of the waveguide.

In the present work, we shall consider the domain $0.1 \leqslant ka \leqslant 1.7$.

The incident wave is the principal mode, which is the only travelling mode in this domain. It is diffracted at the open end, giving rise to radiation into free space. Within the waveguide, besides the reflected mode, all evanescent modes are excited. These modes are usually neglected in approximate treatments of problems of this kind. It was shown in (N) that they give rise to a very small correction in the reflection amplitude for $ka \leqslant 0.1$. We shall compute the evanescent mode correction in the domain

$0.1 \leqslant ka \leqslant 1.7$ by several different approximation methods. The results show that, while the correction to the absolute value of the amplitude remains small in this domain, the correction to the phase becomes appreciable.

The formulation of the problem in term of an infinite system of linear equations will be given in §2. The unknown quantities in this system are the amplitudes of the excited modes.

In §3, we shall derive approximate expressions for the coefficients of the system in the considered interval.

It was shown in (N) that the asymptotic behaviour of the amplitudes of evanescent modes of high order is determined by the behaviour of the field in the neighbourhood of the edges of wedges (where the electric field has a singularity $r^{-1/3}$, r being the distance to the edge). This led to an asymptotic method for the approximate solution of the infinite system of linear equations in the domain $ka \ll 1$. In §4, the asymptotic method will be extended to the domain considered in the present paper. The results show that, as ka increases, the lowest-order evanescent modes become increasingly coupled with the principal mode, so that the asymptotic "Ansatz" has to be modified for these modes.

In §5, we shall introduce a modification in the asymptotic method, to take into account the "non-asymptotic" behaviour of the lowest-order modes. The corrected asymptotic method converges very well up to ka = 1.0 and presumably also for larger values of ka.

In §6, we shall apply the "method of partial systems", in which only the first ℓ modes are considered in the ℓth-order approximation.

In §7, we shall formulate a method which is a combination of the asymptotic method with the method of partial systems.

In the conclusion (§8), we shall discuss the obtained results, the domain of validity of the approximations employed and the agreement between the results of different methods.

2. FORMULATION OF THE PROBLEM

The double wedge and the coordinate system are represented in fig. 1. The incident wave is the principal mode, which travels towards the open end, giving rise to radiation into free space and exciting a reflected mode as well as all evanescent modes within the waveguide.

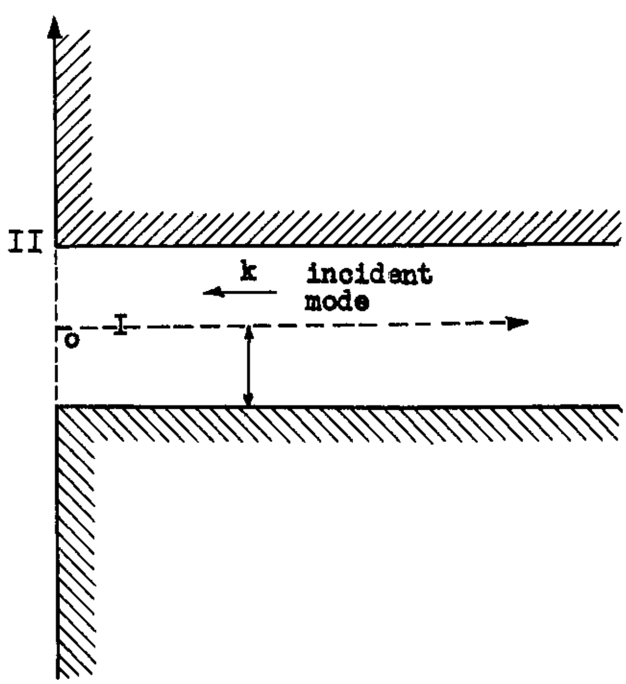

Fig. 1: The coordinate system

The field can be represented by a single scalar function $u(x, y)$, such that

$$\vec{H} = (0,\ 0,\ u(x, y)\exp(-i\omega t)) \tag{2.1}$$

$$\vec{E} = \left(\frac{-1}{ik}\frac{\partial u}{\partial y},\ \frac{1}{ik}\frac{\partial u}{\partial x},\ 0\right)\exp(-i\omega t) \tag{2.2}$$

Hereafter the time factor $\exp(-i\omega t)$ will be omitted. The function $u(x, y)$ must satisfy the following conditions:

(a) $\Delta u + k^2 u = 0$;

(b) $\frac{\partial u}{\partial x}(0,y) = 0$ for $|y| > a$;

(c) $\frac{\partial u}{\partial y}(x,\pm a) = 0$ for $x > 0$;

(d) $u(0_+, y) = u(0_-, y)$ for $|y| < a$;

(e) $\frac{\partial u}{\partial x}(0_+, y) = \frac{\partial u}{\partial x}(0_-, y)$ for $|y| < a$;

(f) the only incoming wave is the incident mode (radiation condition);

(g) grad u is square-integrable over any three-dimensional domain (edge condition).

The general solution is region I, for an incident wave of unit amplitude, is

$$u_I(x, y) = \exp(-ikx) + \sum_{n=0}^{\infty} a_n \cos k_{y_n} y \cdot \exp(ik_{x_n} x) \tag{2.3}$$

where $k_{y_n} = n\pi/a$, and

$$k_{x_n} = (k^2 - k_{y_n}^2)^{\frac{1}{2}} \qquad \text{for} \qquad n \leqslant \frac{ka}{\pi}$$

$$k_{x_n} = i\,(k_{y_n}^2 - k^2)^{\frac{1}{2}} \qquad \text{for} \qquad n > \frac{ka}{\pi}$$

The coefficient a_o is the reflection amplitude; the remaining coefficients are the amplitudes of excitation of the other modes, which are all evanescent for $0 < ka < \pi$.

The general solution in region II is

$$u_{II}(x,y) = \int_0^\infty A(k_y) \cos k_y\, y \exp(-ik_x x)\, dk_y, \qquad (2.4)$$

where

$$k_x = (k^2 - k_y^2)^{\frac{1}{2}} \text{ if } k_y < k \text{ and } k_x = i\,(k_y^2 - k^2)^{\frac{1}{2}}$$

if

$$k_y > k$$

It was shown in (N) that conditions (a) to (g), together with (2.3) and (2.4), lead to an infinite system of linear equations in the unknowns a_n:

$$a_m = \sum_{n=0}^{\infty} K_{mn}\, a_n - K_{mo} \qquad (2.5)$$

Expressions for the coefficients K_{mn} were given in (N). For $m = n = 0$, it was shown that

$$K_{oo} = \frac{1}{2}\left[1 - \mathscr{C}_o(K) + H_1(2K) + \frac{i}{\pi K}\right] \qquad (2.6)$$

where $K = ka$,

$$\mathscr{C}_o = \int_0^{2K} H_o(v)dv,$$

and H_o and H_1 are Hankel's functions of the first kind, of orders zero and one, respectively. This integral can be rewritten as follows [2]:

$$\mathscr{C}_o = K\pi\left[H_o(2K)S_{-1}(2K) + S_o(2K)\cdot H_{-1}(2K)\right], \qquad (2.7)$$

with

$$S_{-1} = \frac{2}{\pi} - S_1, \quad H_{-1} = -H_1,$$

where S_n is Struve's function of order n.

Substituting (2.7) in (2.6), we obtain

$$K_{oo} = \frac{1}{2}\left[1-2K\cdot H_o(2K) + H_1(2K) + \frac{i}{\pi K} + \pi K(H_o(2K)\cdot S_1(2K) - H_1(2K)\cdot S_o(2K))\right] \qquad (2.8)$$

The expressions for the remaining coefficients K_{mn} were given in (N):

$$K_{mn}(K) = (-1)^{m+n}\,\frac{(1-\gamma_n^2)^{\frac{1}{2}}}{2K\epsilon_m}\cdot\frac{\gamma_m\mathscr{S}_m - \gamma_n\mathscr{S}_n}{\gamma_m^2-\gamma_n^2} \qquad (2.9)$$

for $m \neq n$,

where $\gamma_n = n\pi/K$, $\epsilon_m = 2$ for $m = 0$, $\epsilon_m = 1$ for $m > 0$ and

$$\mathscr{S}_m(K) = \int_0^{2K} H_o(v)\sin(\gamma_n v)dv; \qquad (2.10)$$

$$K_{nn}(K) = \frac{1}{2}\left[1-(1-\gamma_n^2)^{\frac{1}{2}}\cdot\mathscr{C}_n\right] + \frac{(1-\gamma_n^2)^{-\frac{1}{2}}}{4K}\times\left[2K\cdot H_o(2K) + \frac{2i}{\pi} + \frac{\mathscr{S}_n}{\gamma_n}\right]$$

$$\text{for } n \neq 0 \qquad (2.11)$$

where

$$\mathscr{C}_n(K) = \int_0^{2K} H_o(v)\cos(\gamma_n v)dv. \tag{2.12}$$

The above formulae reduce the evaluation of the coefficients K_{mn} to that of the functions $\mathscr{S}_n(K)$ and $\mathscr{C}_n(K)$. The approximation methods which were employed, for this purpose, in the considered domain of values of K, will be described in Appendix B.

3. ASYMPTOTIC METHOD

It was shown in (N) that the asymptotic behaviour of the amplitudes of evanescent modes of high order is essentially determined by the behaviour of the field in the neighbourhood of the edge of the wedge. The electric field has a singularity in $r^{-1/3}$ at each edge, where r is the distance to the edge. According to (N), this implies that, for $\gamma_n = n\pi/K \gg 1$, $a_n(K)$ can be represented by an asymptotic expansion of the form:

$$a_n(K) = (-1)^n \left\{ \frac{A_1(K)}{n^{5/3}} + \frac{A_2(K)}{n^{7/3}} + \frac{A_3(K)}{n^{11/3}} + 0\left(n^{-13/3}\right)\right\};\left(n \gg \frac{K}{\pi}\right) \tag{3.1}$$

For $K \ll 1$, the coefficients $A_j(K)$ can be expanded in series of powers and logarithms of K. This method was applied in (N) for $K \lessdot 0.1$, and the convergence was very good in this region. However, for $K \gtrsim 0.1$, these series do not converge well, so that we shall compute $A_j(K)$ by numerical methods for values of K in the interval $0.1 \leqslant K \leqslant 1.7$. We shall employ the representation (3.1) starting from n = 1. For low values of n, the condition $n \gg K/\pi$

will not be fulfilled, so that we have to expect appreciable corrections in the amplitudes of the lowest-order modes. The evaluation of these corrections will be discussed in §4. They will be disregarded in the present section, in order to determine the domain of applicability of the uncorrected asymptotic method. The reliability of the results will also be assessed by comparing them with the results obtained by different approximation methods (§§5,6).

The ℓth-order approximation of the asymptotic method will be defined as follows. We shall denote by $a_n^{(\ell)}$ the right-hand side of (3.1), restricted to its first (ℓ- 1) terms. Replacing a_n by $a_n^{(\ell)}$, for $n \geqslant 1$, in the right-hand side of the first equation of system (2.5) (m=0), we get an expression for $a_o^{(\ell)}$ as a funtion $A_1^{(\ell)}, \ldots, A_{\ell-1}^{(\ell)}$:

$$a_o^{(\ell)}(1-K_{oo}) = \sum_{n=1}^{\infty} K_{on} a_n^{(\ell)} - K_{oo} = A_1^{(\ell)} \sum_{n=1}^{\infty} (-)^n \, n^{-5/3} \, K_{on} +$$

$$+ A_2^{(\ell)} \sum_{n=1}^{\infty} (-)^n \, n^{-7/3} \, K_{on} + \ldots - K_{oo}. \quad (3.2)$$

The coefficient K_{oo} is given by (2.°), and the sums of the series appearing in the last member of (3.2) can be computed for each value of K with the help of the expressions for the coefficients K_{on} given in Appendix B.

Replacing a_n by $a_n^{(\ell)}(n \geqslant 1)$ and a_m by $a_m^{(\ell)}$ in (2.5) for $m \geqslant 1$, we get

$$(-)^m \left[A_1^{(\ell)} m^{-5/3} + \ldots\right] = (a_o^{(\ell)} - 1) K_{mo} +$$

$$+ A_1^{(\ell)} \sum_{n=1}^{\infty} (-)^n n^{-5/3} K_{mn} + \ldots ; \; m \geq 1 \qquad (3.3)$$

The asymptotic expansion (for large m) of the series appearing in the right-hand side will be given in Appendix C. According to results given there, all the terms in the left-hand side of (3.3) are cancelled by corresponding terms appearing in the right-hand side, and there remains a development in terms of $m^{-2} \log m$, m^{-2}, $m^{-4} \log m, m^{-4}, \ldots$, which must vanish identically. Setting the coefficients of the $(\ell - 1)$ first terms of this development equal to zero, and neglecting the remaining ones, we obtain $(\ell - 1)$ linear equations in the unknowns $a_o^{(\ell)}$, $A_1^{(\ell)}$, ..., $A_{\ell-1}^{(\ell)}$, which, together with (3.2), allows us to determine them.

In the present work, we shall not go beyond the 3rd-order approximation, in view of the excessive amount of labour this would require. The system of linear equations which must be solved in the 3rd-order approximation is (cf. Appendix B):

$$1 = \left(a_o^{(3)} - 1\right)\left[K_{oo} - 1\right] + \frac{A_1^{(3)}}{4\pi}\left[\left(\frac{2}{\pi} \log \frac{2\pi}{K} + i(1 - H_o)\right)\xi(8/3) - \frac{2}{\pi}\xi'(8/3) + \right.$$

$$\left. + \frac{K^2}{\pi^2} \cdot \left(-\frac{1}{2\pi} + i\left(\frac{H_o}{2} + H_o''\right)\right)\xi(14/3)\right] + \frac{A_2^{(3)}}{4\pi}\left[+ 2/3\right] \qquad (3.4)$$

$$0 = \left(a_o^{(3)} - 1\right)\frac{iK}{\pi} + A_1^{(3)}\left[\frac{K^2}{2\pi^2}\xi(8/3) + \frac{K^4}{8\pi^4}\xi(14/3) - \xi(2/3)\right] + A_2^{(3)} \cdot$$

$$\cdot \left[+ 2/3\right] \qquad (3.5)$$

$$0 = \left(a_o^{(3)} - 1\right) \frac{K}{2\pi^2} \left[(1 - H_o) - \frac{2i}{\pi} \log \frac{2\pi}{K}\right] + A_1^{(3)} \left[\frac{-K^2}{2\pi^4} \log \frac{2\pi}{K} \xi(8/3) - \right.$$

$$- \frac{K^4}{8\pi^6} \log \frac{2\pi}{K} \xi(14/3) - \frac{iK^2}{2\pi^3} \left(\frac{1}{2} + \frac{i}{2\pi} + H_o''\right) \xi(8/3) + \frac{iK^4}{4\pi^5} \left(\frac{1}{2} + \frac{i}{2\pi} + \right.$$

$$\left. \left. + H_o''\right) \xi(14/3) - \frac{iK^4}{2\pi^5} \left(\frac{3}{8} + \frac{7i}{16\pi} - H_o^{(IV)}\right) \xi(14/3) + \frac{\xi'(2/3)}{\pi^2}\right] +$$

$$+ A_2^{(3)} \left[+ \frac{2}{3}\right] \qquad (3.6)$$

where $\xi(\lambda)$ and $\xi'(\lambda)$ are Riemann's zeta function and its derivative, respectively, and the symbol $[+ 2/3]$ in the coefficients of $A_2^{(3)}$ indicates that they differ from the coefficient of $A_1^{(3)}$ in the same equation only by the replacements: $\xi(\lambda) \rightarrow \xi(\lambda + \frac{2}{3})$, $\xi'(\lambda) \longrightarrow \xi'(\lambda + \frac{2}{3})$. The numerical values of $\xi(\lambda)$ and $\xi'(\lambda)$ for the relevant values of λ are given in Appendix A.

The 2nd-order approximation is obtained by omitting the last equation as well as the terms in A_2 in the first two equations.

The 1st-order approximation follows from (3.4) by omitting the terms in A_1 and A_2:

$$(K_{oo} - 1) \cdot (a_o^{(1)} - 1) = 1, \quad a_o^{(1)} = - K_{oo}/(1 - K_{oo}) \qquad (3.7)$$

This result is obviously equivalent to neglecting all evanescent modes in system (2.5), so that the difference

$$\delta a_o = a_o - a_o^{(1)} \qquad (3.8)$$

where a_o is the exact solution, will be called the evanescent-mode correction. The approximation (3.7) is also identical to the result obtained by Levine and Schwinger's method, with a constant electric field over the aperture as a trial function [3,4].

We shall write

$$a_o = -|a_o| \exp(i\phi) \qquad (3.9)$$

so that the phase ϕ approaches zero for $K \to 0$. The reflection coefficient, i.e. the ratio of the mean reflected energy current to the mean incident energy current, per unit length in the Z direction, is given by

(3.10)

$$r = |a_o|^2 \qquad (3.10)$$

The numerical results obtained by the asymptotic method for several values of K in the domain $0.1 \leqslant K \leqslant 1.7$ are given in Tables I and II. Table I contains the values of $a_o^{(1)}$, $a_o^{(2)}$, $a_o^{(3)}$, $a_1^{(2)} = -A_1^{(2)}$, $A_1^{(3)}$, $A_2^{(3)}$, $a_1^{(3)}$ and $a_2^{(3)}$. Table II contains the values of r and ϕ in the first three approximations.

TABLE I: Values of $a_n^{(l)}$, $A_1^{(3)}$ and $A_2^{(3)}$ obtained by the asymptotic method, for n = 0,1,2; l = 1,2,3, ($a_1^{(2)} = A_1^{(2)}$).

K	$a_0^{(1)}$	$a_0^{(2)}$	$a_0^{(3)}$	$a_1^{(2)}$	$A_1^{(3)}$	$A_2^{(3)}$	$a_1^{(3)}$	$a_2^{(3)}$
0.1	−0.7574 −0.3275i	−0.7601 −0.3207i	−0.7593 −0.3229i	−0.004170 −0.02288i	−0.003426 0.01834i	0.0005250 −0.003085i	0.002901 −0.01525i	−0.0009750 0.005163i
0.4	−0.2997 −0.4261i	−0.3116 −0.4121i	−0.3093 −0.4166i	0.02135 −0.06794i	−0.02060 0.05556i	0.0006731 −0.008388i	0.01992 −0.04717i	−0.006354 0.01534i
0.7	−0.1037 −0.3511i	−0.1215 −0.3346i	−0.1202 −0.3418i	0.03006 −0.1008i	−0.03707 0.08234i	−0.004411 −0.01271i	0.04148 −0.06963i	−0.01255 0.02341i
1.0	−0.009723 −0.2695i	−0.03239 −0.2517i	−0.03218 −0.2634i	0.03187 −0.1307i	−0.05305 0.1038i	−0.01399 −0.01911i	0.06704 −0.08465i	−0.01949 0.02889i
1.1	0.009423 −0.2444i	−0.01478 −0.2264i	−0.01461 −0.2401i	0.03136 −0.1405i	−0.05827 0.1098i	−0.01793 −0.02198i	0.07620 −0.08786i	−0.02191 0.03024i
1.3	0.03566 −0.1982i	0.008495 −0.1803i	0.009418 −0.1982i	0.02912 −0.1602i	−0.06831 0.1204i	−0.02657 −0.02901i	0.09488 −0.09141i	−0.02679 0.03217i
1.4	0.04399 −0.1771i	0.01541 −0.1594i	0.01722 −0.1795i	0.02753 −0.1700i	−0.07304 0.1249i	−0.03115 −0.03320i	0.1042 −0.09173i	−0.02919 0.03276i
1.7	0.05479 −0.1218i	0.02237 −0.1060i	0.02905 −0.1317i	0.02167 −0.1999i	−0.08531 0.1357i	−0.04515 −0.04861i	0.1305 −0.08706i	−0.03583 0.03309i

TABLE II: The reflection coefficient and the phase of the reflection amplitude in the first three approximations of the asymptotic method.

K	$\|a_o^{(1)}\|^2$	$\|a_o^{(2)}\|^2$	$\|a_o^{(3)}\|^2$	$\phi^{(1)}$	$\phi^{(2)}$	$\phi^{(3)}$
0.1	0.6810	0.6806	0.6807	23° 39	22° 88	23° 04
0.4	0.2713	0.2669	0.2692	54° 88	52° 91	53° 40
0.7	0.1340	0.1267	0.1313	73° 55	70° 04	70° 63
1.0	0.07271	0.06439	0.07041	87° 94	82° 67	83° 03
1.1	0.05983	0.05148	0.05786	92° 20	86° 26	86° 51
1.3	0.04056	0.03256	0.03937	100° 21	92° 70	92° 72
1.4	0.03331	0.02565	0.03253	103° 95	95° 52	95° 47
1.7	0.01785	0.01174	0.01820	114° 22	101° 93	102° 42

The values of $|a_0^{(3)}|^2$, $|a_1^{(3)}|^2$ and $|a_2^{(3)}|^2$ as a function of K are plotted in fig. 2, and the phase $\phi^{(3)}$ is plotted in fig. 3. The relative evanescent-mode correction $\delta r/r^{(1)}$ to the reflection coefficient and the relative evanescent-mode correction $\delta\phi/\phi^{(1)}$ to the phase are plotted in fig. 4 and 5, respectively, both in the 2nd-order approximation and in the 3rd-order approximation.

It is clear from Table II or fig. 4 that, from $K \simeq 0.5$ onwards, the 3rd approximation to r is closer to the 1st than to the 2nd one. This effect increases with K, in such a way that, for $K = 1.7$, $\delta r^{(2)}/r^{(1)} \approx 33\%$, whereas $\delta r^{(3)}/r^{(1)} \approx -1.7\%$. Thus, insofar as r is concerned, we see that the asymptotic method does not converge well, at least in the first three approximations, from $K \simeq 0.5$ onwards.

This does not happen with the phase. According to Table II or fig. 5 the 3rd approximation to ϕ is, in general, much closer to the 2nd than to the 1st one. The mean relative deviation $(\phi^{(3)} - \phi^{(2)})/\phi^{(2)}$ in the interval $0.1 \leqslant K \leqslant 1.7$ is of the order of 0.5%, whereas the mean relative deviation $(\phi^{(3)} - \phi^{(1)})/\phi^{(1)}$ in the same interval is of the order of 5%.

It can also be seen in Table I that $a_1^{(3)}$ is not in good agreement with $a_1^{(2)}$ for $K \gtrsim 0.5$. On the other hand, the condition $|A_2^{(3)}| << |A_1^{(3)}|$, which would indicate rapid convergence of the "Ansatz"(3.1), is fulfilled for $K \simeq 1.1$, but not for larger values of K.

Thus, except for the phase, the convergence of the asymptotic method, up to the 3rd order approximation, is not satisfactory for

$K \gtrsim 0.5$ and it gets worse as K increases. These results were to be expected, for, as we have seen, the samllest value of n for which (3.1) represents a good approximation must increase with K, whereas the aymptotic method employs (3.1) already for $n = 1$. The large discrepancy between the 2nd and the 3rd approximation is also related with the fact that, for small m, the terms in m^{-2}, which are neglected in the 2nd approximation, are of the same order as the terms in $m^{-2} \log m$ which are kept.

Physically, we can say that the amplitudes of the lowest-order evanescent modes are determined by the edge singularities only for sufficiently small K, when the aperture, so to speak, consists mainly of edge zones. As K increases, the influence of the edged over the wave function in the central part of the aperture, and consequently over the lowest-order modes, decreases. This is particularly true for the first evanescent mode, the critical frequency of which correspond to $K = \pi$.

According to Table I and fig. 2, $|a_1^{(3)}|$ becomes comparable with $|a_0^{(3)}|$ for $K \gtrsim 1$, and is even larger than $|a_0^{(3)}|$ for $K = 1.7$ on the other hand, $|a_2^{(3)}|$ is very small in the whole interval $0.1 \leq K \leq 1.7$ which agrees with the above considerations.

4. CORRECTION TO THE ASYMPTOTIC METHOD

In order to improve the accuracy of the asymptotic method it is necessary, as we have seen in the previous section, to introduce a correction in the amplitudes of the lowest-order modes.

Let $\alpha_m^{(\ell)} = a_m^{(\ell)} + \Delta a_m^{(\ell)}$ be the exact solution, $a_m^{(\ell)}$ being the

ℓth-order approximation of the asymptotic method. The absolute errors $\Delta a_m^{(\ell)}$ satisfy an infinite system of linear equations which differs from (2.5) only by the inhomogeneous term:

$$\Delta a_m^{(\ell)} = \sum_{n=0}^{\infty} K_{mn} \, \Delta a_n^{(\ell)} + R_m^{(\ell)}, \qquad (4.1)$$

where the absolute residuals $R_m^{(\ell)}$ are given by

$$R_m^{(\ell)} = \sum_{n=0}^{\infty} K_{mn} \, a_n^{(\ell)} - a_m^{(\ell)} - K_{mo} \; . \qquad (4.2)$$

We are interested in the computation of $\Delta a_m^{(\ell)}$ for small values of m. For this purpose, we shall apply to (4.1) the method of partial systems, which consists in extracting finite systems of linear equations from (4.2) by neglecting all equations and unknowns beyond a certain order.

Thus, to compute the correction to the 2nd approximation of the asymptotic method, we shall retain only the first two equations of (4.1) and only the unknowns $\Delta a_o^{(2)}$, $\Delta a_1^{(2)}$; in the correction to the 3rd approximation, we shall retain the first three equations and the unknowns $\Delta a_o^{(3)}$, $\Delta a_1^{(3)}$, $\Delta a_2^{(3)}$.

The residuals (4.2) can be computed by numerical methods. The procedure will be described in Appendix D.

The correction to the asymptotic method has been evaluated only for K = 1. The results are given in Table III. The corrected values of $\delta r^{(2)}/r^{(1)}$ and $\delta\phi^{(2)}/\phi^{(1)}$ for K = 1 are represented by the points ⊙ in figs. 4 and 5 (the arrows represent the value of

TABLE III - Values of $\alpha_o^{(\ell)} = a_o^{(\ell)} + \Delta a_o^{(\ell)}$, obtained in the corrected asymptotic method, for K = 1 and $\ell = 2, 3$.

$a_o^{(2)}$	-0.03239 -0.2517i
$\alpha_o^{(2)}$	-0.03278 -0.2604i
$\lvert a_o^{(2)}\rvert^2$	0.06439
$\lvert \alpha_o^{(2)}\rvert^2$	0.06889
$a_o^{(3)}$	-0.03218 -0.2634i
$\alpha_o^{(3)}$	-0.03425 -0.2628i
$\lvert a_o^{(3)}\rvert^2$	0.07041
$\lvert \alpha_o^{(3)}\rvert^2$	0.07022
$\bar{\phi}^{(2)}$	$82^{o}.83$
$\bar{\phi}^{(3)}$	$82^{o}.57$

the correction). The corrections to the 3rd approximation are not represented because the corrected points practically coincide with the uncorrected ones.

The results show that the corrected asymptotic method converges quite well for K = 1. The corrected value of $r^{(2)}$, in contrast with the uncorrected one, is closer to $r^{(3)}$ than to $r^{(1)}$, thus eliminating the main indication of poor convergence of the asymptotic method. The correction in the phase, where the convergence was already good in the asymptotic method, is very small, and improves the convergence still more. The corrected value of $a_1^{(2)}$ is also much closer to that of $a_1^{(3)}$.

The correction to the 3rd approximation is much smaller than that to the 2nd one, which indicates that the uncorrected 3rd approximation of the asymptotic method is already a good approximation. Note also that $|\Delta a_m^{(3)}/a_m^{(3)}| \ll 1$ and $|R_m^{(3)}/K_{mo}| \ll 1$, (cf. Tables IX and X of Appendix D), which are also indications in the same sense.

Of course, if we define e.g. the corrected 3rd approximation by the corrected values of $a_o^{(3)}$, $a_1^{(3)}$ and $a_2^{(3)}$, the amplitudes of the higher-order modes being given by the Ansatz (3.1), with the uncorrected values of $A_1^{(3)}$ and $A_2^{(3)}$, the corrections in the lowest -order modes will contribute additional terms in m^{-2} log m to the residuals of all the equations of system (3.5), but these contributions are small, in view of the above results.

5. THE METHOD OF PARTIAL SYSTEMS

The method of partial systems can also be applied directly to the system (2.5). In the ℓth-order approximation of this method, we neglect all equations and unknowns in (2.5) beyond the ℓth, and then solve the remaining system of ℓ equations. This method was applied in (N) and led to good results for $K < 0.1$.

The 1st approximation of this method coincides with the 1st approximation of the asymptotic method. The 2nd and 3rd approximations have been computed for K = 0.1, 0.4, 1.0 and 1.7. The resulting amplitudes and phases, denoted by $a_n'^{(\ell)}$ and $\phi_n'^{(\ell)}$, (n = = 0, 1, 2; ℓ = 2,3), are given in Tables IV and V. The corresponding relative evanescent-mode corrections $\delta r/r^{(1)}$ and $\delta\phi/\phi^{(1)}$ are plotted in figs. 4 and 5, respectively. The 2nd and 3rd approximations to $\delta r/r^{(1)}$ practically coincide, so that only the 3rd approximations has been plotted. Both the 2nd and the 3rd approximation to $\delta\phi/\phi^{(1)}$ are shown in fig. 5.

The results show that the method of partial systems still converges well in the considered domain. The maximum relative deviation between the 2nd and 3rd approximation to the reflection coefficient, $|r'^{(3)} - r'^{(2)}|/r'^{(2)}$, is of the order of 0.7%, whereas that in the phase, $|\phi'^{(3)} - \phi'^{(2)}|/\phi'^{(2)}$, is of the order of 2%. The 3rd approximation of the method of partial systems is also in good agreement with the 3rd approximation of the asymptotic method, both for the reflection coefficient and for the phase. For a_1 and a_2, the agreement is good for $K \leqslant 1.0$, but not so good

TABLE IV: Method of partial systems. Values of $a_n^{'(\ell)}$ for n = 0, 1, 2 and ℓ = = 2, 3; $(a_o^{'(1)} = a_o^{(1)})$.

K	$a_o^{'(2)}$	$a_o^{'(3)}$	$a_1^{'(2)}$	$a_1^{'(3)}$	$a_2^{'(3)}$
0.1	-0.7586 -0.3245i	-0.7588 -0.3240i	0.002779 -0.01454i	0.002848 -0.01493i	-0.0007844 0.004130i
0.4	-0.3062 -0.4199i	-0.3064 -0.4199i	0.01912 -0.04486i	0.01940 -0.04558i	-0.006079 0.01500i
1.0	-0.02650 -0.2663i	-0.02969 -0.2651i	0.06927 -0.08508i	0.06988 -0.08664i	-0.01899 0.02779i
1.7	+0.03447 -0.1346i	0.03214 -0.1357i	0.1555 -0.09133i	0.1550 -0.09214i	-0.03200 0.02103i

TABLE V: The reflection coefficient and the phase of the reflection amplitude in the 2nd and 3rd approximations of the method of partial systems.

K	$\phi^{'(2)}$	$\phi^{'(3)}$	$\|a_o^{'(2)}\|^2$	$\|a_o^{'(3)}\|^2$
0.1	23° 16	23° 12	0.6808	0.6808
0.4	53° 90	53° 88	0.2701	0.2702
1.0	84° 32	83° 61	0.07162	0.07115
1.7	104° 38	103° 33	0.01932	0.01944

for K = 1.7. One should expect the convergence of the method of partial systems to become poorer as K increases, because of the increase in the amplitude of the higher-order modes.

6. MIXED METHOD

The asymptotic method should lead to good results for the amplitudes of high-order evanescent modes, to which we can apply the Ansatz (3.1). However, as we have seen, it should not converge very well for the lowest-order modes, unless we apply the correction discussed in §4. The method of partial systems, on the other hand, gives good results for the amplitudes of the lowest-order modes, but computational difficulties prevent its application to determine the amplitudes of high-order ones. It is natural, then, to try to combine the two methods. We shall now introduce a mixed method, which will be applied only to the 3rd approximation. In this approximation, we shall apply the Ansatz (3.1), restricted to its first term, i.e., $a_n'' = (-1)^n A_n''^{-5/3}$, for $n \geqslant 2$ and we shall employ the first three equations of system (2.5) to determine the amplitudes.

Thus, if we denote by double dashes the results obtained by this method, the first two equations are given by

$$a_m'' = K_{mo}\, a_o'' + K_{m1}\, a_1'' + A'' \sum_{n=2}^{\infty} (-1)^n\, n^{-5/3} \cdot K_{mn} - K_{mo}; \qquad (6.1)$$

$$(m = 0,1)$$

and the third equation is

TABLE VI: Mixed method: values of $a_n^{"(\ell)}$, for n = 0, 1, 2; $\ell = 3$ and values of $\phi^{"(3)}$.

K	$a_o^{"(3)}$	$a_1^{"(3)}$	$a_2^{"(3)}$
0.1	- 0.7592 - 0.3231i	0.002849 - 0.01509i	- 0.0009596 0.005068i
0.4	- 0.3067 - 0.4183i	0.01912 - 0.04495i	- 0.006010 0.01488i
1.0	- 0.02870 - 0.2653i	0.07012 - 0.08614i	- 0.01883 0.02752i
1.7	0.02581 - 0.1350i	0.1584 - 0.09773i	- 0.03794 0.03439i

TABLE VII: The reflection coefficient and the phase of the reflection amplitude in the 3rd approximation of the mixed method.

K	$\phi^{"(3)}$	$\lvert a_o^{"(3)}\rvert^2$
0.1	23° 05	0.6808
0.4	53° 47	0.2691
1.0	83° 76	0.07119
1.7	100° 82	0.01890

$$2^{-5/3} A'' = K_{2o} a_o'' + K_{21} a_1'' + A'' \sum_{n=2}^{\infty} (-1)^n n^{-5/3} K_{2n} - K_{2o} . \qquad (6.2)$$

The unknowns a_o'', a_1'' and A'' are determined by (6.1) and (6.2).

The method has been applied for K = 0.1, 0.4, 1.0 and 1.7. The results are given in Tables VI and VII. The corresponding relative evanescent-mode corrections are plotted in figs. 4 and 5.

As ought to be expected, the values obtained by this method fall in between the values obtained in the 3rd approximations of the asymptotic method and of the method of partial systems, though they are closer, as a rule, to the result obtained by the method of partial systems.

* * *

CONCLUSION

The behaviour of the reflection amplitude and the evanescent-mode amplitudes in the domain $0.1 \leqslant K \leqslant 1.7$ has been investigated by three different approximation methods. The agreement among the results obtained in the 3rd approximation of these methods is quite good up to K = 1 and is reasonably good up to K = 1.7.

It was shown in (N) that, in the long-wavelength region ($K \ll 1$), the principal mode becomes practically uncoupled from the evanescent modes. The former suffers strong reflection, while the amplitudes of the latter are determined by the quasi-static behaviour of the field near the edges of the double wedge (edge singularities). The evanescent-mode corrections are very small.

In the intermediate-wavelength region studied in this paper, the coupling between the principal mode and the evanescent ones increases, specially with respect to the lowest-order modes, which become less affected by the edge singularities as K increases. The relative evanescent-mode correction to the reflection coefficient is still quite small ($\lesssim 3\%$), but the correction to the phase reaches $\sim 6\%$ for K = 1 and $\sim 12\%$ for K = 1.7, so that it is much larger than the correction to the reflection coefficient. This result has a simple physical interpretation.

The coupling between the principal mode and the evanescent modes takes place at the open end of the waveguide. The evanescent modes are localized in this region, where they give rise to an energy accumulation. This corresponds to a delay of the incident wave near the open end, which is represented by the correction to the phase. On the other hand, while the evanescent modes store the energy, they do not contribute to the energy flux, so that their effect on the reflection coefficient is much smaller than their effect on the phase.

The accuracy of the various approximation methods can be estimated by comparing them with one another. The simplest method of partial systems, which allows us to obtain only the amplitudes of the first few modes (it is true, however that higher-order modes are physically less important). This method should converge the better, the more rapid the decrease in magntitude of the unknowns.

The convergence in the considered domain is good, but it must become worse as K increases, due to the increase in the amplitudes of the evanescent modes.

The mixed method takes into account in a simple way the effect of higher-order modes, giving rise to a small correction in the amplitudes of lower-order ones, together with an asymptotic expression for the amplitudes. However, only the first few equations of the system are taken into account.

The asymptotic method takes into account all the equations of the system. By construction, the residual obtained by this method should decrease rapidly with the order of the equantion and with the order of approximation. However, the method should not lead to good results for the amplitudes of the first few modes, at least in the lowest-order approximations. In fact, as we have seen, this happens for the 2nd approximation. It is therefore necessary to introduce a correction to the lowest-order amplitudes. This correction was introduced by applying the method of partial systems to the residual system (4.1). As the unknowns $\Delta a_n^{(\ell)}$ in this system should decrease more rapidly with n than the unknowns $a_n^{(\ell)}$ in the original system (2.5), according to the Ansatz (3.1), the method of partial systems should converge better in the case of (4.1). For K = 1, as we have seen, the convergence of the corrected symptotic method is quite good and this should be true a fortiori for smaller values of K. No calculations were made for $K > 1$, but it is to be expected that the convergence becomes worse as K increases: according to fig. 2, $|a_1|^2$ is already of the order of $|a_o|^2$ for $K = 1.7$. In order to improve the convergence of the corrected asymptotic method, one could take a_1 as a new unknown, applying the Ansatz (3.1) only for $n \geqslant 2$. However, this would require the solution of systems of a larger number of equations than those which have been used.

The most reliable approximation methods in the considered domains seem to be the corrected asymptotic method and the mixed method. The correction to the 2nd approximation for $K = 1$ is quite larger (it is $\sim 7\%$ in r and $\sim 1.7\%$ in ϕ), but the correction to the 3rd approximation is very small. The 3rd approximation of the asymptotic method can therefore be employed practically without correction at least for $K \leqslant 1$.

It would be interesting to extend the calculations up to $K = \pi$, which is the critical frequency of the first evanescent mode.

* * *

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APPENDIX A

Riemann's zeta function

Riemann's zeta function $\xi(\lambda)$ is defined, for real positive λ, by

$$\xi(\lambda) = \sum_{p=1}^{\infty} \frac{1}{p^{\lambda}}$$

and its derivative is given by

$$\xi'(\lambda) = -\sum_{p=1}^{\infty} \frac{\log p}{p^{\lambda}} .$$

The values of $\xi(\lambda)$ and $\xi'(\lambda)$ employed in the present work have been obtained from those tabulated in Ref. 5 by applying Everett's interpolation method[6]. For large enough λ, linear interpolation can be applied. The values of $\xi(\lambda)$ and $\xi'(\lambda)$ obtained by Everett's method are contained in Table VIII.

TABLE VIII: Values of Riemann's zeta function and its derivative.

	2/3	4/3	7/3	8/3	10/3	13/3	14/3	16/3
$\xi(\lambda)$	-2.447581	3.600938	1.415156	1.284191	1.147356	1.062564	1.047919	1.028597
$\xi'(\lambda)$	-8.924085	-8.930514	-0.503620	-0.304660	-0.135160	-0.050724	-	-

APPENDIX B

The coefficients K_{mn}.

The coefficients K_{mn} employed in the asymptotic method, in the method of partial systems and in the mixed method are approximations of (2.9) and (2.11), except for K_{oo}, which is computed directly from (2.8). For $0.1 \leqslant K \leqslant 1.7$, and $n \geqslant 3$, the functions and $\mathcal{C}_n(K)$ can be asymptotically expanded in powers of γ_n^{-1}.

For this purpose, let us rewrite (2.10) in the form:

$$\mathcal{J}_n^{(K)} = \int_0^\infty H_o(v)\, \sin(\gamma_n\, v)dv - \int_{2K}^\infty H_o(v)\, \sin(\gamma_n\, v)dv \qquad (B.1)$$

The first integral in (B.1) is given by [2]

$$\int_0^\infty H_o(v)\, \sin(\gamma_n v)dv = \frac{1}{\sqrt{\gamma_n^2-1}} \left(1 - \frac{2i}{\pi} \log \sigma_n\right),\ \gamma_n > 1 \qquad (B.2)$$

where

$$\sigma_n = \gamma_n + (\gamma_n^2 - 1)^{\frac{1}{2}}\ .$$

The asymptotic expansion of the second integral in (B.1) for $\gamma_n \gg 1$ is obtained by means of repeated partial integration:

$$\int_{2K}^\infty H_o(v)\, \sin(\gamma_n v)dv = \frac{1}{\gamma_n} \left[\sum_{S=0}^{2p} \frac{H_o^{(2S)}(2K)}{(i\ \gamma_n)^{2S}} + \underline{0}\ (\gamma_n^{-2p-2}) \right], \qquad (B.3)$$

where $H_o^{(2S)}$ denotes the derivative of order 2S of H_o.

Expanding $(\gamma_n^2 - 1)^{-\frac{1}{2}}$ and $\log \sigma_n$ in (B.2) in inverse powers of γ_n, and substituting (B.2) and (B.3), we obtain the asymptotic expansion of $\mathcal{S}_n(K)$ for $\gamma_n \gg 1$.

The asymptotic expansion of $\mathcal{C}_n(K)$ is obtained by an entirely similar procedure:

$$\mathcal{C}_n(K) = (1-\gamma_n^2)^{-\frac{1}{2}} + \frac{1}{\gamma_n}\left[\sum_{S=0}^{p} (-1)^s \frac{H_o^{(2S+1)}(2K)}{\gamma_n^{2S+1}} + \underline{O}\,(\gamma_n^{-2p-3})\right], \quad \text{(B.4)}$$

For n = 1 or n = 2, these expansions do not converge well, so that we write

$$\int_{2K}^{\infty} H_o(v) \sin(\gamma_n v)dv = \int_{2K}^{\lambda} H_o(v) \sin(\gamma_n v)dv + \int_{\lambda}^{\infty} H_o(v) \sin(\gamma_n v)dv \quad \text{(B.5)}$$

where λ is chosen sufficiently large for the asymptotic expansion of the integral from λ to ∞ to converge well; the integral from 2K to λ is computed by numerical integration. The same method is applied to $\mathcal{C}_n(K)$.

Substituting the above results in (2.9) and (2.11), we get the asymptotic expansion of the coefficients K_{mn}. The first few terms of the expansions are indicated below (where γ_n has been replaced by $n\pi/K$):

$$K_{on}(K) = \frac{(-1)^n}{4\pi}\left\{\left[\frac{2}{\pi}\log\frac{2\pi}{K} + i\,(1 - H_o(2K)\right]\frac{1}{n} + \frac{2}{\pi}\frac{\log n}{n} + \right.$$

$$\left. + \frac{1}{n^3}\frac{K^2}{\pi^2}\left[\frac{-1}{2\pi} + \frac{iH_o(2K)}{2} + iH_o''(2K)\right] + \underline{O}\,(n^{-3}\log n)\right\} \quad \text{(B.6)}$$

$$K_{mo}(K) = \frac{(-1)^m K}{2\pi^2}\left\{\left[1 - H_o(2K) - \frac{2i}{\pi}\log\frac{2\pi}{K}\right]\frac{1}{m^2} - \frac{2i}{\pi}\frac{\log m}{m^2} + \underline{O}(m^{-4}\log m)\right\} \quad (B.7)$$

$$K_{mn}(K) = (-1)^{m+n}\left\{\frac{1}{\pi^2}\frac{n \log m/n}{m^2 - n^2} - \frac{\log m}{m^2}\left[\frac{K^2}{2\pi^4}\left(\frac{1}{n} + \frac{K^2}{4\pi^2 n^3}\right)\right] + \right.$$

$$+ \frac{1}{m^2}\left[\frac{-K^2}{2\pi^4}\log\frac{2\pi}{K}\left(\frac{1}{n} + \frac{K^2}{4\pi^2 n^3}\right) - \frac{iK^2}{2\pi^3}\left(\left(\frac{1}{2} + \frac{i}{2\pi} + H_o''(2K)\right) \times \right.\right.$$

$$\left.\left.\left.\times\left(\frac{1}{n} - \frac{K^2}{2\pi^2 n^3}\right) + \left(\frac{3}{8} + \frac{7i}{16\pi} - H_o^{IV}(2K)\right)\frac{K^2}{\pi^2 n^3}\right)\right] + \underline{O}\left(m^{-4}\log m\right)\right\} \quad (B.8)$$

In order to derive the system of equations (3.4), (3.5), (3.6), we proceed as follows.

We consider the expansion (3.1) reduced to its first two terms:

$$a_n^{(3)}(K) = (-1)^n\left[\frac{A_1^{(3)}(K)}{n^{5/3}} + \frac{A_2^{(3)}(K)}{n^{7/3}}\right], \quad n \geq 1 \quad (B.9)$$

Substituting (B.6) and (B.9) in (2.5) for m = 0, and taking into account the definitions of the zeta function and its derivative (cf. Appendix A), we get equation (3.4).

Substituting (B.7), (B.8) and (B.9) in (2.5), for $m \neq 0$ we get:

$$(-1)^m\left[\frac{A_1^{(3)}(K)}{m^{5/3}} + \frac{A_2^{(3)}(K)}{m^{7/3}}\right] = \left(a_o^{(3)}(K) - 1\right)\frac{(-1)^m \cdot K}{2\pi^2}\left\{\left[1 - H_o(2K) - \right.\right.$$

$$- \frac{2i}{\pi} \log \left.\frac{2\pi}{K}\right] \frac{1}{m^2} - \frac{2i}{\pi} \frac{\log m}{m^2} \Bigg\} + \sum_{n=1}^{\infty} (-1)^{m+n} \left\{ \frac{1}{\pi^2} \frac{n \log m/n}{m^2 - n^2} - \right.$$

$$- \frac{\log m}{m^2} \left[\frac{K^2}{2\pi^2} \left(\frac{1}{n} + \frac{k^2}{4\pi^2 n^3} \right) \right] + \frac{1}{m^2} \left[\frac{-K^2}{2\pi^2} \log \frac{2\pi}{K} \left(\frac{1}{n} + \frac{K^2}{4\pi^2 n^3} \right) - \right.$$

$$- \frac{iK^2}{2\pi^3} \left(\left(\frac{1}{2} + \frac{i}{2\pi} + H_o''(2K) \right) \times \left(\frac{1}{n} - \frac{K^2}{2\pi^2 n^3} \right) + \left(\frac{3}{8} + \frac{7i}{16\pi} - H_o^{IV}(2K) \right) \times \right.$$

$$\left. \left. \left. \times \frac{K^2}{\pi^2 n^3} \right) \right] \right\} \times (-1)^n \times \left[\frac{A_1^{(3)}(K)}{n^{5/3}} + \frac{A_2^{(3)}(K)}{n^{7/3}} \right] \qquad (B.10)$$

In the right-hand side of (B.10) there appear the functions $\boxed{H}(2/3, m)$ and $\boxed{H}(4/3, m)$, where $\boxed{H}(\lambda, m)$ is defined in Appendix C. Replacing these functions by their asymptotic expansions for large m, given by (C.3), we reproduce the left-hand side of (B.10) and there remains an asymptotic expansion in terms of $m^{-2}\log m, m^{-2}$, $m^{-4} \log m$, m^{-4}, $m^{-6}\log m$, m^{-6} which must vanish identically. Setting the coefficients of $m^{-2}\log m$ and m^{-2} equal to zero in this expansion, and neglecting higher-order terms, we get (3.5) and (3.6).

* * *

APPENDIX C

The function $\boxed{H}(\lambda, m)$

The function $\boxed{H}(\lambda,m)$ is defined by

$$\boxed{H}(\lambda,m) = \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{\log m - \log n}{n^{\lambda}(m^2 - n^2)} \qquad (C.1)$$

It satisfies the recurrence relation

$$\boxed{H}(\lambda+2,m) = m^{-2}\boxed{H}(\lambda,m) + \xi(\lambda+2)(m\pi)^{-2}\log m + \xi'(\lambda+2)(m\pi)^{-2} \qquad (C.2)$$

It was shown in (N) that, if $0<\lambda<2$, $\lambda \neq 1$, the following asymptotic expansion is valid:

$$\boxed{H}(\lambda,m) = \frac{1}{4\cos^2(\lambda\pi/2)m^{\lambda+1}} + \frac{\xi(\lambda)\log m}{(\pi m)^2} + \frac{\xi'(\lambda)}{(\pi m)^2} -$$

$$- 8(2\pi)^{\lambda-1}\Bigg\{\sin(\lambda\pi/2)\cdot\xi(3-\lambda)\cdot\Gamma(3-\lambda)\times\frac{\log m}{(\pi m)^4} + \Bigg[\frac{\pi}{2}\cos(\lambda\pi/2) +$$

$$+ \sin(\lambda\pi/2)\times\left(\log 2\pi - \frac{\xi'(3-\lambda)}{\xi(3-\lambda)} - \frac{\Gamma'(3-\lambda)}{\Gamma(3-\lambda)}\right)\Bigg] \times \frac{\xi(3-\lambda)\cdot\Gamma(3-\lambda)}{(2\pi m)^4} -$$

$$- \sin(\lambda\pi/2)\cdot\xi(5-\lambda)\cdot\Gamma(5-\lambda)\frac{\log m}{(2\pi m)^6} - \Bigg[\frac{\pi}{2}\cos(\lambda\pi/2) + \sin(\lambda\pi/2)\times$$

$$\times\left(\log 2\pi - \frac{\xi'(5-\lambda)}{\xi(5-\lambda)} - \frac{\Gamma'(5-\lambda)}{\Gamma(5-\lambda)}\right)\Bigg] \times \frac{\xi(5-\lambda)\cdot\Gamma(5-\lambda)}{(2\pi m)^6} + \ldots\ldots\Bigg\} \qquad (C.3)$$

where $\zeta(\lambda)$ is Riemann's zeta function and $\Gamma(\lambda)$ is the gamma function.

Together with (C.2), this determines the asymptotic expansion of $\boxed{H}(\lambda, m)$ for all positive non-integral values of λ.

* * *

APPENDIX D

The residual $R_m^{(\ell)}$ for $K = 1$

The residuals $R_m^{(\ell)}$ are defined by (4.2):

$$R_m^{(\ell)} = \sum_{n=0}^{\infty} K_{mn}\, a_n^{(\ell)} - a_m^{(\ell)} - K_{mo}. \qquad (D.1)$$

As $R_m^{(1)} = 0$, we have to compute only $R_m^{(2)}$ and $R_m^{(3)}$. We shall indicate the procedure only for $R_1^{(3)}$, since it is entirely similar for the other ones. According to (D.1),

$$R_1^{(3)} = K_{10}\left[a_o^{(3)} - 1\right] + \left[K_{11} - 1\right] a_1^{(3)} + K_{12}\, a_2^{(3)} + \sum_{n=3}^{\infty} K_{1n}\, a_n^{(3)}. \qquad (D.2)$$

The amplitudes $a_n^{(3)}(K=1)$ for $n \geqslant 1$ are given by (B.10), with the values of $A_1^{(3)}$, $A_2^{(3)}$ given in Table I. The coefficients $K_{mn}(K=1)$ for $m \leqslant 2$ and $n \leqslant 2$ are given in Table IX.

For larger values of m or n, we can employ the asymptotic expansions given in appendix B. However, we must carry these expansions up to terms in m^{-6} in order to get the residuals for $K = 1$ with sufficient precision.

The last term of (D.2) contains series of the type

$$\sum_{n=3}^{\infty} n^{-(2p+4/3)} \times \frac{\log n}{(1 - n^2)} \qquad (D.3)$$

where p = 1, 2, ..., as well as series which are reducible to $\xi(\lambda)$ and $\xi'(\lambda)$, with $\lambda = 2p + 10/3$, p = 1, 2, ... The series (D.3) can be summed with the help of (C.3). The results have been checked by an independent calculation of (D.3) by means of the Euler-Maclaurin expansion. The obtained agreement was good.

The calculated values of $R_m^{(3)}$ for K = 1 are listed in Table X.

TABLE IX: The coefficient $K_{mn}(K)$, for K = 1,0 and m, n = 0, 1, 2.

K_{oo}	0.075478 0.246735i
K_{o1}	-0.134662 -0.063456i
K_{o2}	0.084411 0.031061i
K_{1o}	-0.042614 0.090431i
K_{11}	0.033791 -0.010507i
K_{12}	-0.038143 0.005286i
K_{20}	0.010014 -0.027216i
K_{21}	-0.018297 0.002538i
K_{22}	0.022864 -0.001174i

TABLE X: Residuals $R_n^{(\ell)}$ of the corrected asymptotic method.

$R_o^{(1)}$	0
$R_o^{(2)}$	-0.0000079 -0.0001268i
$R_1^{(2)}$	0.03519 0.03919i
$R_o^{(3)}$	-0.0000051 -0.0000806i
$R_1^{(3)}$	0.006338 -0.009306i
$R_2^{(3)}$	-0.001530 0.002911i

REFERENCES

1. H. M. NUSSENZVEIG, Phil. Trans. Roy. Soc. (London) A 252, 1 (1959).

2. G. N. WATSON, "Theory of Bessel Functions", (Cambridge University Press, 1948), p. 350, (3), (4).

   A. ERDÉLYI, "Higher Transcendental Functions",(N.York, Mc Graw Hill, 1953), vol. II, p. 42, (84).

3. L. LEWIN, "Advanced Theory of Waveguides", (Iliffe and Sons, London, 1951), p. 128.

4. N. MARCUVITZ, Waveguide handbook, (Mc Graw-Hill, N. York, 1951), p. 184.

5. H. B. DWIGHT, "Mathematical Tables", (Mc Graw-Hill, N. York, 1941).

6. R. HARTREE, "Numerical Analysis", (Oxford Press, 1952), p. 68.

7. G. VALIRON, Theory des Fonctions, (Masson et Cie, 1948), p. 214.

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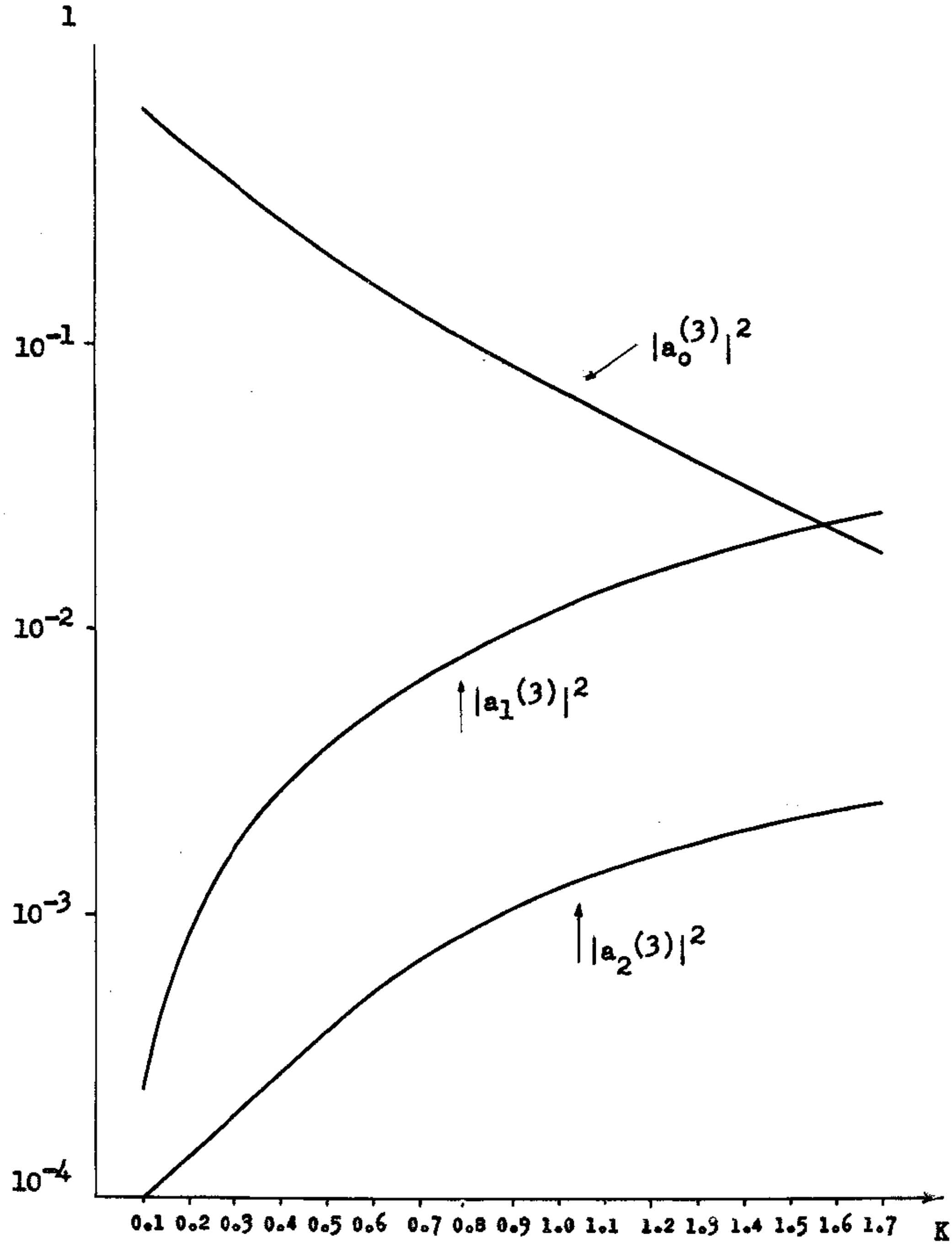

Fig. 2: Behaviour of $|a_o|^2$, $|a_1|^2$ and $|a_2|^2$ as a function of K in the third approximation of the asymptotic method.

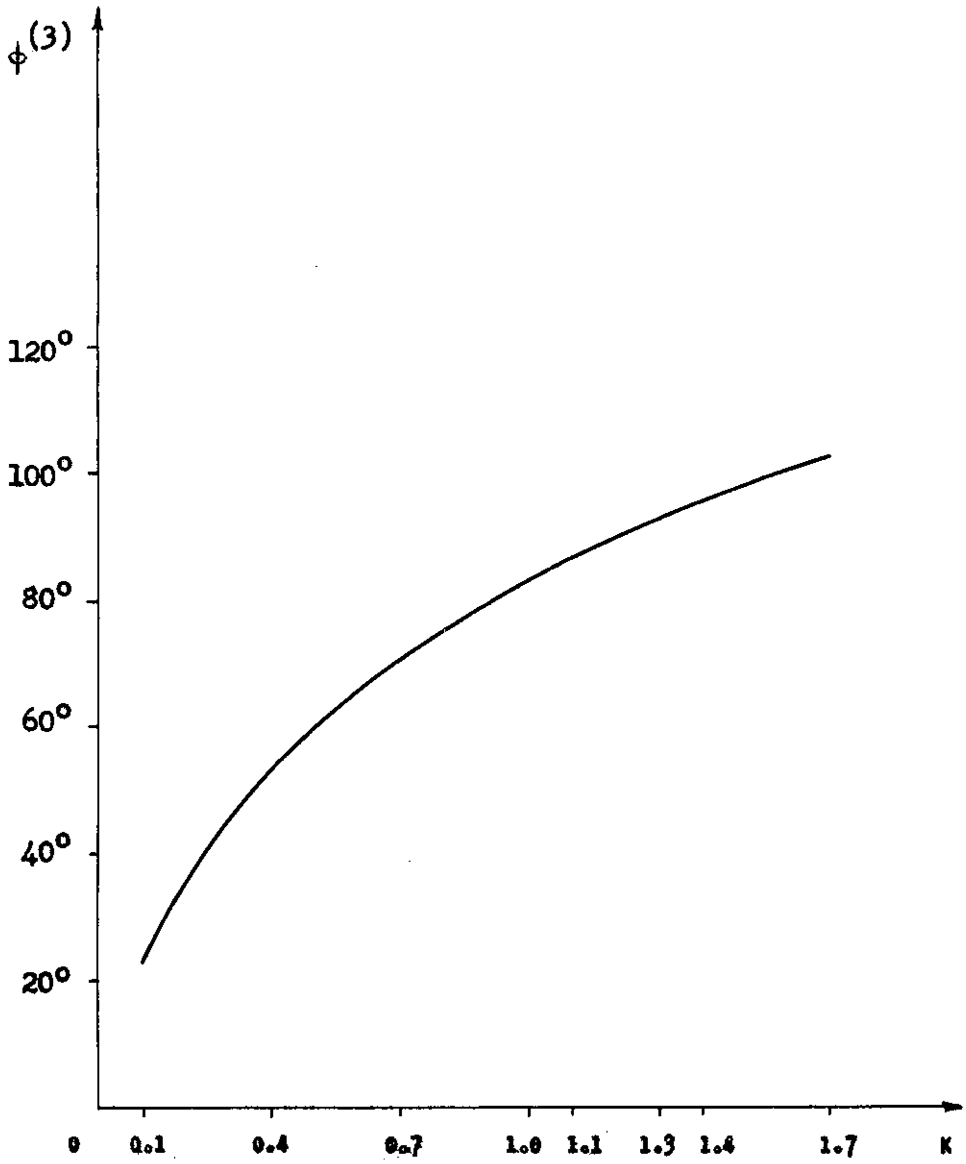

Fig. 3: Phase of the reflection amplitude as a function of K in the third approximation of the asymptotic method.

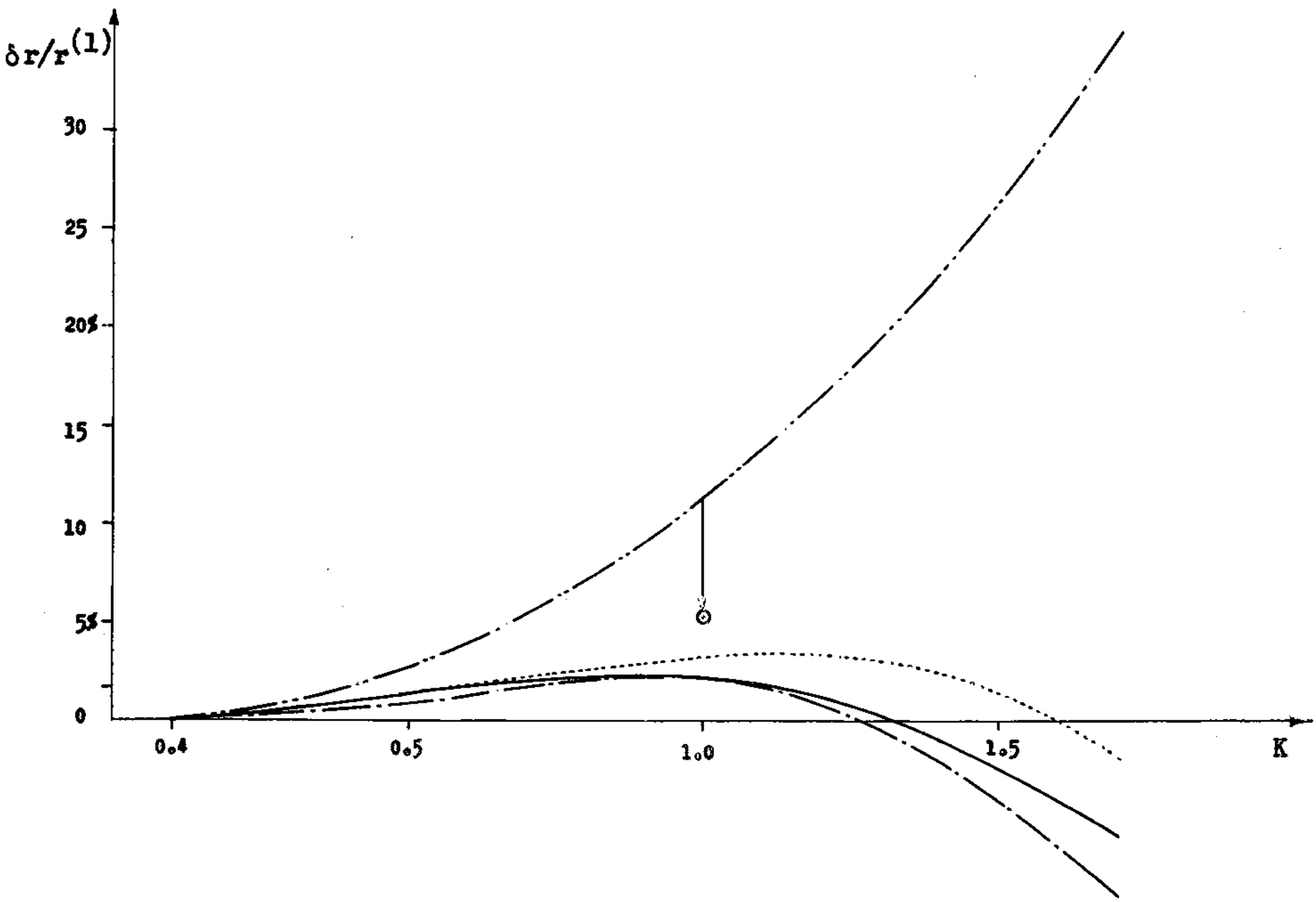

Fig. 4: Relative evanescent-mode correction $\delta r/r^{(1)}$ to the reflection coefficient in different approximations.

—··—··—··—	2nd approximation, asymptotic method;
—·—·—·—·—	3rd approximation, method of partial systems;
···············	3rd approximation, asymptotic method;
————————	3rd approximation, mixed method;
– – – – – – – –	2nd approximation, method of partial systems;
⊙	corrected asymptotic method.

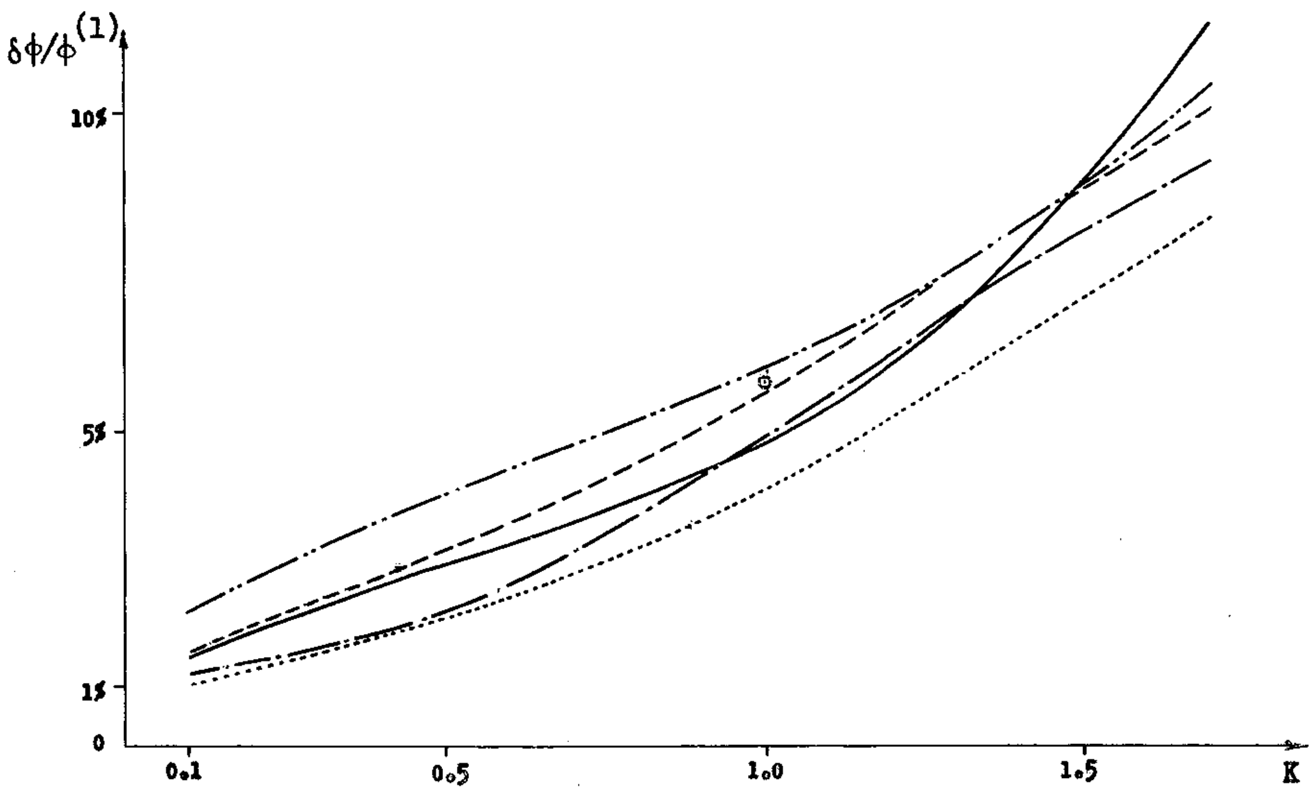

Fig. 5: Relative evanescent-mode correction $\delta\phi/\phi^{(1)}$ to the phase in different approximations. Same conventions as in fig. 4.