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RADIATION FIELD OF AN OSCILLATING DIPOLE - I

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RADIATION FIELD OF AN OSCILLATING DIPOLE* - I*

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The problem of the emission of radiation can be formulated in a closed form in the special case in which the radiation source can be considered to be an harmonic oscillator (magnetic dipole). In the first part of the present paper an approximate solution of the problem, neglecting the reaction of the radiation field and corresponding closely to the classical theory, is obtained. Radiation reaction is found to be closely connected to the behaviour of the system near resonance, which will be dealt with in the second part.

THE HAMILTONIAN AND THE WAVE FUNCTION OF THE SYSTEM

We will study the quantum radiation field produced by a magnetic oscillating dipole, that is, by an alternating electric current over a small closed circuit. Let us suppose this circuit to

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have capacity C, self-inductance $\mathcal{L}$, and resistence R = 0. If I is the current, $\frac{d^2I}{dt^2} + \frac{I}{\mathcal{L}C} = 0$. This equation can be substituted by Hamilton equations, with hamiltonian $H = \frac{1}{\omega_o^2} \cdot \frac{1}{2}\left(\frac{I^2}{C} + \mathcal{L}\dot{I}^2\right)$, where $\omega_o = \frac{1}{\sqrt{\mathcal{L}C}}$ is the frequency of the oscillation. We want to write this hamiltonian of the circuit in the form

$$H = \frac{1}{2}\left(\frac{P^2}{M} + M\omega_o^2 Q^2\right)$$

and then we must put $\frac{P}{\sqrt{M}} = \frac{I}{\omega_o\sqrt{C}}$, $\sqrt{M}\,\omega_o Q = -\frac{\sqrt{\mathcal{L}}}{\omega_o}\dot{I}$ (1)

The hamiltonian of the system constituted by the circuit and its field is

$$H = H_{circuit} + H_{field} + H_{interaction} =$$

$$= \frac{1}{2}\left(\frac{P^2}{M} + M\omega_o^2 Q^2\right) + \frac{1}{2}\sum_\lambda\left(p_\lambda^2 + \omega_\lambda^2 q_\lambda^2 - \hbar\omega_\lambda\right) + H_{int.} \quad (2)$$

where the q 's and p 's are the coördinates and momenta of the field oscillators, defined by

$$\vec{A} = \sum_\lambda b_\lambda \vec{A}_\lambda + b_\lambda^* \vec{A}_\lambda^*$$

$$\vec{A}_\lambda = \frac{\hbar c}{\sqrt{2\pi\hbar\omega_\lambda L^3}}\,\vec{a}_\lambda\, e^{i\vec{k}_\lambda\cdot\vec{r}} \quad (3)$$

$$b_\lambda = \sqrt{\frac{\omega_\lambda}{2\hbar}}\left(q_\lambda + \frac{p_\lambda}{i\omega_\lambda}\right)$$

L^3 is the volume of the cubic box which encloses the field, $\vec{A}$ is the potential vector of the magnetic field, each $\vec{a}_\lambda$ is a unit vector, and the star indicates complex conjugate.

The interaction energy between a current of density $\vec{J}$

and a magnetic field is given by $E_{int.} = \frac{1}{2c} \int \vec{J} \cdot \vec{A}\, dv$. Let us suppose the circuit to be constituted by a thin strip of width ℓ and negligible thickness, bent to form a cylinder of radius ρ_o and height ℓ. If the current is uniformly distributed, and $\vec{I}$ is the vector with the modulus and the direction of the total current, the vector $\vec{J}$ can be written

$$\vec{J} = \vec{I}\, \frac{1}{\ell}\, \delta(\rho - \rho_o) \qquad \text{for } |z| \leq \ell/2$$

$$= 0 \qquad \text{for } |z| \leq \ell/2$$

where ρ and z (along the simmetry axis of the circuit) are cylindrical coordinates. Using the expansion of $\vec{A}$ given in (3) we get:

$$E_{int.} = \sum_{\lambda}' \eta_\lambda \frac{P}{\sqrt{M}}\, p_\lambda$$

with

$$\eta_\lambda = \frac{\pi \rho_o}{\sqrt{\pi L^3}} \times \frac{\omega_o \sqrt{C}}{\omega_\lambda} \times \frac{k_\lambda}{\sqrt{k_{\lambda x}^2 + k_{\lambda y}^2}} \times$$

$$\times \frac{\sin(\frac{\ell}{2} k_{\lambda z})}{(\frac{\ell}{2} k_{\lambda z})}\; \vec{a}_\lambda \cdot \left(\frac{\vec{\mu}}{\mu} \wedge \frac{\vec{k}_\lambda}{k_\lambda} \right) J_1 \left(\rho_o \sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2} \right) \quad (4)$$

where J_1 is the Bessel function of first kind and first order; $\vec{\mu}$, magnetic moment of the circuit, is a vector directed along the z axis, and of modulus $\mu = \frac{1}{c} \pi \rho_o^2 I$. η_λ is a dimensionless quantity, and is small if ρ_o is small compared with L.

For $\sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2}\, \rho_o \ll 1$ and $\frac{\ell}{2} k_{\lambda_z} \ll 1$ it behaves like

$$\eta_\lambda \sim \frac{\omega_o \sqrt{C}}{2\sqrt{\pi L^3}}\, \vec{a}_\lambda \cdot \left(\frac{\vec{\mu}}{\mu} \wedge \frac{\vec{k}_\lambda}{k_\lambda}\right) \frac{1}{c} \times \pi \rho_o^2 \tag{5}$$

For $\sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2}\, \rho_o \gg 1$, we have

$$\eta_\lambda \sim \frac{\sqrt{2\pi\rho_o}}{\sqrt{\pi L^3}} \frac{\omega_o\sqrt{C}}{c}\, \vec{a}_\lambda \cdot \left(\frac{\vec{\mu}}{\mu} \wedge \frac{\vec{k}_\lambda}{k_\lambda}\right) \frac{\sin\left(\frac{\ell}{2} k_{\lambda_z}\right)}{\left(\frac{\ell}{2} k_{\lambda_z}\right)} \times$$

$$\times \frac{1}{\left(k_{\lambda_x}^2 + k_{\lambda_y}^2\right)^{3/4}} \cos\left(\rho_o \sqrt{k_{\lambda_x}^2 + k_{\lambda_y}^2} - \frac{3\pi}{4}\right) \tag{5'}$$

The hamiltonian of the system constituted by the circuit and the field is then

$$H = \frac{1}{2}\left(\frac{P^2}{M} + M\omega_o^2 Q^2\right) + \sum_\lambda \frac{1}{2}\left(p_\lambda^2 + \omega_\lambda^2 q_\lambda^2 - h\omega_\lambda\right) + \sum_\lambda \eta_\lambda \frac{P}{\sqrt{M}}\, p_\lambda \tag{6}$$

Here we have the hamiltonian of a system of coupled oscillators. There exists a linear canonical transformation which transforms H to principal axes:

$$q'_\mu = \sum_\nu a_{\mu\nu} q'_\nu \qquad \mu, \nu = 0, 1, 2 \ldots$$

$$H = \frac{1}{2}\sum_\mu \left(p'^2_\mu + \bar{\omega}_\mu^2 q'^2_\mu\right) + \text{const.} \tag{7}$$

eliminating the coupling term. The solution of the Schroedinger equation with hamiltonian (7) is of the form

$$\prod_\mu \mathcal{H}_{n_\mu}(q'_\mu)\, e^{-\frac{i}{\hbar}\bar{E}_\mu t} \tag{8}$$

Such a solution would solve in closed form the problem of the radiation field produced by the circuit considered. But we do not know explicitely the solution, that is, we cannot determine the $\overline{\omega_\mu}$. The field oscillators do not act directly one upon the other, but only through the circuit. If the "mass" and the "momentum" of the oscillator that represents the circuit are big enough compared with the "masses" and "momenta" of the field oscillators, the reaction of these upon those is very weak if the field oscillators are far from resonance, changing very little the motion of the circuit. This small alteration in the circuit motion due to a given field oscillator, would produce a smaller alteration in the motion of the other field oscillators. If we do not consider the contribution due to the resonance, the coeficients $a_{\mu\nu}$ with $\mu \neq \nu$ and μ or $\nu \neq o$ are then very small and can be neglected with good approximation.

In this approximation we can solve separately the problem of one field oscillator coupled with the circuit, that is, we can work with the hamiltonian

$$H_\lambda = \frac{1}{2}\left(\frac{P^2}{M} + M\,\omega_o^2\,Q^2\right) + \frac{1}{2}\left(p_\lambda^2 + \omega_\lambda^2\,q_\lambda^2 - \frac{1}{2}\hbar\,\omega_\lambda\right) + \eta_\lambda \frac{P}{\sqrt{M}}\,p_\lambda \tag{9}$$

We have to find the canonical transformation that eliminates the coupling term and reduces the problem to that of two independent oscillators. This transformation is (omitting the index λ):

$$p' = -\,\omega_o\sqrt{M}\,Q\cos\varphi + \omega\,q\sin\varphi$$

$$p' = -\,\omega_o\sqrt{M}\,Q\sin\varphi - \omega\,q\cos\varphi$$

$$Q' = \frac{P}{\omega_o \sqrt{M}} \cos \varphi - \frac{p}{\omega} \sin \varphi \qquad (10)$$

$$q' = \frac{P}{\omega_o \sqrt{M}} \sin \varphi + \frac{p}{\omega} \cos \varphi$$

where

$$\mathrm{tg}\, 2\varphi = \frac{2\,\eta\,\omega\,\omega_o}{\omega^2 - \omega_o^2} \qquad (11)$$

The hamiltonian becomes

$$H = \frac{1}{2}\left(P'^2 + \overline{\omega}_o^2 Q'^2 + p'^2 + \overline{\omega}^2 q'^2 - \hbar\omega\right) \qquad (12)$$

with

$$\overline{\omega}^2 = \omega_o^2 \sin^2\varphi + \omega^2 \cos^2\varphi + 2\,\eta\,\omega\,\omega_o \sin\varphi\cos\varphi$$

$$\overline{\omega}_o^2 = \omega_o^2 \cos^2\varphi + \omega^2 \sin^2\varphi - 2\,\eta\,\omega\,\omega_o \sin\varphi\cos\varphi \qquad (13)$$

As η is small enough, we will have, far from resonance:

$$\begin{cases} \varphi \simeq \dfrac{\eta\,\omega\,\omega_o}{\omega^2 - \omega_o^2} \\ \overline{\omega}_o \simeq \omega_o \\ \overline{\omega} \simeq \omega \end{cases} \qquad \begin{cases} P' \simeq -\,\omega_o \sqrt{M}\,Q \\ Q' \simeq \dfrac{P}{\omega_o \sqrt{M}} \end{cases} \qquad (14)$$

The Schröedinger equation for (12) has solutions of the form:

$$\psi_{N,n} = C_{N,n}\, H_N\left(\frac{Q'}{\sqrt{\hbar/\bar{\omega}_o}}\right) e^{-\frac{1}{2}\left(\frac{Q'}{\sqrt{\hbar/\bar{\omega}_o}}\right)^2} H_n\left(\frac{q'}{\sqrt{\hbar/\bar{\omega}}}\right) \times$$

$$\times\, e^{-\frac{1}{2}\left(\frac{q'}{\hbar/\bar{\omega}}\right)} e^{-\frac{i}{\hbar}\left(\bar{E}_n + \bar{E}_N\right) t} \qquad (15)$$

where H indicates the Hermite polynomials.

We want a solution in which the current in the circuit has the probability distribution of a Gauss packet oscillating in time. Let us call

$$y = \frac{Q'}{\sqrt{\hbar/\bar{\omega}_o}} \simeq \frac{P}{\sqrt{M\hbar\bar{\omega}_o}}$$

$$x = \frac{q'}{\sqrt{\hbar/\bar{\omega}}} = \frac{1}{\sqrt{\hbar/\bar{\omega}}}\left(\frac{P}{\omega_o\sqrt{M}} \sin\psi + \frac{p}{\omega}\cos\psi\right) = c\,P + dp$$

The general solution can be written in the form

$$\phi = \left(\sum_n c_n H_n(x)\, e^{-x^2/2}\, e^{-\frac{i}{\hbar}\bar{E}_n t}\right) \times$$

$$\times \left(\sum_N C_N H_N(y)\, e^{-y^2/2}\, e^{-\frac{i}{\hbar}\bar{E}_N t}\right) = \varphi(x)\,\psi(y) \qquad (16)$$

We know that $\sum_N z^N \frac{H_N(y)}{N!} = e^{-z^2 + 2zy}$; choosing $C_N = \left(\frac{B}{2}\right)^N \times \frac{1}{N!}$

where B is a constant we have

$$\psi(y) = \sum_N \left(\frac{B}{2}\right)^N \times \frac{1}{N!} H_N(y)\, e^{-y^2/2}\, e^{-\frac{i}{\hbar}\bar{E}_N t} =$$

$$= \left(e^{-\frac{y^2}{2}}\, e^{-\frac{i}{2}\bar{\omega}_o t}\right) e^{-\left(\frac{B}{2} e^{-i\bar{\omega}_o t}\right)^2 + B e^{-i\omega_o t} y}$$

As x is a slowly varying function of P (and so of y) in the region far from resonance, the probability distribution in P will be given approximately by

$$\psi(y)\, \psi^*(y) = e^{-y^2 - 2\frac{B^2}{4} \cos 2\bar{\omega}_o t + 2 B y \cos \bar{\omega}_o t} =$$

$$= \text{const.}\; e^{-(y - B \cos \bar{\omega}_o t)^2}$$

which is the oscillating Gauss packet we want. The wave function is then

$$\psi = \left(\sum_n c_n H_n(x)\, e^{-\frac{x^2}{2}}\, e^{-\frac{i}{\hbar}\bar{E}_n t}\right) \times$$

$$\times \left(e^{-\frac{y^2}{2}}\, e^{-\frac{i}{2}\bar{\omega}_o t}\, e^{-\left(\frac{B}{2} e^{-i\bar{\omega}_o t}\right)^2 + B e^{-i\omega_o t} y}\right)$$

Let us take the field oscillators in the fundamental states. The wave function is

$$\psi = \text{const.}\; e^{-\frac{x^2}{2}} \left(e^{-\frac{y^2}{2} - \frac{i}{2}\bar{\omega}_o t - \left(\frac{B}{2} e^{-i\bar{\omega}_o t}\right)^2 + B e^{-i\bar{\omega}_o t}\, y} \right) \tag{17}$$

$$\psi = \text{const.}\; e^{-\frac{1}{2}(cP + dp)^2} f(P,t)$$

This ψ gives the amplitude of probability of finding at the time t the field oscillator with momentum p and the circuit oscillator with momentum P. We want to hnow the amplitude of probability of finding the field oscillator with momentum p for any value of P, that is, we want to know the behaviour of the field oscillator after taking the average over the quantum distribution in the circuit. By the fact that $x = cP + dp$ is, far from resonance, a slowly varying function of P (because c is very small), we have, in our approximation :

$$\phi(p) = \int_{-\infty}^{+\infty} (\text{const.})\; e^{-\frac{1}{2}(cP + dp)^2} f^*(P) f(P)\, dP$$

$$\phi(p) = (\omega\hbar\pi)^{-1/4} \exp\left[-\frac{1}{2}\left(\frac{p}{\sqrt{\hbar\omega}} - B\frac{\eta\,\omega\sqrt{\omega\,\omega_o}}{\omega^2 - \omega_o^2}\cos\omega_o t \right)^2 \right] \tag{18}$$

The average values of q and p over $\phi(p)$ are

$$\bar{q} = 0 \quad \text{and} \quad \bar{p} = \sqrt{\hbar\,\omega_o}\; B\,\frac{\eta\,\omega^2}{\omega^2 - \omega_o^2}\cos\omega_o t \tag{19}$$

THE RADIATION FIELD

The average value of the potential vector is given by

$$\vec{A} = \sum_\lambda \left(\bar{b}_\lambda \vec{A}_\lambda + \bar{b}_\lambda^* \vec{A}_\lambda^*\right) = \sum_\lambda \frac{\rho_\lambda}{i\,\omega_\lambda} \sqrt{\frac{\omega_\lambda}{2\hbar}} \left(\vec{A}_\lambda - \vec{A}_\lambda^*\right) =$$

$$= \frac{1}{2i} \frac{c\sqrt{\hbar}}{\sqrt{\pi L^3}} \sqrt{\omega_o} \cos \omega_o t \sum_\lambda \frac{B_\lambda \eta_\lambda \omega_\lambda}{\omega_\lambda^2 - \omega_o^2} \vec{a}_\lambda \left(e^{i\vec{k}_\lambda \cdot \vec{r}} - e^{-i\vec{k}_\lambda \cdot \vec{r}}\right),$$

The Fourier expansion of the classical field is

$$\vec{A} = \sum_\lambda \frac{8\pi^2}{\sqrt{\pi L^3}\; 2\omega_o \sqrt{C}} \eta_\lambda \frac{\left(\frac{\vec{\mu}}{\mu} \wedge \frac{\vec{k}_\lambda}{k_\lambda}\right) \cdot k_\lambda}{\vec{a}_\lambda \cdot \left(\frac{\vec{\mu}}{\mu} \wedge \frac{\vec{k}_\lambda}{k_\lambda}\right)} \times$$

$$\times \frac{I_o \cos \omega_o t}{k_o^2 - k_\lambda^2}\, i \left(e^{i\vec{k}_\lambda \cdot \vec{r}} - e^{-i\vec{k}_\lambda \cdot \vec{r}}\right)$$

being then $\vec{a}_\lambda$ a vector parallel to $\frac{\vec{\mu}}{\mu} \wedge \frac{\vec{k}_\lambda}{k_\lambda}$, that is,

$$\frac{\vec{k}_\lambda}{k_\lambda} \wedge \frac{\vec{\mu}}{\mu} = \vec{a}_\lambda \left\{ \vec{a}_\lambda \cdot \left(\frac{\vec{k}_\lambda}{k_\lambda} \quad \frac{\vec{\mu}}{\mu}\right)\right\}$$

Comparing the classical and the quantum fields we get

$$B_\lambda = B = \frac{8\pi^2 I_o}{\omega_o \sqrt{\hbar \omega_o} \sqrt{C}} \qquad (20)$$

The approximations made are based on the assumption that $\operatorname{tg} 2\varphi \ll 1$, that is, $\frac{2\eta\,\omega\,\omega_o}{\omega^2 - \omega_o^2} \ll 1$. Looking at the value of η given by (4) we see that the larger the box, the closer we could go to resonance, satisfying these conditions of the approximation. But we have to remember that the larger the box, the greater the number of levels that exists near resonance; if the box is very large we have to include the effects of resonance, which have been neglected.

EXPANSION OF THE WAVE FUNCTION IN EIGENFUNCTIONS OF THE HARMONIC OSCILLATOR

$$\phi(p) = \frac{1}{(\omega\hbar\pi)^{1/4}} \exp\left[-\frac{1}{2}\left(\frac{p}{\sqrt{\hbar\omega}} - b\right)^2\right] =$$

$$= \mathcal{H}_0\left(\frac{p}{\sqrt{\hbar\omega}} - b\right) = \sum_n c_n \mathcal{H}_n\left(\frac{p}{\sqrt{\hbar\omega}}\right)$$

where $\mathcal{H}$ indicates the normalized eigenfunctions of the harmonic oscillator, and

$$b = \frac{1}{\sqrt{\hbar\omega}} \times \frac{8\pi^2 I_0}{\omega_0\sqrt{c}} \frac{\eta\,\omega^2}{\omega^2 - \omega_0^2} \cos\omega_0 t \qquad (21)$$

We get

$$c_n = \frac{b^n e^{-b^2/4}}{\sqrt{2^n \times n!}} \qquad (22)$$

$p_n = c_n^{\,2}$ represents the probability of finding the field oscillator of fundamental frequency ω (ω_λ, remembering the indices) in the state of energy $n\hbar\omega$.

We see that the quantum field presents small fluctuations around the average value, which equals the classical field. The quantum treatment implies fluctuations of the field and its energy density and approaches classical results only in the limit of strong fields.

The results expressed by (22) refer to the stationay states of the system: the field is enclosed in a box, and the circuit emits and absorbs photons. The probabilities p_n are periodic functions of time.

THE WAVE FUNCTION OF THE WHOLE FIELD

The formulae we have derived above are valid as long as the induced vibrations in the field are so weak that they do not appreciably alter the motion of the circuit. This will be the case as long as our box is so small that no vibrations of the field exist in the immediate neighbourhood of resonance. They may also be used outside of the domain of resonance, in cases in which we can suppose that the oscillating currents in the circuit can be described with sufficient approximation by our wave packet (17), e.g. neglecting a small damping of the system during a limited period of time. A more complete treatment of the problem requires, however, a detailed study of the field in the resonance region, which will be attempted in the second part of this paper.

In the assumed approximations the wave function of the whole field is

$$\Phi \left(p_1, p_2 \dots p_\lambda \dots \right) = \prod_\lambda \varphi \left(p_\lambda \right) = \prod_\lambda \mathcal{H}_0 \left(\frac{p}{\sqrt{\hbar \omega_\lambda}} - b_\lambda \right) \qquad (23)$$

If we expand this function in terms of the eigenfunctions of the field oscillator:

$$\phi\,(p_1, p_2 \dots p_\lambda \dots) = c_{00\dots 0\dots}\,\mathcal{H}_0^1\,\mathcal{H}_0^2 \dots +$$

$$c_{10\dots 0}\,\mathcal{H}_1^1\,\mathcal{H}_0^2 \dots + c_{010\dots}\,\mathcal{H}_0^1\,\mathcal{H}_1^2\,\mathcal{H}_0^3 \dots + \dots$$

the coefficients represent the amplitudes of probability of finding the fiel oscillators in the different states. We have

$$c_{00\dots 0} = \prod_\lambda c_0^\lambda \quad , \quad c_{10\dots 0} = c_1^1 \prod_{\lambda \neq 1} c_0^\lambda \quad \text{, and so on, where}$$

$$c_n^\lambda = \frac{b_\lambda^n \, e^{-\frac{b_\lambda^2}{4}}}{\sqrt{2^n \times n!}}$$

$p_{00\dots 0\dots}$ = probability of finding all field oscillators in the ground state

$$= \prod_\lambda \left(c_0^\lambda\right)^2 = e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}$$

$p_{10\dots 0,\dots}$ = probability of finding the oscillator of frequency ω_1 in the first excited state and all the other ones in the ground state

$$= \left(c_1^1\right)^2 \prod_{\lambda \neq 1} \left(c_0^\lambda\right)^2 = \frac{1}{2}\, b_1^2 \; e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}$$

Analogously: $p_{010\dots 0\dots} = \frac{1}{2}\, b_2^2 \; e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}$, and so on.

As each $p^\lambda = \left(c^\lambda\right)^2$ is smaller than 1 and there is an infinite number of factors, these probabilities are zero; but a meaning can be given to the relative values of the probabilities and in the probabilities of finding the oscillators with frequencies contained in finite intervals $\Delta\omega_\lambda$ in the several states.

VIRTUAL PHOTONS

(Results which the perturbation method furnishes)

If we develop separately the wave function of each oscillator in series of eigenfunctions, the wave function of the whole field is

$$\phi = \prod_\lambda \left(c_0^\lambda \mathcal{H}_0\left(\frac{p_\lambda}{\sqrt{\hbar\omega_\lambda}}\right) + c_1^\lambda \mathcal{H}_1\left(\frac{p_\lambda}{\sqrt{\hbar\omega_\lambda}}\right) + \ldots\ldots \right) \tag{24}$$

The probability of finding zero virtual photons in all the field (all oscillators in the ground state) is

$$W(0) = \left|\prod_\lambda c_0^\lambda\right|^2 = e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}$$

The probability of finding any one photon in the field, that is, one, no matter which, of the oscillators in the first excited state and all the others in the ground state, is

$$W(1) = \sum_k \left|c_1^k \prod_{\lambda\neq k} c_0^\lambda\right|^2 = e^{-\frac{1}{2}\sum_\lambda b_\lambda^2} \sum_k \left|\frac{c_1^k}{c_0^k}\right|^2 =$$

$$= \left(\frac{1}{2}\sum_k b_k^2\right) e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}$$

Analagously

$$W(2) = \frac{1}{2!} \sum_{k,k'} \left| c_1^k c_1^{k'} \prod_{\lambda \neq k,k'} c_o^\lambda \right|^2 = \frac{1}{2!} \left(\frac{1}{2} \sum_k b_k^2 \right)^2 e^{-\frac{1}{2} \sum_\lambda b_\lambda^2}$$

$$W(n) = \frac{1}{n!} \left(\frac{1}{2} \sum_\lambda b_\lambda^2 \right)^n e^{-\frac{1}{2} \sum_\lambda b_\lambda^2} \tag{25}$$

The number of photons has then a Poisson distribution. The average number is

$$\bar{n} = \left(\frac{1}{2} \sum_\lambda b_\lambda^2 \right) \tag{26}$$

In this way we can estimate the number of photons in the field. In the case of a dipole

$$\rho_o \sqrt{k_x^2 + k_y^2} \ll 1 \quad \text{and} \quad \ell k_z \ll 1$$

this number diverges; in fact, using (5) we get

$$\bar{n} = \frac{(8\pi^2)^2}{2\pi L^3} \left(\pi \rho_o^2 \right)^2 \frac{I_o^2 \cos^2 \omega_o t}{\hbar c^3} \sum_\lambda \frac{k_\lambda^3}{\left(k_\lambda^2 - k_o^2 \right)^2}$$

The sum diverges like $\int^\infty k \, d\, k$.

AVERAGE ENERGY IN THE FIELD

$$\bar{E} = \sum_k \hbar \omega_k \left| c_1^k \prod_{\lambda \neq k} c_o^\lambda \right|^2 +$$

$$+ \frac{1}{2!} \sum_{k,k'} \left(\hbar \omega_k + \hbar \omega_{k'} \right) \left| c_1^k c_1^{k'} \prod_{\lambda \neq k,k'} c_o^\lambda \right|^2 + \ldots =$$

$$= e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}\left[\left\{\sum_k \hbar\,\omega_k\left(\frac{1}{2}\,b_k^2\right)\right\} + \right.$$

$$\left. + \frac{1}{2!}\times 2\left\{\sum_k \hbar\,\omega_k\left(\frac{1}{2}\,b_k^2\right)\right\}\left(\sum_{k'}\frac{1}{2}\,b_{k'}^2\right) + \ldots\right] =$$

$$= e^{-\frac{1}{2}\sum_\lambda b_\lambda^2}\left\{\sum_k \hbar\,\omega_k\left(\frac{1}{2}\,b_k^2\right)\right\}\times$$

$$\times\left[1 + \frac{1}{1!}\left(\sum_\lambda\frac{1}{2}\,b_\lambda^2\right) + \frac{1}{2!}\left(\sum_\lambda\frac{1}{2}\,b_\lambda^2\right)^2 + \ldots\right]$$

$$\bar{E} = \sum_k \hbar\,\omega_k\left(\frac{1}{2}\,b_k^2\right) \tag{27}$$

In the case of dipole, this becomes

$$\bar{E} = \frac{(8\pi^2)^2}{2\pi L^3}\left(\pi\rho_0^2\right)^2\frac{I_0^2\cos^2\omega_0 t}{c^2}\sum_\lambda\frac{k_\lambda^4}{\left(k_\lambda^2 - k_0^2\right)^2}$$

which diverges like $\int^\infty k^2\,d\,k$ as in the classical case.

PROBABILITIES IN A GIVEN INTERVAL OF FREQUENCIES

The probability of finding zero virtual photons in Δk_λ is $\underset{\Delta k_\lambda}{W}(0) = \prod_{\lambda\text{ in }\Delta k_\lambda} |c_0^\lambda|^2 = e^{-\frac{1}{2}\sum' b_\lambda^2}$

where $\sum'$ indicates sum over all the frequencies in the interval $\triangle k_\lambda$. The probability of finding n photons in $\triangle k_\lambda$ is

$$\underset{\triangle k_\lambda}{W}(n) = \frac{1}{n!}\left[\frac{1}{2}\sum{}' b_\lambda^2\right] e^{-\frac{1}{2}\sum{}' b_\lambda^2} \tag{28}$$

The average number of photons in the interval is $\bar{n}_{\triangle k_\lambda} = \frac{1}{2}\sum{}' b_\lambda^2$

This number is large when b_λ is large: this happens in the region of resonance. The average energy in $\triangle k_\lambda$ is

$$\bar{E}_{\triangle k_\lambda} = \frac{1}{2}\sum{}' \hbar\,\omega_\lambda b_\lambda^2 .$$

For large values of n the maximum in the probability distributions (25) and (28) occurs approximately for

$$n = \bar{n} = \frac{1}{2} \sum b_\lambda^2 .$$

ANOTHER DESCRIPTION OF THE FIELD.
THE EVOLUTION OF THE SYSTEM WITH TIME

We dealt with the stationary case. We have to see how to describe in this formalism the non-stationary case. We have to choose an initial configuration and to study how it develops in time. In order to study the radiation process we could choose as simplified initial conditions: the field not coupled with the circuit, and this in an excited state

$$\varphi(0) = \mathcal{H}_N\left(\frac{p}{\sqrt{M h \omega_0}}\right) \prod_\lambda \mathcal{H}_0\left(\frac{p_\lambda}{\sqrt{\hbar\,\omega_\lambda}}\right) \tag{29}$$

To know how the system changes in time, we expand $\varphi(0)$ in terms of the stationary solutions of the problem:

$$\varphi(0) = \mathcal{H}_N\left(\frac{P}{\sqrt{M\hbar\omega_0}}\right)\prod_\lambda \mathcal{H}_0\left(\frac{p_\lambda}{\sqrt{\hbar\omega_\lambda}}\right) = \sum_{L,\ell_\lambda} d_{L,\ell_\lambda}\,\mathcal{H}_L\left(\frac{P}{\sqrt{M\hbar\omega_\lambda}}\right)\times$$

$$\prod_\lambda \mathcal{H}_{\ell_\lambda}\left(\frac{p_\lambda}{\sqrt{\hbar\omega_\lambda}} + \alpha_\lambda\frac{P}{\sqrt{M\hbar\omega_0}}\right) \tag{30}$$

where $\alpha_\lambda = \sqrt{\frac{\omega_\lambda}{\omega_0}}\,\sin\varphi_\lambda$

$$\varphi(t) = \sum_{L,\ell_\lambda}\left(d_{L,\ell_\lambda}\,e^{-\frac{i}{\hbar}tE_{L,\ell_\lambda}}\right)\mathcal{H}_L\left(\frac{P}{\sqrt{M\hbar\omega_0}}\right)\times$$

$$\prod_\lambda \mathcal{H}_{\ell_\lambda}\left(\frac{p_\lambda}{h} + \alpha_\lambda\frac{P}{\sqrt{M\hbar\omega_0}}\right) \tag{31}$$

d_{L,ℓ_λ} represents the amplitude of probability of finding the circuit in the L-th excited state and ℓ_λ photons of frequency ω_λ in the field. Calculating in this way the emission process we have a different description of the field; we may call "real photons" these states of excitation.

The description of the initial condition as has been done, defining them in terms of the states of the non-coupled system, is insatisfactory. For example, if initially, we have the circuit in the fundamental state and the field is the non-coupled vacuum, that is, if

$$\varphi(0) = \mathcal{H}_0\left(\frac{P}{\sqrt{M\hbar\omega_0}}\right)\prod_\lambda \mathcal{H}_0\left(\frac{p_\lambda}{\sqrt{\hbar\omega_\lambda}}\right) \tag{32}$$

we get

$p_{0;0}$ = probability of finding zero photons in the field and the circuit in the fundamental state = $\left(1 + \frac{1}{4} \sum \alpha_\lambda^2\right)^{-1}$

$p_{2;0}$ = probability of finding two photons, no matter which, and the circuit in the fundamental state = $\left(\frac{1}{y} \sum \alpha_\lambda^2\right)^2 \left(1 + \frac{1}{y} \sum \alpha_\lambda^2\right)^{-3}$

$p_{4;0} = 3 \times 1 \times \left(\frac{1}{y} \sum \alpha_\lambda^2\right)^4 \left(1 + \frac{1}{y} \sum \alpha_\lambda^2\right)^{-5}$, and so on.

So we have the probabilities of finding photons and excited states in a system which was initially in the state called "vacuum". The creation of these photons and excited states is due to the coupling.

Let us estimate the value of $\sum \alpha_\lambda^2$. We have

$$\sum \alpha_\lambda^2 = \sum \frac{\omega_\lambda}{\omega_0} \sin^2 \varphi_\lambda = \sum_\lambda \frac{\omega_\lambda}{\omega_0} \frac{\eta_\lambda^2 \, \omega_\lambda^2 \, \omega_0^2}{\left(\omega_\lambda^2 - \omega_0^2\right)^2} =$$

$$= \frac{\omega_0 (\pi \rho_0)^2}{\pi L^3} \frac{\omega_0^2 c}{c^3} \sum_\lambda \frac{k_\lambda^3}{\left(k_{\lambda x}^2 + k_{\lambda y}^2\right)\left(k_\lambda^2 - k_0^2\right)^2} \times$$

$$\frac{\sin^2\left(\frac{\ell}{2} k_{\lambda z}\right)}{\left(\frac{\ell}{2} k_{\lambda z}\right)^2} J_1^2\left(\rho_0 \sqrt{k_{\lambda x}^2 + k_{\lambda y}^2}\right)$$

If L is very large we may transform the sum in an integral; using asymptotic expressions for the Bessel function, assuming that $k_0 \rho_0 \ll 1$ amd that ℓ is smaller than ρ_0, and neglecting resonance effects, we get

$$\sum_{\lambda} \alpha_{\lambda}^{2} \approx \frac{1}{2\pi^2} \times \frac{C\rho_0}{\gamma^2} \left(k_0 \rho_0\right)^2 = \frac{1}{2\pi^2} \left(\frac{\rho_0}{\gamma}\right)^2 \frac{\rho_0}{c^2 \mathcal{L}} \qquad (33)$$

If the values of the capacity, frequency and dimensions of the circuit are such that $\sum \alpha_{\lambda}^{2} \ll 1$ the probabilities of appearing photons when we have initially vacuum are very small, and the probability of the system to go to the state called "coupled vacuum" is very near 1. In this case of weak coupling there is sense in saying that (32) represents the vacuum, and we are entilled to use (29) as initial condition for the system: if we consider high excitation levels in the circuit at the initial state, that is, if we want, as in the classical case, to describe the emission of a large number of photons the contributions due to the coupling are of small importance, and our results will approximate those obtained by perturbation methods (virtual photons). We may expect that the probabilities of the different emission processes calculated in this way will be affected by errors of the order of the probabilities of these processes when the initial state is the vacuum, that is we have to note that some of the photons that appears in the field are due to the coupling; these error can be neglected in cases of weak coupling $\left(\sum \alpha_{\lambda}^{2} \ll 1.\right)$.

Our particular problem, excluding resonance, is then solved, at least in principle, in cases of weak coupling. There exists the possibility of applying this solution to the cases of strong coupling but this problem has not get been solved. The question of resonance will be discussed in the second part of this paper.

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