ISSN 0101-9201

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
Rio de Janeiro

Notas de Física

CBPF-NF-011/11
July 2011

The vanishing volume of $D = 4$ superspace

Guillaume Bossard, P.S. Howe, K.S. Stelle and Pierre Vanhove

Ministério da
Ciência e Tecnologia

CBPF-NF-011/11
CPHT-RR045.0511
IHES/P/11/14
Imperial/TP/11/KSS/02
IPHT-T-11/124
KCL-MTH-11-11
NSF-KITP-11-069

The vanishing volume of $D = 4$ superspace

Guillaume Bossard,[a] **P.S. Howe,**[b] **K.S. Stelle,**[c,d,e] **Pierre Vanhove**[d,f,g]

[a] *Centre de Physique Théorique, Ecole Polytechnique, CNRS 91128 Palaiseau Cedex, France*

[b] *Department of Mathematics, King's College, University of London Strand, London WC2R 2LS, UK*

[c] *Theoretical Physics Group, Imperial College London Prince Consort Road, London SW7 2AZ, UK*

[d] *Kavli Institute for Theoretical Physics, University of California Santa Barbara CA 93106, USA*

[e] *TEO/CBPF, Rua Dr. Xavier Sigaud 150 cep 22290-180, Rio de Janeiro (RJ), Brazil*

[f] *Institut des Hautes Etudes Scientifiques Le Bois-Marie, F-91440 Bures-sur-Yvette, France*

[g] *Institut de Physique Théorique, CEA, IPhT, F-91191 Gif-sur-Yvette, France CNRS, URA 2306, F-91191 Gif-sur-Yvette, France*

E-mail: bossard@cpht.polytechnique.fr, paul.howe@kcl.ac.uk, k.stelle@imperial.ac.uk, pierre.vanhove@cea.fr

Abstract: The volume of on-shell $D = 4$, $\mathcal{N} = 8$ superspace is shown to vanish. Despite this, it is shown that there is a fully supersymmetric and duality-invariant candidate $\nabla^8 R^4$ counterterm corresponding to an anticipated seven-loop logarithmic divergence in $D = 4$. We construct this counterterm explicitly and also give the complete nonlinear extension of the 1/8-BPS $\nabla^6 R^4$ invariant. Similar results are derived for $\mathcal{N} = 4, 5$ & 6.

Keywords: supergravity; supersymmetric invariants; ultraviolet divergences

Contents

1 Introduction 1

2 Superspace formalism 4

3 Full superspace integrals 12

4 Invariants in extended superspace 13

5 Conclusion 20

A On-shell extended Superspace 23

B Kinematic structure 25

1 Introduction

The problem of ultraviolet divergences in supergravity has attracted the attention of theorists since the origins of the theory. Early on, it was realised that candidate supersymmetric counterterms, non-vanishing subject to the classical equations of motion, exist starting from the 3-loop level in $D = 4$, where they would generically be of (curvature)^4 structure [1–4]. It was noted at the time that, with respect to the full "on-shell" supersymmetry of the $\mathcal{N} > 4$ extended supergravities, these (curvature)^4 counterterms need to be expressed as subsurface integrals over the full superspace, *i.e.* as "BPS", or "F-term" invariants. But according to the understanding at the time of the possible linearly realisable "off-shell" supersymmetry, which is less than the full on-shell degree, they appeared to be expressible as full superspace integrals of the linearly realisable off-shell supersymmetry and thus were not thought to violate applicable non-renormalisation theorems [5]. Thus, despite their BPS subsurface-integral structure, the ultraviolet divergences of $D = 4$ supergravity looked set to begin at the 3-loop level, provided the maximal off-shell linearly realisable supersymmetry corresponded to just half the full on-shell degree. Should the linearly realisable supersymmetry turn out to be more than half the full on-shell degree, *e.g.* through a harmonic superspace formulation, the divergence onset loop order would correspondingly rise [6, 7].

In case the non-renormalisation theorems for BPS invariants turned out to be stronger than anticipated, it was also noted in the 1980s that full-superspace integral invariants would in any case be available starting at the 7-loop or 8-loop order [3, 8]. The constraints of continuous duality symmetries such as $E_{7(7)}$ for the maximal $\mathcal{N} = 8$ theory were recognised to be important as well. The 8-loop full-superspace counterterm was recognised to be manifestly duality invariant. But it was also anticipated that a duality-invariant counterterm could exist already at 7 loops, where naïve power counting in $D = 4$ gives an expectation of a dimension 16 counterterm, corresponding to the $\int d^{32}\theta$ full-superspace integral for maximal supergravity. An obvious candidate for such a dimension-16 duality invariant non-BPS counterterm was the full volume of the $\mathcal{N} = 8$ superspace, $\int d^4xd^{32}\theta E$, where E is the Berezinian determinant of the supervielbein.

For the $\mathcal{N} \leq 3$ lesser extended supergravities, it was recognised that the volume of superspace vanishes subject to the classical supergravity field equations, for a series of specific reasons. For $\mathcal{N} = 1$, the volume of superspace gives the dimension-2 supergravity action [9], and thus vanishes on-shell for the "non-gauged" Poincaré supergravities without a cosmological constant. Indeed, in the "new minimal" auxiliary-field formulation, the superspace volume vanishes even off-shell [10, 11]. In the $\mathcal{N} = 2$ case, the vanishing of the superspace volume was expected because the corresponding dimension-4 (curvature)^2 invariants are constrained by the Gauss-Bonnet identity to be equivalent, up to a total divergence, to quadratic expressions in the Ricci tensor or its trace, thus leading to on-shell vanishing counterterms. The vanishing of the $\mathcal{N} = 2$ superspace volume was confirmed explicitly in [12] by reducing it to a chiral integral. In the $\mathcal{N} = 3$ case, it was similarly known that there are no dimension-6 counterterms (corresponding to (curvature)^3 structures) that are non-vanishing subject to the classical equations of motion [13]. Aside from these rather transparent low-$\mathcal{N}$ supergravity cases, however, there seemed to be no particular reason why the superspace volume should vanish for the higher-$\mathcal{N}$ extended supergravities.

In the meantime, computational techniques have improved dramatically, and much more is now known from explicit calculations about the ultraviolet divergences of supergravity (see [14] for a recent review). The result is that, despite the anticipation of first maximal supergravity divergences at 3 loops, ultraviolet cancellations turn out to continue unabated in $D = 4$ and also in $D = 5$ at the 3-loop [15, 16] and at the 4-loop levels [17].[1] This clearly required revisiting the analysis of the non-renormalisation theorems. Indeed, although the earlier 1980s non-renormalisation analysis had relied upon the known off-shell linearly realisable degree of supersymmetry, it turns out that the full on-shell supersymmetry imposes further constraints that were not initially recognised. Even though the full on-shell supersymmetry involves nonlinear transformations and is thus subject to complicated transformation renormalisations, the corresponding Ward identities, expressed using BRST algebraic renormalisation techniques, show that the (curvature)^4 counterterm previously anticipated at the $D = 4$ 3-loop level, is actually ruled out [7]. Similarly, the BPS counterterms expected at loop orders up to 6 were brought under suspicion.

Another aspect of the BPS counterterms that was missed in the original 1980s analysis is their delicacy with respect to the continuous duality symmetries. Originally, the only analysis that could be carried out used linearised $\mathcal{N} = 8$ supersymmetry transformations, focusing purely on the leading 4-particle level of the candidate counterterms. At this lead-

[1] Owing to the on-shell conditions, no non-vanishing 4-loop divergences could have appeared in the $\mathcal{N} = 8$, $D = 4$ four-graviton amplitude. Moreover, nonlinear $\nabla^2 R^4$ and R^5 invariants were ruled out in Ref. [18]. A discussion of the kinematic structure of four-point counterterms in $D = 4$ non-maximal supergravity will be given in Appendix B.

ing order, the (curvature)^4 candidate passed the only available test of duality invariance, namely invariance under constant shifts of the 70 scalar fields. This happened because at the 4-point level all scalar fields in the invariant are covered by derivatives [5]. But little was known at the time about the full nonlinear structure of the (curvature)^4 candidate. This became much clearer recently, however, through relations between counterterms obtained via dimensional reduction, starting from field-theory limits of string-theory amplitudes [19–21] or purely within supergravity [22]. The result is that, contrary to the initial 1980s impression that the (curvature)^4 counterterm might be $E_{7(7)}$ invariant, it in fact turns out to fail this test at the nonlinear level, owing to scalar-field "dressings" of the purely gravitational (curvature)^4 term. There are just two other linearised BPS invariants in $D = 4, \mathcal{N} = 8$ supergravity, at 5 and 6-loops [18, 23, 24], but these also turn out to be incompatible with $E_{7(7)}$ invariance at the nonlinear level [22, 25]. So, none of the $D = 4$ F-term invariants can correspond to divergences, because it is now known that $E_{7(7)}$ can be preserved in the quantum perturbation theory [26]. These duality invariance requirements end up invalidating the previously-thought-acceptable BPS counterterms at 3 through 6 loops in $D = 4$ maximal supergravity [20–22, 25].

Consequently, the candidate counterterm at the 7-loop order assumes a greater importance than it was previously accorded: it is now the leading candidate for a $D = 4$ maximal supergravity divergence. So the question of its structure becomes of key importance, and in particular the question whether it can in fact be written as the full-superspace volume of the maximal theory. This is the question that we will address in the present paper.

We will prove two main results: firstly, that the volume of superspace actually vanishes on-shell for any $\mathcal{N}$, and secondly, despite this, that there are nevertheless duality-symmetric invariants of the same dimension, schematically of the form $\nabla^{2\mathcal{N}-8}R^4$. These invariants correspond to possible counterterms at the $(\mathcal{N} - 1)$-loop level.

We do not currently know any obvious *a priori* reason why the $D = 4$ superspace volumes should vanish. The proof that it does relies on harmonic superspace methods to reduce full superspace integrals to integrals over superspaces with four fewer odd coordinates. A quick way of understanding the result is to consider how one might integrate an unconstrained scalar superfield Φ over the reduced superspace using an appropriate projection operator. For example, in off-shell minimal $\mathcal{N} = 1$ supergravity the chiral projection operator is $\bar{D}^2 + S$, where $\bar{D}^2 := \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}$ and S is a chiral superfield whose leading component is the complex auxiliary scalar. So the integral of Φ over the full superspace is equal to the integral of $(\bar{D}^2 + S)\Phi$ over chiral superspace. If we take $\Phi = 1$, this integral, which is just the volume of superspace, need not vanish. On the other hand, on-shell one has $S = 0$, and so the volume vanishes on-shell. In the new-minimal formalism, however, a $U(1)$ connection is included in the covariant derivative, the superfield S vanishes and the projector is simply $\bar{D}^2$ so that the volume vanishes even off-shell [10, 11].

In $\mathcal{N}$-extended on-shell supergravity it turns out, as we shall see, that one can select one undotted and one dotted covariant spinor covariant derivative, say D^1_α and $\bar{D}_{\dot{\alpha}\mathcal{N}}$, that anticommute with each other when acting on scalar superfields. There are therefore G-analytic superfields that are, by definition, annihilated by these derivatives and which can be thought of as generalised chiral superfields. It turns out, as we shall prove later, that

$(D^1)^2$ commutes with $(\bar{D}_{\mathcal{N}})^2$ and that the associated projection operators are $(D^1)^2(\bar{D}_{\mathcal{N}})^2$. This means that we can integrate a scalar superfield Φ over a superspace with four fewer odd coordinates and that, as a corollary, the volume of the full superspace must vanish because we can write it as a sub-superspace integral of $(D^1)^2(\bar{D}_{\mathcal{N}})^2$ acting on the constant superfield integrand $\Phi = 1$.

In Section 2, we define $(\mathcal{N}, 1, 1)$ harmonic superspace for supergravity theories and show that the volume of superspace vanishes for all $\mathcal{N}$. In Section 3, we show that full superspace integrals can be reduced to integrals with respect to the $(\mathcal{N}, 1, 1)$ harmonic superspace measure. In Section 4, using this harmonic measure, we construct fully supersymmetric and duality-invariant $(\mathcal{N}-1)$-loop counterterms of general structure $\nabla^{2(\mathcal{N}-4)}R^4$. In addition, we construct nonlinear versions of the non-duality invariant $1/\mathcal{N}$-BPS supersymmetry invariants of general structure $\nabla^{2(\mathcal{N}-5)}R^4$ and clarify the classification of duality invariant $\mathcal{N}$-loop candidate counterterms. Section 5 contains our conclusions. Conventions and details about extended on-shell superspace are given in Appendix A. Appendix B contains an analysis of the kinematic structure of the derivative expansion appearing in the four-graviton amplitudes in supergravity.

2 Superspace formalism

2.1 Standard superspace

In $D = 4$, $\mathcal{N}$-extended superspace, M, is a supermanifold with 4 even and $4\mathcal{N}$ odd dimensions; local coordinates are denoted by $z^M = (x^m, \theta^{\underline{\mu}}, \bar{\theta}^{\underline{\dot{\mu}}})$ where x^m are the even, spacetime, coordinates and the thetas are the odd coordinates. The preferred basis forms are $E^A := (E^a, E^{\underline{\alpha}}, E^{\underline{\dot{\alpha}}})$ with $E^{\underline{\alpha}} = E_i^{\alpha}$, $E^{\underline{\dot{\alpha}}} = E^{\dot{\alpha}i}$. The index i runs from 1 to $\mathcal{N}$, α and $\dot{\alpha}$ are two-component spinor indices and underlined indices combine internal and spinor ones. The structure group, under which the preferred frames transform, is $SL(2, \mathbb{C}) \times U(\mathcal{N})$, with the former factor acting on the vector index a in the usual way. The connection, torsion and curvature are defined as usual with

$$\begin{aligned} T^A &= DE^A := dE^A + E^B \Omega_B{}^A \ , \\ R_A{}^B &= d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B \ . \end{aligned} \tag{2.1}$$

Because the structure group is purely even it follows that the mixed, even-odd, components of the connection one-forms, $\Omega_A{}^B$, and the curvature two-forms, $R_A{}^B$, are zero. The dimension-zero torsion does not involve the connection and takes the same form as it does in flat superspace, namely

$$\begin{aligned} T_{\alpha\beta}^{i\,j\,c} &= 0 \ , \\ T_{\alpha\dot{\beta}j}^{i\ \ c} &= -i\delta_j^i (\sigma^c)_{\alpha\dot{\beta}} \ . \end{aligned} \tag{2.2}$$

These equations, together with the conventional constraints that allow one to choose the connection and the vectorial basis E_a [27], determine the conformal constraints that were discussed in [28], to which paper we refer for further details. (We also collect some useful results in Appendix A.) The Bianchi identities corresponding to these constraints were solved in detail in [28]; we note here that the dimension one-half torsion components are zero except for

$$T_{\alpha\beta}^{ij\dot\gamma k} = \varepsilon_{\alpha\beta}\bar\chi^{\dot\gamma ijk} \tag{2.3}$$

and its complex conjugate. The leading component of the field $\chi_{\alpha ijk}$ (in which roman-index sequences like ijk are understood to be totally antisymmetric) denotes the 56 spin-one-half fields in the supergravity multiplet for $\mathcal{N} = 8$; there are additional independent spinors χ_α^{ijklm} for $\mathcal{N} = 5, 6$. The on-shell theory is completed at dimension one by specifying a number of superfields in terms of the physical component fields [29]. In addition to the geometrical fields, there are also spin-one field strengths and the scalars, the latter entering via a coset sigma model $K\backslash G$ which for $\mathcal{N} = 8$ is $(SU(8)/\mathbb{Z}_2)\backslash E_{7(7)}$.

A key point about the above equations is that they are compatible with at most one D_α and one $\bar D_{\dot\alpha}$ being in involution; indeed, one could say that the dimension-zero torsion constraints are representation-preserving [27] for fields that are annihilated by such a set of derivatives. However, we clearly cannot pick out such a pair in ordinary superspace without breaking $U(\mathcal{N})$ symmetry, and for this reason we need to enlarge the setting to harmonic superspace.

2.2 Harmonic superspace

Harmonic superspace (and the closely related projective superspace) is ordinary superspace augmented by an additional bosonic space that parametrises sets of mutually anticommuting fermionic derivatives [30–32]. In general, this space is the flag manifold

$$\mathbb{F}_{p,q}(\mathcal{N}) \cong \big(U(p)\times U(\mathcal{N}-q-p)\times U(q)\big)\backslash U(\mathcal{N}) \tag{2.4}$$

which parametrises the possible sets of p undotted and q dotted spinorial derivatives that anticommute on scalar fields [33]. In our case, we need $p = q = 1$ which gives $\mathbb{F}_{1,1}(\mathcal{N})$. One way of working on such a coset space is to consider functions on the group $K \cong U(\mathcal{N})$ that are equivariant with respect to the isotropy group $H \cong U(p)\times U(\mathcal{N}-q-p)\times U(q)$, as advocated in the work of [31, 34]. In supergravity,[2] $U(\mathcal{N})$ is a gauge group so this means that in the equivariant formalism we should work on the principal $U(\mathcal{N})$ bundle which we will call P. We denote an element of $U(\mathcal{N})$ by $u^I{}_i$ where the local gauge group acts to the right and the isotropy group acts to the left. The inverse is denoted $u^i{}_I$. We can split the I index according to the structure of the isotropy group: $I = (1, r, \mathcal{N})$, and we use u or its inverse to convert K indices to H ones. In particular, for the fermionic derivatives, we have

$$\begin{aligned} D_\alpha^I &= u^I{}_i D_\alpha^i = (D_\alpha^1, D_\alpha^r, D_\alpha^{\mathcal{N}}) \\ \bar D_{\dot\alpha I} &= u^i{}_I \bar D_{\dot\alpha i} = (\bar D_{\dot\alpha 1}, \bar D_{\dot\alpha r}, \bar D_{\dot\alpha\mathcal{N}})\ . \end{aligned} \tag{2.5}$$

[2] Other aspects of $D = 4, \mathcal{N} = 2$ supergravity, including off-shell Poincaré supergravity, have been studied in harmonic superspace, *cf.* for example [35, 36], and more recently in projective superspace [37, 38]. It has so far proved difficult to extend the off-shell Poincaré formalism to $\mathcal{N} > 2$ [39].

One can immediately see that D^1_α and $\bar{D}_{\dot{\alpha}\mathcal{N}}$ anticommute among themselves, at least as far as the torsion is concerned, owing to the antisymmetry of $\chi_{\alpha ijk}$. (The curvature terms will be discussed shortly). In addition to the superspace derivatives, we also have the group derivatives $D^I{}_J$ which are simply the right-invariant vector fields on K; they obey the Lie algebra commutation relations for $U(\mathcal{N})$ and act in a simple fashion on u,

$$D^I{}_J u^K{}_k = \delta^K_J u^I{}_k \ . \tag{2.6}$$

(There is also a trace term for $SU(8)$.) These derivatives split into those corresponding to the isotropy algebra $\mathfrak{h}$, $(D^1{}_1, D^r{}_s, D^{\mathcal{N}}{}_{\mathcal{N}})$ and the remainder corresponding to the coset directions $\mathfrak{f}$, where the Lie algebra $\mathfrak{k}$ of K splits into $\mathfrak{k} \cong \mathfrak{h} \oplus \mathfrak{f}$. Since the coset space is complex, the latter divide into two complex conjugate sets: $(D^1{}_r, D^r{}_{\mathcal{N}}, D^1{}_{\mathcal{N}})$ and $(D^r{}_1, D^{\mathcal{N}}{}_r, D^{\mathcal{N}}{}_1)$.

In the principal bundle P there is a Lie-algebra-valued one-form ω that combines the Maurer–Cartan form on the group with the $U(\mathcal{N})$ connection on the base:

$$\omega = du\, u^{-1} + u\Omega u^{-1} \ . \tag{2.7}$$

A complete set of basis forms is then given by adding to these the basis vielbein forms on the supermanifold M. The dual basis vector fields are the right-invariant vectors fields on K together with the horizontal lifts of the basis vectors on M, $\tilde{E}_A$. The latter are given by

$$\tilde{E}_A = E_A - \Omega_A{}^I{}_J D^J{}_I \ . \tag{2.8}$$

The set of vector fields $(D^1_\alpha, \bar{D}_{\dot{\alpha}\mathcal{N}}, D^r{}_1, D^{\mathcal{N}}{}_1, D^{\mathcal{N}}{}_r)$ span a CR structure in the principal bundle P, *i.e.* an involutive, complex distribution that has a null intersection with the complex conjugate set. The proof of this is given in [40]; it depends on the details of the curvature tensor. The number of odd vector fields in this set cannot be increased for $\mathcal{N} = 5, 6, 8$, although one can have $(2,1)$ structures in $\mathcal{N} = 3, 4$, and a $(2,2)$ structure in $\mathcal{N} = 4$.

Instead of working on P it will turn out to be useful for the normal coordinate discussion to work directly on harmonic superspace M_H. This is the associated fibre bundle with fibre the coset space $F \cong H\backslash K$, where F is the flag manifold, *i.e.* $\mathbb{F}_{1,1}(\mathcal{N})$, described above. To derive a convenient basis of forms on this space, one simply needs to split ω into its isotropy and coset components, $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{f}}$. The latter will be interpreted as a vertical vielbein while the former is a connection for H. The form basis is completed by the vielbein forms from the base, but we have to contract the fermionic ones with u or u^{-1} so that they are not acted on by K directly. Thus, $E^\alpha_I = E^\alpha_i u^i{}_I$ while $E^{\dot{\alpha}I} = u^I{}_i E^{\dot{\alpha}i}$. The resulting space has the structure group $SL(2,\mathbb{C}) \times H$, although one should note that there has not been a choice of $U(\mathcal{N})$ gauge. One can work out the components of the torsion from the equation

$$d\omega + \omega^2 = uRu^{-1} \ , \tag{2.9}$$

where R is the $\mathfrak{k} \cong \mathfrak{u}(\mathcal{N})$ component of the curvature, simply by decomposing it into its isotropy and coset components. We have

$$\begin{aligned} D\omega_{\mathfrak{f}} &= -(\omega_{\mathfrak{f}} \wedge \omega_{\mathfrak{f}})_{\mathfrak{f}} + (uRu^{-1})_{\mathfrak{f}} \\ d\omega_{\mathfrak{h}} + \omega_{\mathfrak{h}}^2 &= -(\omega_{\mathfrak{f}} \wedge \omega_{\mathfrak{f}})_{\mathfrak{h}} + (uRu^{-1})_{\mathfrak{h}} \ , \end{aligned} \tag{2.10}$$

where D here denotes the exterior derivative which is covariant with respect to H. In these equations, we have fixed the gauge with respect to the isotropy group acting on K so that u should here be considered as a function of local coordinates, t say, on F. It will be useful to introduce a quantity $h(I)$ such that

$$h(1) = 1 \ , \quad h(r) = 0 \ , \quad h(\mathcal{N}) = -1 \ . \tag{2.11}$$

The coset indices are then pairs I, J such that $h(I) \neq h(J)$ while the H-indices are pairs I, J with $h(I) = h(J)$. The vielbein $V^I{}_J$ on F is $(du\, u^{-1})_{\mathfrak{f}}$, and the corresponding quantity on M_H is $\tilde{V}^I{}_J$ which is given by $\omega_{\mathfrak{f}} = (du\, u^{-1} + u\Omega u^{-1})_{\mathfrak{f}}$. Thus,

$$\begin{aligned} V^I{}_J &= du^I{}_i\, u^i{}_J \\ \tilde{V}^I{}_J &= du^I{}_i\, u^i{}_J + u^I{}_i \Omega^i{}_j u^j{}_J \ , \end{aligned} \tag{2.12}$$

where in both of these expressions $h(I) \neq h(J)$. The full set of basis forms is thus $\tilde{E}^{\underline{A}} = (\tilde{V}^I{}_J, E^a, E^\alpha_I, E^{\dot{\alpha} I})$. The torsion 2-form on M_H, $\tilde{T}^{\underline{A}}$, is given by

$$\tilde{T}^a = T^a \ , \quad \tilde{T}^\alpha_I = T^\alpha_i u^i{}_I + E^\alpha_J \wedge \tilde{V}^J{}_I \ , \quad \tilde{T}^I{}_J = -\tilde{V}^I{}_K \wedge \tilde{V}^K{}_J + u^I{}_i R^i{}_j u^j{}_J \ , \tag{2.13}$$

where $h(I) \neq h(J) \neq h(K)$.

We denote the vector fields on F dual to $V^I{}_J$ by $d^I{}_J$; they are only defined for $h(I) \neq h(J)$. The complete set of vector fields dual to the basis forms consists of the $d^I{}_J$ together with the horizontal lifts of the basis vector fields of M which we shall call $\tilde{E}_A$ with the understanding that the internal indices are capitalised. The full set is denoted $\tilde{E}_{\underline{A}} = (\tilde{E}_A, d^I{}_J)$. One has

$$\tilde{E}_A = E_A - \Omega_{A,}{}^I{}_J d^J{}_I \ . \tag{2.14}$$

The combination $\Omega^I{}_J d^J{}_I$ (where the sum runs only over indices for which $h(I) \neq h(J)$) can be rewritten as $\Omega^i{}_j K^j{}_i$, where the $K^i{}_j$ are the Killing vector fields on F that generate the right action of K on the coset. The graded commutator of two basis vector fields is

$$[\tilde{E}_{\underline{A}}, \tilde{E}_{\underline{B}}] = C_{\underline{A}\,\underline{B}}{}^{\underline{C}} \tilde{E}_{\underline{C}} := \left(\tilde{\Omega}_{\underline{A},\underline{B}}{}^{\underline{C}} - (-1)^{\underline{AB}} \tilde{\Omega}_{\underline{B},\underline{A}}{}^{\underline{C}} - \tilde{T}_{\underline{A}\,\underline{B}}{}^{\underline{C}} \right) \tilde{E}_{\underline{C}} \ . \tag{2.15}$$

In particular, for two fermionic indices, for example undotted ones, we have

$$\{\tilde{E}^I_\alpha, \tilde{E}^J_\beta\} = T^{IJ\dot{\gamma}K}_{\alpha\beta} \tilde{E}_{\dot{\gamma}K} - R^{I\,J\,K}_{\alpha\beta,\ \ L} d^L{}_K + \text{connection terms} \ . \tag{2.16}$$

The term involving the curvature here is a torsion term from the point of view of harmonic superspace. Note that the connection terms refer to $SL(2,\mathbb{C}) \times H$ and so do not mix the indices $(1, r, \mathcal{N})$. This formula, together with those for mixed and undotted spinor indices, allows one to show that the subset of vector fields

$$\hat{E}_{\hat{A}} := \{\tilde{E}^1_\alpha, \tilde{E}_{\dot{\alpha}\mathcal{N}}, d^1{}_r, d^r{}_{\mathcal{N}}, d^1{}_{\mathcal{N}}\} \ , \qquad 2 \leq r \leq \mathcal{N} - 1 \tag{2.17}$$

is in involution,

$$\{\hat{E}_{\hat{A}}, \hat{E}_{\hat{B}}\} = C_{\hat{A}\hat{B}}{}^{\hat{C}}\, \hat{E}_{\hat{C}} \ , \tag{2.18}$$

and is preserved under the action of the structure group $SL(2,\mathbb{C}) \times U(1) \times U(\mathcal{N}-2) \times U(1)$. The vector fields $(d^1{}_r, d^r{}_\mathcal{N}, d^1{}_\mathcal{N})$ indeed close under commutation (they obey the commutation relations of a Heisenberg algebra) and can be thought of as being in essence the components of the anti-holomorphic Dolbeault exterior derivative $\bar{\partial}$ on the coset. It is obvious that they commute with $\tilde{E}^1_\alpha$ and $\tilde{E}_{\dot{\alpha}\mathcal{N}}$ because the relation (2.6) is also valid for the $d^I{}_J$:

$$d^I{}_J u^K{}_k = \delta^K_J u^I{}_k \qquad (\text{where } h(I) \neq h(J)) \ . \tag{2.19}$$

It is also clear that the torsion term vanishes for the commutator of any two of these odd basis vector fields owing to the total antisymmetry of $\chi_{\alpha ijk}$ in the ijk indices.

The curvature term also has the desired properties, as one can see from [28]. Setting $(\tilde{E}^i_\alpha, \tilde{E}_{\dot{\alpha}\mathcal{N}}) := \tilde{E}_{\hat{\alpha}}$, we need to show that [40]

$$R_{\hat{\alpha}\hat{\beta},}{}^1{}_\mathcal{N} = R_{\hat{\alpha}\hat{\beta},}{}^1{}_r = R_{\hat{\alpha}\hat{\beta},}{}^r{}_\mathcal{N} = 0 \ , \tag{2.20}$$

because these components of the curvature tensor couple to the derivatives $(d^r{}_1, d^\mathcal{N}{}_r, d^\mathcal{N}{}_1)$ in the commutator $\{\tilde{E}_{\hat{\alpha}}, \tilde{E}_{\hat{\beta}}\}$. It follows that this is indeed the case because

$$R^{ij}_{\alpha\beta,}{}^k{}_l = \delta^i_l N^{jk}_{\alpha\beta} + \delta^j_l N^{ik}_{\alpha\beta} \ , \tag{2.21}$$

while

$$\begin{aligned} R^i_{\alpha\dot{\beta}j,}{}^k{}_l &= -J^{ik}_{\alpha\dot{\beta},jl} + \delta^i_j H^k_{\alpha\dot{\beta}l} + \frac{1}{2}\delta^k_l H^i_{\alpha\dot{\beta}j} - \delta^k_j H^i_{\alpha\dot{\beta}l} - \delta^i_l H^k_{\alpha\dot{\beta}j} \\ &\quad - \Big(\frac{1}{2}\delta^i_j\delta^k_l + \delta^k_j\delta^i_l + \delta^i_l\delta^k_j\Big) G_{\alpha\dot{\beta}} \ , \end{aligned} \tag{2.22}$$

where the tensors G, H, J and N are given in Appendix A. (The curvature with two dotted spinor indices is the complex conjugate of the one with two undotted indices.) They are all bilinears in the fermion fields χ, $\bar{\chi}$. To see more explicitly that the curvature has the desired properties, consider first the undotted vector fields $\tilde{E}^1_\alpha, \tilde{E}^1_\beta$. In order for these to be part of the involutive set (2.17), we require

$$R^{11}_{\alpha\beta,}{}^1{}_\mathcal{N} = R^{11}_{\alpha\beta,}{}^1{}_r = R^{11}_{\alpha\beta,}{}^r{}_\mathcal{N} = 0 \ . \tag{2.23}$$

It is obvious that these conditions are satisfied owing to the presence of the Kronecker deltas in (2.21). A similar discussion is valid for the case of two dotted indices by complex conjugation. For the mixed index case, we need to show that

$$R^1_{\alpha\dot{\beta}\mathcal{N},}{}^1{}_\mathcal{N} = R^1_{\alpha\dot{\beta}\mathcal{N},}{}^1{}_r = R^1_{\alpha\dot{\beta}\mathcal{N},}{}^r{}_\mathcal{N} = 0 \ . \tag{2.24}$$

The tensor J is not a problem because it is antisymmetric on both its upper and lower indices, while the terms involving G and H cannot be non-zero, again because of the Kronecker deltas.

The explicit form of the involution equations (2.18) is therefore

$$\begin{aligned} \{\tilde{E}^1_\alpha, \tilde{E}^1_\beta\} &= 2\Omega^1_{(\alpha\beta)}{}^\gamma\, \tilde{E}^1_\gamma + 2\Omega^1_{(\alpha}{}^1{}_1\, \tilde{E}^1_{\beta)} - 2\, N^{1r}_{\alpha\beta}\, d^1{}_r \\ \{\tilde{E}_{\dot{\alpha}\mathcal{N}}, \tilde{E}_{\dot{\beta}\mathcal{N}}\} &= -2\Omega_{(\dot{\alpha}\mathcal{N}\,\dot{\beta})}{}^{\dot{\gamma}}\tilde{E}_{\dot{\gamma}\mathcal{N}} - 2\Omega_{(\dot{\alpha}\mathcal{N}}{}^\mathcal{N}{}_\mathcal{N}\, \tilde{E}_{\dot{\beta})\mathcal{N}} - 2\,\bar{N}_{\dot{\alpha}\dot{\beta}\,\mathcal{N}\,r}\, d^r{}_\mathcal{N} \\ \{\tilde{E}^1_\alpha, \tilde{E}_{\dot{\beta}\mathcal{N}}\} &= -\Omega^1_{\alpha\,\dot{\beta}}{}^{\dot{\gamma}}\, \tilde{E}_{\dot{\gamma}\mathcal{N}} + \Omega_{\dot{\beta}\mathcal{N}\,\alpha}{}^\gamma\, \tilde{E}^1_\gamma - \Omega^{1}_\alpha{}^\mathcal{N}{}_\mathcal{N}\, \tilde{E}_{\dot{\beta}\mathcal{N}} + \Omega_{\dot{\beta}\mathcal{N}}{}^1{}_1\, \tilde{E}^1_\alpha \\ &\quad -\frac{1}{6}\, C^{(1)}_{\alpha\dot{\beta}}\, d^1{}_\mathcal{N} + \frac{1}{2}\, C^{(2)1}_{\alpha\dot{\beta}\ \ r}\, d^r{}_\mathcal{N} + \frac{1}{2}\, C^{(3)r}_{\alpha\dot{\beta}\ \ \mathcal{N}}\, d^1{}_r \ , \end{aligned} \tag{2.25}$$

where $N^{ij}_{\alpha\beta}$ is given in (A.6), $\bar{N}_{\dot{\alpha}\dot{\beta}\,ij}$ is its complex conjugate and where

$$
\begin{aligned}
C^{(1)}_{\beta\dot{\alpha}} &= \begin{cases} \bar{\chi}^{rst}_{\dot{\alpha}}\chi_{\beta\,rst} & \text{for } \mathcal{N}=8 \text{ and } \mathcal{N}\le 4 \\ \bar{\chi}^{rst}_{\dot{\alpha}}\chi_{\beta\,rst} - \frac{2}{5}\chi^{rstuv}_{\beta}\bar{\chi}_{\dot{\alpha}\,rstuv} - 4\chi^{1\mathcal{N}rst}_{\beta}\bar{\chi}_{\dot{\alpha}1\mathcal{N}rst} & \text{for } \mathcal{N}=5,6 \end{cases} \\
C^{(2)1}_{\beta\dot{\alpha}\ \ r} &= \begin{cases} \bar{\chi}^{1st}_{\dot{\alpha}}\chi_{\beta\,rst} & \text{for } \mathcal{N}=8 \text{ and } \mathcal{N}\le 4 \\ \bar{\chi}^{1st}_{\dot{\alpha}}\chi_{\beta\,rst} + \frac{1}{3}\chi^{1stuv}_{\beta}\bar{\chi}_{\dot{\alpha}\,rstuv} & \text{for } \mathcal{N}=5,6 \end{cases} \\
C^{(3)r}_{\beta\dot{\alpha}\ \ \mathcal{N}} &= \begin{cases} \bar{\chi}^{rst}_{\dot{\alpha}}\chi_{\beta\,\mathcal{N}st} & \text{for } \mathcal{N}=8 \text{ and } \mathcal{N}\le 4 \\ \bar{\chi}^{rst}_{\dot{\alpha}}\chi_{\beta\,\mathcal{N}st} + \frac{1}{3}\chi^{rstuv}_{\beta}\bar{\chi}_{\dot{\alpha}\,\mathcal{N}stuv} & \text{for } \mathcal{N}=5,6\,. \end{cases}
\end{aligned}
\tag{2.26}
$$

We define a Grassmann-, or G-analytic, field on M_H to be one that is annihilated by $D_{\hat{\alpha}}$ and a harmonic-, or H-analytic, field to be one that is annihilated by $d^1{}_{\mathcal{N}}, d^1{}_r, d^r{}_{\mathcal{N}}$. Since the coset F is a complex compact manifold, it follows that H-analytic fields, which are analytic in the usual sense on F, have short harmonic expansions, and since we are on-shell, our superfields will be of this type. For G-analyticity, we note that the derivatives $D_{\hat{\alpha}}$ will contain connection terms with respect to the structure group $SL(2,\mathbb{C})\times H$, and that there may be restrictions on the representations under which they can transform. Indeed, Lorentz scalar G-analytic fields can only be charged with respect to a certain $U(1)$ subgroup of H in such a way that they carry sets of indices with the same number of upper 1 and lower $\mathcal{N}$ indices and no others. This restriction follows from the fact that the anticommutator of $D_{\hat{\alpha}}$ and $D_{\hat{\beta}}$ will involve the curvature $R_{\hat{\alpha}\hat{\beta}}$ with values in $\mathfrak{h}$. The proof that this restriction is required again requires details of the $U(\mathcal{N})$ curvature tensor. We have

$$R^{11}_{\alpha\beta,}{}^{1}{}_{1} = R^{11}_{\alpha\beta,}{}^{\mathcal{N}}{}_{\mathcal{N}} = 0\ , \tag{2.27}$$

and similarly for two dotted indices, while

$$R^{1}_{\alpha\dot{\beta}\mathcal{N},}{}^{1}{}_{1} = R^{1}_{\alpha\dot{\beta}\mathcal{N},}{}^{\mathcal{N}}{}_{\mathcal{N}} = -\frac{1}{2}H^{1}_{\alpha\dot{\beta}\mathcal{N}}\ . \tag{2.28}$$

It is the latter equation that shows the need to match the upper 1 and lower $\mathcal{N}$ indices for G-analytic fields. The tensor appearing on the right-hand side of this equation will play a key role in the following so we give it its own name,

$$B_{\alpha\dot{\beta}} := 2H^{1}_{\alpha\dot{\beta}\mathcal{N}}\ . \tag{2.29}$$

Explicitly, we have

$$B_{\alpha\dot{\beta}} = \begin{cases} \bar{\chi}^{1ij}_{\dot{\beta}}\chi_{\alpha\,\mathcal{N}ij} & \text{for } \mathcal{N}=4,5,8 \\ \bar{\chi}^{1ij}_{\dot{\beta}}\chi_{\alpha\,6ij} + \frac{1}{3}\chi^{1ijkl}_{\alpha}\bar{\chi}_{\dot{\beta}\,6ijkkl} & \text{for } \mathcal{N}=6\ , \end{cases} \tag{2.30}$$

where i,j,k,l are $(S)U(\mathcal{N})$ indices.

The field $B_{\alpha\dot{\alpha}}$ is also G-analytic, and since it also carries Lorentz indices there is an additional integrability condition that it has to satisfy, namely

$$R_{\hat{\alpha}\hat{\beta},\gamma}{}^{\epsilon}B_{\epsilon\dot{\beta}} - R_{\hat{\alpha}\hat{\beta},\dot{\beta}}{}^{\dot{\epsilon}}B_{\gamma\dot{\epsilon}} = 0\ . \tag{2.31}$$

The fact that this is true follows from the explicit forms for these curvatures,

$$R^{1}_{\alpha\dot\beta\mathcal{N},\gamma}{}^{\delta} = \frac{1}{2}\,\delta^{\delta}_{\alpha}\,B_{\gamma\dot\beta} - \frac{1}{4}\,\delta^{\delta}_{\gamma}\,B_{\alpha\dot\beta}\ , \qquad R^{1}_{\alpha\dot\beta\mathcal{N},\dot\gamma}{}^{\dot\delta} = \frac{1}{2}\,\delta^{\dot\delta}_{\dot\beta}\,B_{\alpha\dot\gamma} - \frac{1}{4}\,\delta^{\dot\delta}_{\dot\gamma}\,B_{\alpha\dot\beta}\ , \tag{2.32}$$

and

$$R^{1\,1}_{\alpha\beta,\gamma}{}^{\delta} = R^{1\,1}_{\alpha\beta,\dot\gamma}{}^{\dot\delta} = R_{\dot\alpha\mathcal{N}\dot\beta\mathcal{N},\gamma}{}^{\delta} = R_{\dot\alpha\mathcal{N}\dot\beta\mathcal{N},\dot\gamma}{}^{\dot\delta} = 0\ . \tag{2.33}$$

2.3 Normal coordinates

We are now going to evaluate on-shell the volume of $\mathcal{N}$-extended superspace. In principle, one could do this explicitly but it would be extremely tedious. Instead, we shall make use of the normal coordinate method, introduced for superspace in [41] and further developed in [42], to rewrite the volume integral as an integral over $4(\mathcal{N}-1)$ odd coordinates using the harmonic superspace formalism. Our discussion follows that of [43] where the volume of $\mathcal{N}=2$ superspace was reduced to a chiral integral by this method.

Although the conditions for the existence of normal coordinates [41]

$$\zeta^{\hat A} := \{\zeta^{\alpha} := \delta^{\alpha}_{\mu}\theta^{\mu}_{i}u^{i}{}_{1}\,, \bar\zeta^{\dot\alpha} := \delta^{\dot\alpha}_{\dot\mu}u^{\mathcal{N}}{}_{i}\,\bar\theta^{\dot\mu\,i}, z^{r}{}_{1}, z^{\mathcal{N}}{}_{r}, z^{\mathcal{N}}{}_{1}\} \tag{2.34}$$

associated to the vector fields (2.17) are satisfied, one must take into account the fact that these vector fields are only defined on the complexified tangent space, and one must therefore consider the associated normal coordinate expansion as a 'holomorphic' expansion in complex coordinates, rather than describing strictly the expansion in coordinates parametrising geodesics normal to a submanifold.

Nevertheless, the conditions assumed in [43] for the expansion in normal coordinates of the superspace vielbein Berezinian are satisfied (since the vector fields $\hat E_{\hat A}$ are in involution and span a representation of the structure group), and accordingly the harmonic superspace vielbein Berezinian $\tilde E = E \times V(t)$, where $V(t)$ is the determinant of the vielbein (2.12) over $\mathbb{F}_{1,1}(\mathcal{N})$, satisfies the flow equation

$$\zeta^{\hat A}\partial_{\hat A}\ln\tilde E = (-1)^{\underline{A}}\big(\Omega_{\underline{A}\,\hat B}{}^{\underline{A}}\zeta^{\hat B} - \zeta^{\hat B}T_{\hat B\underline{A}}{}^{\underline{A}}\big) + (-1)^{\hat M}\delta^{\hat A}_{\hat M}\big(E_{\hat A}{}^{\hat M} - \delta^{\hat M}_{\hat A}\big)\ , \tag{2.35}$$

where we have introduced the notation $\partial_{\hat A} := \partial/\partial_{\zeta^{\hat A}}$. One computes that

$$(-1)^{\underline{A}}T_{\hat B\underline{A}}{}^{\underline{A}} = 0\ . \tag{2.36}$$

Note, moreover, that the same formula applies to the flag manifold $\mathbb{F}_{1,1}(\mathcal{N})$ itself for the expansion of the vielbein determinant $V(t)$ in terms of the normal coordinates $z^{R} := (z^{r}{}_{1}, z^{\mathcal{N}}{}_{r}, z^{\mathcal{N}}{}_{1})$. Since $V(t)$ does not depend on the fermionic variables by construction, one can decompose

$$\zeta^{\hat A}\partial_{\hat A}\ln\tilde E = z^{R}\partial_{R}\ln V(t) + \zeta^{\hat A}\partial_{\hat A}\ln E\ , \tag{2.37}$$

and, removing the pure harmonic component of equation (2.35), one computes that the superspace vielbein Berezinian $E(x,\theta)$ satisfies the flow equation

$$\begin{aligned}\zeta^{\hat A}\partial_{\hat A}\ln E \,=\, & -\big(E^{1}_{\beta}{}^{M}\Omega_{M\,\alpha}{}^{\beta} + E^{1}_{\alpha}{}^{M}\Omega_{M}{}^{1}{}_{1}\big)\zeta^{\alpha} + \big(E_{\dot\beta\mathcal{N}}{}^{M}\Omega_{M\,\dot\alpha}{}^{\dot\beta} + E_{\dot\alpha\mathcal{N}}{}^{M}\Omega_{M}{}^{\mathcal{N}}{}_{\mathcal{N}}\big)\zeta^{\dot\alpha}\\ & -\delta^{\alpha}_{\mu}\big(E^{1}_{\alpha}{}^{\mu}_{1} - \delta^{\mu}_{\alpha}\big) - \delta^{\dot\alpha}_{\dot\mu}\big(E_{\dot\alpha\mathcal{N}}{}^{\dot\mu\mathcal{N}} - \delta^{\dot\mu}_{\dot\alpha}\big)\ .\end{aligned} \tag{2.38}$$

The right-hand side is left invariant by the derivatives $d^1{}_r, d^r{}_{\mathcal{N}}, d^1{}_{\mathcal{N}}$. To show this, we first note that, thanks to (2.23), (2.24) and (2.27), the normal-coordinate gauge condition

$$\left(\Omega_{\hat{\alpha}\underline{A}}{}^{\underline{B}}\right)\big|_{\zeta=0} = 0 \tag{2.39}$$

extends to arbitrary $\zeta^{\hat{A}}$ for the components

$$\Omega_{\hat{\alpha}}{}^1{}_r = \Omega_{\hat{\alpha}}{}^r{}_{\mathcal{N}} = \Omega_{\hat{\alpha}}{}^1{}_{\mathcal{N}} = 0 \ . \tag{2.40}$$

It follows that one can neglect the harmonic components in $\Omega_M{}^1{}_1$ and $\Omega_M{}^{\mathcal{N}}{}_{\mathcal{N}}$ when checking that the right-hand side of (2.38) is left invariant by $d^1{}_r, d^r{}_{\mathcal{N}}, d^1{}_{\mathcal{N}}$. We conclude that the super-vielbein Berezinian $E(x,\theta)$ does not depend on the coordinates $z^r{}_1, z^{\mathcal{N}}{}_r, z^{\mathcal{N}}{}_1$ and thus one can consider consistently its normal-coordinate expansion in terms of the Grassmann variables $\zeta^{\hat{\alpha}} := (\zeta^\alpha, \zeta^{\dot{\alpha}})$ alone, *i.e.*

$$\zeta^{\hat{\alpha}}\partial_{\hat{\alpha}} \ln E = -\Omega_{\hat{\beta}\,\hat{\alpha}}{}^{\hat{\beta}}\zeta^{\hat{\alpha}} - \delta^{\hat{\alpha}}_{\hat{\mu}}\left(E_{\hat{\alpha}}{}^{\hat{\mu}} - \delta^{\hat{\alpha}}_{\hat{\mu}}\right) \ , \tag{2.41}$$

At this point, the computation goes exactly as in [43], and one deduces that the flow equation can be rewritten as[3]

$$\begin{aligned}\zeta^{\hat{\alpha}}\partial_{\hat{\alpha}} \ln E = &\ \frac{1}{3} R_{\hat{\gamma}\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}|_{\zeta=0}\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}} + \frac{1}{45} R_{\hat{\eta}\hat{\alpha}\hat{\beta}}{}^{\hat{\rho}} R_{\hat{\rho}\hat{\gamma}\hat{\delta}}{}^{\hat{\eta}}|_{\zeta=0}\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}}\zeta^{\hat{\delta}} \\ &+ \frac{5}{12} D_{\hat{\gamma}} R_{\hat{\delta}\hat{\alpha}\hat{\beta}}{}^{\hat{\delta}}|_{\zeta=0}\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}} - \frac{3}{40} D_{\hat{\alpha}} D_{\hat{\beta}} R_{\hat{\rho}\hat{\gamma}\hat{\delta}}{}^{\hat{\rho}}|_{\zeta=0}\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}}\zeta^{\hat{\delta}} \ .\end{aligned} \tag{2.42}$$

Note that one can consider the Riemann tensor to be that of M (with appropriate harmonic projections), since those of its components that are torsion components on M_H do not contribute to this equation. The curvature components appearing in (2.42) are expressible in terms of $B_{\alpha\dot{\alpha}}$ (2.30), as one can see from (2.28), (2.32) and (2.33). The G-analyticity conditions of $B_{\alpha\dot{\beta}}$, *i.e.* $D^1_\gamma B_{\alpha\dot{\beta}} = \bar{D}_{\dot{\gamma}\mathcal{N}} B_{\alpha\dot{\beta}} = 0$, imply that the second line in (2.42) vanishes for all $\mathcal{N}$. Therefore, the flow equation takes the form

$$\zeta^{\hat{\alpha}}\partial_{\hat{\alpha}} \ln E = -\frac{1}{3} B_{\alpha\dot{\beta}}\zeta^\alpha\bar{\zeta}^{\dot{\beta}} + \frac{1}{18} B_{\alpha\dot{\beta}} B_{\alpha\dot{\alpha}}\zeta^\alpha\zeta^\beta\bar{\zeta}^{\dot{\alpha}}\bar{\zeta}^{\dot{\beta}} \ . \tag{2.43}$$

Integrating this equation, we conclude that, for all $\mathcal{N}$, the supervielbein Berezinian has the expansion

$$E(\hat{x},\zeta,\bar{\zeta}) = \mathcal{E}(\hat{x})\left(1 - \frac{1}{6} B_{\alpha\dot{\beta}}\zeta^\alpha\zeta^{\dot{\beta}}\right) \ , \tag{2.44}$$

where $\hat{x}$ stands for all the harmonic superspace coordinates aside from $\zeta^{\hat{\alpha}}$.

In the end, we are not forced to consider the expansion of the fibre determinant $V(t)$ in normal coordinates, and so we can avoid dealing with the issue of reality of the "holomorphic" expansion in the variables z^R. Moreover, the expansion of $E(\hat{x},\zeta,\bar{\zeta})$ is manifestly real with respect to the twisted anti-involution [31, 40]

$$(u^1{}_i)^* = u^i{}_{\mathcal{N}} \ , \quad (u^{\mathcal{N}}{}_i)^* = -u^i{}_1 \ , \quad (u^I{}_i)^* = u^i{}_I \ , \tag{2.45}$$

[3] The summation convention is such that for fermion bilinears one has $\phi^{\hat{\alpha}}\psi_{\hat{\alpha}} = \phi^\alpha_1\psi^1_\alpha + \bar{\phi}^{\mathcal{N}}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_{\mathcal{N}}$.

preserving G-analyticity, and so one is ensured that the integral is real.

We conclude that the superspace volume, subject to the vacuum equations of motion, vanishes for all $\mathcal{N}$:

$$\begin{aligned}\mathcal{V}_{\mathcal{N}} &= \kappa^{2(\mathcal{N}-2)} \int d^4x\, d^{4\mathcal{N}}\theta\, E(x,\theta) \\ &= \kappa^{2(\mathcal{N}-2)} \int d\mu_{(\mathcal{N},1,1)}\, d^4\zeta \left(1 - \frac{1}{6} B_{\alpha\dot{\beta}}\zeta^\alpha\zeta^{\dot{\beta}}\right) = 0 \ ,\end{aligned} \tag{2.46}$$

where κ^2 is Newton's constant (in four dimensions) and we have introduced the $1/\mathcal{N}$-BPS harmonic measure $d\mu_{(\mathcal{N},1,1)}$ defined as

$$d\mu_{(\mathcal{N},1,1)} := d^4x\, d^{4\mathcal{N}-6}t\, d^{2(\mathcal{N}-1)}\theta\, d^{2(\mathcal{N}-1)}\bar{\theta}\, \mathcal{E}(\hat{x})V(t) \ . \tag{2.47}$$

At the linearised level, this reduces to the measure discussed in [18]. In the next section we will discuss some properties of this measure.

3 Full superspace integrals

Let us now interpret formula (2.44). The normal-coordinate expansion of a generic scalar superfield Φ (not necessarily of mass dimension 0) is [43]

$$\Phi = \exp\left(\zeta^{\hat{A}} D_{\hat{A}}\right) \Phi\Big|_{\zeta=0} \ . \tag{3.1}$$

However, because Φ does not depend on the harmonic variables and because the covariant derivatives in the harmonic direction commute with the Grassmann covariant derivatives, this expansion reduces to

$$\Phi = \exp\left(\zeta^{\hat{\alpha}} D_{\hat{\alpha}}\right) \Phi\Big|_{\zeta=0} \ . \tag{3.2}$$

The expansion of the vielbein Berezinian is such that

$$\left(1 - \frac{1}{6}B_{\alpha\dot{\beta}}\zeta^\alpha\bar{\zeta}^{\dot{\beta}}\right) \exp\left(\zeta^{\hat{\alpha}} D_{\hat{\alpha}}\right) \Phi\Big|_{\zeta=0} = \frac{1}{2}\Big\{\exp\left(\zeta^\alpha D^1_\alpha\right), \exp\left(\zeta^{\dot{\alpha}} \bar{D}_{\dot{\alpha}\mathcal{N}}\right)\Big\}\Phi\Big|_{\zeta=0} \tag{3.3}$$

and so it plays the role of a normal-ordering operator. It follows that

$$\begin{aligned}\int d^4x d^{4\mathcal{N}}\theta\, E(x,\theta)\, \Phi &= \frac{1}{2}\int d\mu_{(\mathcal{N},1,1)}\, d^4\zeta\, \Big\{\exp\left(\zeta^\alpha D^1_\alpha\right), \exp\left(\zeta^{\dot{\alpha}} \bar{D}_{\dot{\alpha}\mathcal{N}}\right)\Big\}\Phi\Big|_{\zeta=0} \\ &= \frac{1}{4}\int d\mu_{(\mathcal{N},1,1)} \left((D^1)^2(\bar{D}_{\mathcal{N}})^2\, \Phi\right)\Big|_{\zeta=0} \ ,\end{aligned} \tag{3.4}$$

where $(D^1)^2 := \varepsilon^{\alpha\beta}D^1_\alpha D^1_\beta$ and $(\bar{D}_{\mathcal{N}})^2 := \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\alpha}\mathcal{N}}\bar{D}_{\dot{\beta}\mathcal{N}}$ and where we have used the commutation property

$$[(D^1)^2, (\bar{D}_{\mathcal{N}})^2] = 0 \ . \tag{3.5}$$

Therefore, the form of the Berezinian derived in the previous section implies that any full superspace integral can be rewritten as an integral over the harmonic measure (2.47).

Conversely, using this measure one can define supersymmetric invariants for any G-analytic integrand. The integrand in (3.4) is indeed G-analytic with the correct $U(1)$-charges.

We note further that this confirms the vanishing of the full superspace volume, because it can be thought of as the integral of $\Phi = 1$ over the full superspace.

In the following section, we will use the harmonic measure to construct non-vanishing supersymmetric duality invariants.

4 Invariants in extended superspace

The result that the extended superspace volumes all vanish might be considered disturbing, since one expects the existence of a duality invariant of this dimension from the linearised supersymmetry analysis [25]. Nevertheless, we shall see that such invariants do indeed exist as $1/\mathcal{N}$-BPS integrals.

4.1 $(\mathcal{N}-1)$-loop supersymmetric & duality invariants

By integrating G-analytic quartic expression in the fermions over the harmonic measures $d\mu_{(\mathcal{N},1,1)}$, we obtain a set of fully supersymmetric duality-invariant integrals

$$I^{\mathcal{N}} := \kappa^{2(\mathcal{N}-2)} \int d\mu_{(\mathcal{N},1,1)}\, B_{\alpha\dot{\beta}}\, B^{\alpha\dot{\beta}} \ . \tag{4.1}$$

One can check that the integrand of (4.1) is the unique duality-invariant G-analytic scalar superfield at this dimension for $\mathcal{N} = 4, 5, 8$. This is also the G-analytic duality-invariant scalar operator of smallest mass dimension. We will show that this reduces to the quartic invariant $\int d^{4\mathcal{N}}\theta\, (W_{ijkl}\bar{W}^{ijkl})^2 \sim (\partial^{\mathcal{N}-4} C\bar{C})^2$ in the linearised approximation.

For $\mathcal{N} = 6$ there is an additional integral

$$I^6_2 := \kappa^8 \int d\mu_{(6,1,1)}\, \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} \Big(J_{\alpha\dot{\beta}\,6i}^{\ \ \ 1i} J_{\beta\dot{\alpha}\,6j}^{\ \ \ 1j} + \frac{4}{3}\, J_{\alpha\dot{\beta}\,6j}^{\ \ \ 1i} J_{\beta\dot{\alpha}\,6i}^{\ \ \ 1j} \Big) \ , \tag{4.2}$$

which we will show to correspond to an independent combination of $\int d^{24}\theta\, (W_{ijkl}\bar{W}^{ijkl})^2$ and the additional linearised quartic invariant $\int d^{24}\theta\; W_{ijkl}\bar{W}^{klmn}W_{mnpq}\bar{W}^{pqij}$. These two invariants contribute to the two inequivalent forms of $(\partial^2 C\bar{C})^2$.

These expressions are non-vanishing, fully supersymmetric and duality-invariant candidate counterterms that could correspond to $(\mathcal{N}-1)$-loop logarithmic divergences in four-dimensional $\mathcal{N}$-extended supergravity.

Importantly, these invariants cannot be rewritten as full superspace integrals because there is no duality-invariant dimension-zero scalar superfield Φ such that the integrand of (4.1) is given by $(D^1)^2(\bar{D}_{\mathcal{N}})^2\Phi$. We will see below that such a scalar can be found at the linearised order but that it does not extend to the full theory in a duality-invariant way.

• For $\mathcal{N} = 4, 6$ and $\mathcal{N} = 8$, we evaluate the integral in (4.1) in the linearised approximation. First of all, we note that in this approximation the scalar superfield W_{ijkl} satisfies the linear constraints

$$D^p_\alpha W_{ijkl} = 2\delta^p_{[i}\chi_{\alpha\, jkl]} \ , \quad \bar{D}_{\dot{\alpha}p} W_{ijkl} = \bar{\chi}_{\dot{\alpha}\, pijkl} \ , \tag{4.3}$$

and similarly for its complex conjugate $\bar{W}^{ijkl}$. For $\mathcal{N}=8$, $\bar{W}^{ijkl}=\frac{1}{24}\varepsilon^{ijklmnpq}W_{mnpq}$. As a direct consequence, W_{ijkl} and $\bar{W}^{ijkl}$ also satisfy the quadratic constraints

$$D^1_\alpha D^1_\beta W_{ijkl}=D^1_\alpha D^1_\beta \bar{W}^{ijkl}=\bar{D}_{\dot\alpha\mathcal{N}}\bar{D}_{\dot\beta\mathcal{N}}W_{ijkl}=\bar{D}_{\dot\alpha\mathcal{N}}\bar{D}_{\dot\beta\mathcal{N}}\bar{W}^{ijkl}=0\ . \tag{4.4}$$

The components $W_{1rs\mathcal{N}}$ and their complex conjugates satisfy in particular

$$D^1_\alpha W_{1rs\mathcal{N}}=\frac{1}{2}\chi_{\alpha\,\mathcal{N}rs}\ ,\quad D^1_\alpha\bar{W}^{1rs\mathcal{N}}=0\ ,\quad \bar{D}_{\dot\alpha\mathcal{N}}W_{1rs\mathcal{N}}=0\ ,\quad \bar{D}_{\dot\alpha\mathcal{N}}\bar{W}^{1rs\mathcal{N}}=-\frac{1}{2}\bar{\chi}^{1rs}_{\dot\alpha}\ . \tag{4.5}$$

It follows, in the linearised approximation, that for $\mathcal{N}=4,5$ and 8, one has

$$(D^1)^2(\bar{D}_\mathcal{N})^2\big(W_{1rs\mathcal{N}}\bar{W}^{1rs\mathcal{N}}\big)^2=\frac{1}{4}B_{\alpha\dot\beta}B^{\alpha\dot\beta}\ . \tag{4.6}$$

The integration over the harmonic variables is done using the measure $du:=d^{4\mathcal{N}-6}t\,V(t)$ with respect to which one has the relations

$$\int du\,1=1\ ,\quad \int du\,u^i{}_1u^1{}_j=\int du\,u^i{}_\mathcal{N}u^\mathcal{N}{}_j=\frac{1}{\mathcal{N}}\delta^i_j\ ,\quad \int du\,u^i{}_1u^\mathcal{N}{}_j=0\ , \tag{4.7}$$

and

$$\begin{aligned}\int du\,u^{i_1}{}_1u^{i_2}{}_1u^1{}_{k_1}u^1{}_{k_2}u^{j_1}{}_\mathcal{N}u^{j_2}{}_\mathcal{N}u^\mathcal{N}{}_{l_1}u^\mathcal{N}{}_{l_2}&=\frac{4}{(\mathcal{N}-1)\mathcal{N}^2(\mathcal{N}+2)(\mathcal{N}+3)}\\ \times\Big((\mathcal{N}+2)\,\delta^{(i_1}_{(k_1}\delta^{i_2)}_{k_2)}\delta^{(j_1}_{(l_1}\delta^{j_2)}_{l_2)}-4\,\delta^{(i_1}_{(k_1}\delta^{i_2)}_{(l_1|}\delta^{(j_1}_{|k_2)}\delta^{j_2)}_{l_2)}&+\frac{2}{\mathcal{N}+1}\,\delta^{(i_1}_{(l_1}\delta^{i_2)}_{l_2)}\delta^{(j_1}_{(k_1}\delta^{j_2)}_{k_2)}\Big)\ .\end{aligned} \tag{4.8}$$

Using this result, we find that

$$\begin{aligned}I^\mathcal{N}&=\kappa^{2(\mathcal{N}-2)}\int d^4x\,du\,d^{4\mathcal{N}}\theta\,(W_{1rs\mathcal{N}}\bar{W}^{1rs\mathcal{N}})^2\\ &=\frac{\kappa^{2(\mathcal{N}-2)}}{(\mathcal{N}^2-1)\mathcal{N}^2}\int d^4x\,d^{4\mathcal{N}}\theta\,(I_1+2I_2+I_3)\ ,\end{aligned} \tag{4.9}$$

where

$$I_1=(W_{ijkl}\bar{W}^{ijkl})^2\ ,\quad I_2=W_{ijkm}\bar{W}^{ijkn}W_{rstn}\bar{W}^{rstm}\ ,\quad I_3=W_{ijrs}\bar{W}^{ijmn}W_{pqmn}\bar{W}^{ijpq}\ . \tag{4.10}$$

We have $I_3=I_1/6$ and $I_2=I_1/4$ for $\mathcal{N}=4\,\&\,5$, and $I_3=I_1/12$ and $I_2=I_1/8$ for $\mathcal{N}=8$. We conclude that, in the linearised approximation for $\mathcal{N}=4,5$ and 8, $I^\mathcal{N}$ evaluates to yield the full superspace integrals analysed in [8, 25]:

$$\begin{aligned}I^\mathcal{N}&=\kappa^{2(\mathcal{N}-2)}\frac{5-\delta_{\mathcal{N},8}}{3(\mathcal{N}^2-1)\mathcal{N}^2}\int d^4x\,d^{4\mathcal{N}}\theta\,(W_{ijkl}\bar{W}^{ijkl})^2\\ &\sim\kappa^{2(\mathcal{N}-2)}\int d^4x\,(\partial^{\mathcal{N}-4}(\bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}C_{\alpha\beta\gamma\delta})\partial^{\mathcal{N}-4}(\bar{C}^{\dot\alpha\dot\beta\dot\gamma\dot\delta}C^{\alpha\beta\gamma\delta})+\text{s.s.c.})\ .\end{aligned} \tag{4.11}$$

As shown in detail in Appendix B, these linearised expressions are unique. Because $\mathcal{N}$-extended supergravity admits an enhanced $SU(2,2|\mathcal{N})$ superconformal symmetry in the linearised approximation, one can use superconformal representation theory to determine the number of independent integrands defined as functions of the scalar superfields [18, 24]. A $U(\mathcal{N})$ scalar monomial in $(W\bar{W})^n$ is a superconformal primary operator of conformal weight $2n$, and zero R-charge whereas the only short such superconformal primary operators are necessarily of conformal weight 2 (or zero) [24, 44]. So it follows that any independent $U(\mathcal{N})$ scalar monomial of order four in W gives rise to a non-trivial superspace integral in the linearised approximation which is not a total derivative, and which can be shown to include $(\partial^{\mathcal{N}-4}C\bar{C})^2$ type terms.[4] To see this property explicitly in $\mathcal{N}=8$ supergravity, it is convenient to consider a formulation in $(8,4,4)$ linearised harmonic superspace. We note here that, although this harmonic superspace formulation cannot be extended to the non-linear level, it is perfectly well defined in the linear approximation [40]. Using the linear constraints on W_{ijkl}, one computes that

$$(D^1)^2(D^2)^2(D^3)^3(D^4)^2(\bar{D}_5)^2(\bar{D}_6)^2(\bar{D}_7)^2(\bar{D}_8)^2\ (W_{ijkl}\bar{W}^{ijkl})^2 \sim (\partial^2 W_{1234})^4\ , \tag{4.12}$$

because $SU(8)$ considerations imply that the result must be quartic in the $(8,4,4)$ G-analytic superfield W_{1234} ; and this expression cannot be a total derivative because $(W\bar{W})^2$ is a long primary operator. It is straightforward to check that the contractions of the derivatives are uniquely fixed by Lorentz invariance up to a total derivative. Using the property that the derivatives commute with integration over the fermionic variables, together with the fact that $(W_{1234})^4$ integrates in $(8,4,4)$ superspace to yield the linearised $(C\bar{C})^2$ invariant [18], one concludes that

$$\begin{aligned}\int d^4x\, d^{32}\theta (W_{ijkl}\bar{W}^{ijkl})^2 &\sim \int d\mu_{(8,4,4)}(\partial^2 W_{1234})^4 \\ &\sim \int d^4x\big((\partial^2 C\partial^2\bar{C})^2 + \text{s.s.c.}\big)\ ,\end{aligned} \tag{4.13}$$

which clearly coincides with the invariant exhibited in [45].

• For the $\mathcal{N}=6$ case, one must consider in addition the components W_{rstu} and their complex conjugates, which satisfy

$$D^1_\alpha W_{rstu} = 0\ ,\quad D^1_\alpha \bar{W}^{rstu} = \chi^{1rstu}_\alpha\ ,\quad \bar{D}_{\dot\alpha 6} W_{rstu} = \bar{\chi}_{\dot\alpha\, 6rstu}\ ,\quad \bar{D}_{\dot\alpha 6}\bar{W}^{rstu} = 0\ . \tag{4.14}$$

Note that W_{rstu} with $2\le r,s,t,u\le \mathcal{N}-1$ vanishes identically for $\mathcal{N}<6$, and is equal to $\frac{1}{2}\varepsilon_{rstuvw}\bar{W}^{1vw8}$ for $\mathcal{N}=8$. In $\mathcal{N}=6$, one has in the linearised approximation

$$(D^1)^2(\bar{D}_6)^2\Big(W_{1rs6}\bar{W}^{1rs6} + \frac{1}{12}W_{rstu}\bar{W}^{rstu}\Big)^2 = \frac{1}{4}B_{\alpha\dot\beta}B^{\alpha\dot\beta}\ . \tag{4.15}$$

The invariant (4.1) evaluates to give

$$\begin{aligned}I^6_1 := I^6 &= \kappa^8\int d\mu_{(6,1,1)}\Big(4W_{1rs6}\bar{W}^{1rs6} + \frac{1}{3}W_{rstu}\bar{W}^{rstu}\Big)^2 \\ &= \frac{\kappa^8}{945}\int d^4x\, d^{24}\theta\,\Big(23(W_{ijkl}\bar{W}^{ijkl})^2 + 12W_{ijkl}\bar{W}^{klpq}W_{pqmn}\bar{W}^{mnij}\Big)\ ,\end{aligned} \tag{4.16}$$

[4] The leading ∂^4F^4 term in the analogous $(W_{ij}\bar{W}^{ij})^2$ integrand in $\mathcal{N}=4$ abelian super Yang–Mills theory was evaluated explicitly in [18].

while the invariant (4.2) evaluates to yield

$$
\begin{aligned}
I_2^6 &= \kappa^8 \int d\mu_{(6,1,1)} \Bigl((W_{1rs6} \bar{W}^{1rs6})^2 + \frac{4}{3} W_{1rs6} W_{1tu6} \bar{W}^{1tr6} \bar{W}^{1su6} \Bigr)^2 \\
&= \frac{\kappa^8}{30240} \int d^4x \, d^{24}\theta \, \Bigl(23 (W_{ijkl} \bar{W}^{ijkl})^2 - 58 W_{ijkl} \bar{W}^{klpq} W_{pqmn} \bar{W}^{mnij} \Bigr) \, .
\end{aligned} \tag{4.17}
$$

These two invariants clearly define the supersymmetrisation of two different combinations of the two linearised independent $(\partial\bar{C}\partial C)^2$ structures that exist for $\mathcal{N}=6$ (see Appendix B for details). Since pure $\mathcal{N}=6$ supergravity is a strict truncation of $\mathcal{N}=8$ theory, the four-graviton amplitudes are different in these theories.

4.2 $\nabla^{2(\mathcal{N}-5)} R^4$ invariants

Using the $(\mathcal{N},1,1)$-measure, by integrating G-analytic functions of the scalar fields generalising the ones given in [18], we can construct nonlinear versions of the $1/\mathcal{N}$-BPS invariants of general structure $\nabla^{2(\mathcal{N}-5)} R^4$. These will be invariant under supersymmetry and the the corresponding R-symmetry groups K, but not under the continuous duality symmetries G as were the $\nabla^{2(\mathcal{N}-4)} R^4$ invariants of the last section.

• For $\mathcal{N}=8$, let us define the superfield $(SU(8)/\mathbb{Z}_2)\backslash E_{7(7)}$ representative in the fundamental **56** representation decomposed as $\mathbf{28} \oplus \overline{\mathbf{28}}$ of $SU(8)$

$$
\mathcal{V} := \begin{pmatrix} U_{ij}{}^{\mathcal{I}\mathcal{J}} & V_{ij\mathcal{K}\mathcal{L}} \\ \bar{V}^{kl\mathcal{I}\mathcal{J}} & \bar{U}^{kl}{}_{\mathcal{K}\mathcal{L}} \end{pmatrix} \, , \tag{4.18}
$$

where $\mathcal{I}$, $\mathcal{J}\dots$ stand for the rigid $SU(8)$ indices while the i, j, indices stand for local $SU(8)$ indices as used throughout this paper. The derivative D^i_α acts on these superfields as follows

$$
\begin{aligned}
D^k_\alpha U_{ij}{}^{\mathcal{I}\mathcal{J}} &= 2\delta^k_{[i} \chi_{\alpha\, j]pq} \bar{V}^{pq\mathcal{I}\mathcal{J}} \, , \quad & D^k_\alpha \bar{U}^{ij}{}_{\mathcal{I}\mathcal{J}} &= \frac{1}{12} \varepsilon^{ijklmnpq} \chi_{\alpha\, lmn} V_{pq\mathcal{I}\mathcal{J}} \, , \\
D^k_\alpha V_{ij\mathcal{I}\mathcal{J}} &= 2\delta^k_{[i} \chi_{\alpha\, j]pq} \bar{U}^{pq}{}_{\mathcal{I}\mathcal{J}} \, , \quad & D^k_\alpha \bar{V}^{ij\mathcal{I}\mathcal{J}} &= \frac{1}{12} \varepsilon^{ijklmnpq} \chi_{\alpha\, lmn} U_{pq}{}^{\mathcal{I}\mathcal{J}} \, ,
\end{aligned} \tag{4.19}
$$

and similarly for $\bar{D}_{\dot{\alpha} i}$ by complex conjugation. It follows that the superfields $U_{8r}{}^{\mathcal{I}\mathcal{J}}$, $V_{8r\mathcal{I}\mathcal{J}}$, $\bar{U}^{1r}{}_{\mathcal{I}\mathcal{J}}$ and $\bar{V}^{1r\mathcal{I}\mathcal{J}}$ are all G-analytic. There are *a priori* several combinations of these superfields that are of the right $U(1)$ weight and that are left invariant under the rigid $SU(8)$ symmetry, but we are going to see that they are all equivalent because of $E_{7(7)}$ identities, consistently with the property that there is a unique $SU(8)$-invariant G-analytic function of the scalar superfield in the linearised approximation. A first set of conditions arises from the fact that [46]

$$
\mathcal{V}^{-1} = \begin{pmatrix} \bar{U}^{ij}{}_{\mathcal{I}\mathcal{J}} & -V_{kl\mathcal{I}\mathcal{J}} \\ -\bar{V}^{ij\mathcal{K}\mathcal{L}} & U_{kl}{}^{\mathcal{K}\mathcal{L}} \end{pmatrix} \, . \tag{4.20}
$$

This implies that the G-analytic superfields satisfy

$$
U_{8i}{}^{\mathcal{I}\mathcal{J}} \bar{U}^{1j}{}_{\mathcal{I}\mathcal{J}} = V_{8i\mathcal{I}\mathcal{J}} \bar{V}^{1j\mathcal{I}\mathcal{J}} \, , \quad U_{8i}{}^{\mathcal{I}\mathcal{J}} V_{8j\mathcal{I}\mathcal{J}} = U_{8j}{}^{\mathcal{I}\mathcal{J}} V_{8i\mathcal{I}\mathcal{J}} \, , \quad \bar{U}^{1i}{}_{\mathcal{I}\mathcal{J}} \bar{V}^{1j\mathcal{I}\mathcal{J}} = \bar{U}^{1j}{}_{\mathcal{I}\mathcal{J}} \bar{V}^{1i\mathcal{I}\mathcal{J}} \, . \tag{4.21}
$$

Using the fact that, for any element X of the complex Lie algebra $\mathfrak{e}_7$, $\mathcal{V}^{-1}X\mathcal{V}$ is also an element of $\mathfrak{e}_7$, one deduces further identities satisfied by $U_{ij}{}^{\mathcal{I}\mathcal{J}}$ and $V_{ij\mathcal{I}\mathcal{J}}$ [46]. In particular, taking the $\mathfrak{sl}(8,\mathbb{C})\subset\mathfrak{e}_7$ element

$$X := \left(\begin{array}{cc} 2\delta^{[k}_{[i}\, u^1{}_{j]}u^{l]}{}_8 & 0 \\ 0 & -2\delta^{[i}_{[k}\, u^1{}_{l]}u^{j]}{}_8 \end{array}\right) , \tag{4.22}$$

one obtains

$$\begin{aligned} U_{8i}{}^{\mathcal{I}\mathcal{J}}\bar U^{1i}{}_{\mathcal{I}\mathcal{J}} &= V_{8i\mathcal{I}\mathcal{J}}\bar V^{1i\mathcal{I}\mathcal{J}} = 0 , \\ U_{8i}{}^{\mathcal{I}\mathcal{J}}\bar U^{1i}{}_{\mathcal{K}\mathcal{L}} + V_{8i\mathcal{K}\mathcal{L}}\bar V^{1i\mathcal{I}\mathcal{J}} &= \frac{2}{3}\delta^{[\mathcal{I}}_{[\mathcal{K}}\Big(U_{8i}{}^{\mathcal{J}]\mathcal{P}}\bar U^{1i}{}_{\mathcal{L}]\mathcal{P}} + V_{8i\mathcal{L}]\mathcal{P}}\bar V^{1i\mathcal{J}]\mathcal{P}}\Big) , \\ \bar U^{1i}{}_{\mathcal{I}\mathcal{J}}V_{8i\mathcal{K}\mathcal{L}} + \bar U^{1i}{}_{\mathcal{K}\mathcal{L}}V_{8i\mathcal{I}\mathcal{J}} &= -\frac{1}{12}\varepsilon_{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{L}\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}}U_{8i}{}^{\mathcal{M}\mathcal{N}}\bar V^{1i\mathcal{P}\mathcal{Q}} . \end{aligned} \tag{4.23}$$

Using the converse, *i.e.* the fact that for any element Y of the complex Lie algebra $\mathfrak{sl}(8,\mathbb{C})\subset\mathfrak{e}_7$, $\mathcal{V}Y\mathcal{V}^{-1}$ is an element of $\mathfrak{e}_7$, one obtains similarly

$$U_{8i}{}^{\mathcal{I}\mathcal{K}}V_{8j\mathcal{J}\mathcal{K}} + U_{8j}{}^{\mathcal{I}\mathcal{K}}V_{8i\mathcal{J}\mathcal{K}} = \frac{1}{4}\delta^{\mathcal{I}}_{\mathcal{J}}U_{8i}{}^{\mathcal{K}\mathcal{L}}V_{8j\mathcal{K}\mathcal{L}} , \tag{4.24}$$
$$U_{8s}{}^{\mathcal{I}\mathcal{K}}\bar U^{1r}{}_{\mathcal{J}\mathcal{K}} + V_{8s\mathcal{J}\mathcal{K}}\bar V^{1r\mathcal{I}\mathcal{K}} = \frac{1}{4}\delta^{\mathcal{I}}_{\mathcal{J}}V_{8s\mathcal{K}\mathcal{L}}\bar V^{1r\mathcal{K}\mathcal{L}} + \frac{1}{6}\delta^r_s\big(U_{8i}{}^{\mathcal{I}\mathcal{K}}\bar U^{1i}{}_{\mathcal{J}\mathcal{K}} + V_{8i\mathcal{J}\mathcal{K}}\bar V^{1i\mathcal{I}\mathcal{K}}\big) .$$

Using these identities, one shows that all G-analytic $SU(8)$ invariant functions of the scalar superfields are determined as functions of one single expression which reproduces the unique 1/8 BPS integrand defined in [18] in the quartic approximation, *viz.*

$$\mathcal{F}^{11}_{88}(\mathcal{V}) := u^1{}_iu^1{}_ju^k{}_8u^l{}_8\,\bar V^{im\mathcal{I}\mathcal{J}}\bar V^{jn\mathcal{K}\mathcal{L}}V_{km\mathcal{K}\mathcal{L}}V_{ln\mathcal{I}\mathcal{J}} , \tag{4.25}$$

so that

$$\int d\mu_{(8,1,1)}\,\mathcal{F}^{11}_{88}(\mathcal{V}) \sim \int d^4x\, e\Big(f^8_6(\phi)\nabla^3R^2\cdot\nabla^3R^2 + \text{s.s.c.}\Big) , \tag{4.26}$$

where ∇^kR^2 is the rank $k+4$ symmetric traceless tensor obtained by acting with k covariant derivatives on the Bel–Robinson tensor, and $f^8_6(\phi)$ is the (appropriately normalised) $SU(8)$ invariant function of the 70 scalar fields discussed in [22, 25]. This provides a nonlinear supersymmetric $SU(8)$, but not $E_{7(7)}$, invariant form for the 1/8-BPS coupling $(\nabla^3R^2)^2$ in $\mathcal{N}=8$ supergravity.

• For $\mathcal{N}=6$, we define the superfield $U(6)\backslash SO^*(12)$ representative in the vector representation $\mathbf{12}$ decomposed as the $\mathbf{6}^{(-1)}\oplus\overline{\mathbf{6}}^{(1)}$ of $U(6)$

$$\mathcal{V} := \left(\begin{array}{cc} U_i{}^{\mathcal{I}} & V_{i\mathcal{J}} \\ -\bar V^{j\mathcal{I}} & \bar U^j{}_{\mathcal{J}} \end{array}\right) , \tag{4.27}$$

similarly as for $\mathcal{N}=8$. In this case, it is enough to use the property that $\mathcal{V}$ preserves the Kähler metric

$$\mathcal{G} := \left(\begin{array}{cc} 0 & \delta^{\mathcal{J}}_{\mathcal{K}} \\ \delta^{\mathcal{L}}_{\mathcal{I}} & 0 \end{array}\right) , \tag{4.28}$$

i.e. $\mathcal{V}\mathcal{G}\mathcal{V}^T = \mathcal{G}$, to find that there is a unique G-analytic superfield of the correct $U(1)$ weight left invariant by the rigid $U(6)$, *i.e.*

$$\mathcal{F}^{11}_{66}(\mathcal{V}) := u^1{}_i u^1{}_j u^k{}_6 u^l{}_6 \, \bar{V}^{i\mathcal{I}} \bar{V}^{j\mathcal{J}} V_{k\mathcal{I}} V_{l\mathcal{J}} \ . \tag{4.29}$$

The resulting integral is of the form

$$\int d\mu_{(6,1,1)} \, \mathcal{F}^{11}_{66}(\mathcal{V}) \sim \int d^4x \, e \Big(f^6_4(\phi) \nabla R^2 \cdot \nabla R^2 + \text{s.s.c.} \Big) \ . \tag{4.30}$$

This provides a nonlinear supersymmetric $U(6)$, but not $SO^*(12)$, invariant for the 1/6-BPS coupling $(\nabla R^2)^2$ in $\mathcal{N} = 6$ supergravity.

• For $\mathcal{N} = 5$, we define the superfield $U(5)\backslash SU(5,1)$ representative in the fundamental representation $\mathbf{6}$ decomposed as the $\mathbf{1}^{(5)} \oplus \mathbf{5}^{(-1)}$ of $U(5)$

$$\mathcal{V} := \begin{pmatrix} U & V_{\mathcal{I}} \\ V^i & U^i{}_{\mathcal{I}} \end{pmatrix} \ . \tag{4.31}$$

In the same way as above, the unique G-analytic superfield of the right $U(1)$ weight that is left invariant by the rigid $U(5)$ is

$$\mathcal{F}^{11}_{55}(\mathcal{V}) := u^1{}_i u^1{}_j u^k{}_5 u^l{}_5 \, V^i V^j \bar{V}_k \bar{V}_l \ . \tag{4.32}$$

The resulting integral is of the form

$$\int d\mu_{(5,1,1)} \, \mathcal{F}^{11}_{55}(\mathcal{V}) \sim \int d^4x \, e \big(f^5_3(\phi) R^2 \cdot R^2 + \text{s.s.c.} \big) \ . \tag{4.33}$$

This provides a nonlinear supersymmetric $U(5)$, but not $U(5,1)$, invariant for the 1/5-BPS coupling R^4 in $\mathcal{N} = 5$ supergravity.

4.3 Duality-invariant full-superspace integrals

The vanishing of the superspace volume implies that the first duality-invariant full superspace integrals available as invariant candidate counterterms will start from the $\mathcal{N}$-loop order for $\mathcal{N}$-extended supergravity.

For the $\mathcal{N} = 8$ case, the candidate counterterm contributing to four-point amplitudes is the invariant discussed in [3, 8]

$$I_{(\chi\bar{\chi})^2} := \kappa^{14} \int d^4x \, d^{32}\theta \, E(x,\theta) \, \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \chi_{\alpha\, ijk} \bar{\chi}^{ijk}_{\dot{\alpha}} \chi_{\beta\, mnp} \bar{\chi}^{mnp}_{\dot{\beta}} \ . \tag{4.34}$$

It can be computed to give rise to a $(\partial^5 C\bar{C})^2$ contribution in the linearised approximation,

$$I_{(\chi\bar{\chi})^2} \sim \kappa^{14} \int d^4x \, e \big((\nabla^5 R^2)^2 + \text{s.s.c.} \big) \ . \tag{4.35}$$

At the same dimension, there are also the duality invariants

$$I_{\chi^2\bar{\chi}^2} := \kappa^{14} \int d^4x \, d^{32}\theta \, E(x,\theta) \, \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \chi_{\alpha\, ijm} \bar{\chi}^{ijn}_{\dot{\alpha}} \chi_{\beta\, pqn} \bar{\chi}^{pqm}_{\dot{\beta}} \tag{4.36}$$

and

$$I_{\chi^2 M} := \kappa^{14} \int d^4x\, d^{32}\theta\, E(x,\theta)\, \varepsilon^{\alpha\gamma}\varepsilon^{\delta\beta}\, \varepsilon^{ijklmnpq}\chi_{\alpha\, ijk}\chi_{\beta\, lmn}M_{\gamma\delta\, pq}\ , \tag{4.37}$$

where $M_{\alpha\beta\, ij}$ is the dimension-one superfield for the vector field-strengths, *viz*

$$F_{\alpha\beta,\dot\alpha\dot\beta\, ij} = -i\varepsilon_{\dot\alpha\dot\beta}\, M_{\alpha\beta\, ij} + \frac{i}{72}\varepsilon_{\alpha\beta}\, \varepsilon_{ijklmnpq}\bar\chi_{\dot\alpha}^{klm}\bar\chi_{\dot\beta}^{npq}\ . \tag{4.38}$$

Using the relation

$$\begin{aligned}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\, \varepsilon^{jklmnpqr}\, D^i_\alpha\Big(\chi_{\gamma\, ijk}\chi_{\beta\, lmn}\chi_{\delta\, pqr}\Big) &= 9\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\varepsilon^{ijklmnpq}M_{\alpha\gamma\, ij}\chi_{\beta\, klm}\chi_{\delta\, npq} \\ &\quad -90\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\chi_{\alpha\, i[jk}\chi_{\beta\, lmn]}\bar\chi_{\dot\alpha}^{i[jk}\bar\chi_{\dot\beta}^{lmn]}\end{aligned} \tag{4.39}$$

and (2.36), one shows that

$$I_{\chi^2 M} = 10\kappa^{14} \int d^4x\, d^{32}\theta\, E(x,\theta)\, \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\chi_{\alpha\, i[jk}\chi_{\beta\, lmn]}\bar\chi_{\dot\alpha}^{i[jk}\bar\chi_{\dot\beta}^{lmn]}\ , \tag{4.40}$$

because the difference is the superspace integral of a total superspace derivative. We conclude that at mass dimension 18 there are only two nonlinear supersymmetric duality invariants. These invariants are fully $E_{7(7)}$ invariant because they are constructed from a full superspace integral of the superfield entering in the superspace torsion. They are independent as can easily be seen from the inequivalent $SU(8)$ structures in (4.34) and (4.40).

Since at the linearised order there is only one kinematic structure $(\partial^5 C\bar C)^2$ contributing to the 4-point amplitude [25], one expects that the second invariant $I_{\chi^2 M}$ will only start contributing at 8-loop order from the five-point amplitude

$$I_{\chi^2 M} \sim \kappa^{14} \int d^4x\, e(\nabla^8 R^5 + \text{s.s.c.})\ . \tag{4.41}$$

This can be proved using the analysis in [25] which states that the superconformal representation theory of $SU(2,2|8)$ implies that there is only one linearised invariant of this dimension that contributes first at four points, and only one complex (two real) linearised invariant that contributes first at five points. They are the only invariants of this dimension that are left invariant by a shift of the scalar fields in the linearised approximation. However, the parity-odd linearised five-point invariant does not extend at the non-linear level to a duality-invariant full superspace integral, because the imaginary part of $I_{\chi^2 M}$ is the integral of a total derivative and thus vanishes. It is possible that there exists a duality-invariant parity-odd invariant which would be defined as the $(8,1,1)$ harmonic superspace integral of a G-analytic superfield of mass-dimension 4. We will not investigate this possibility further because such an invariant would be ruled out as a possible counterterm by the odd parity.

To understand why the invariant associated to the cubic integrand (4.37) indeed starts contributing only from five points, it is relevant to compare it to the linearised Konishi operator $W_{ijkl}\bar W^{ijkl}$. They both satisfy the quadratic constraint

$$\varepsilon^{\alpha\beta}D^i_\alpha D^j_\beta L = \varepsilon^{\dot\alpha\dot\beta}\bar D_{\dot\alpha i}\bar D_{\dot\beta j}L = 0\ , \tag{4.42}$$

in the linearised approximation [18]. Their superspace integrals therefore vanish in the linearised approximation. However, computing the G-analytic descendent of the naïve nonlinear equivalent of the Konishi operator, *i.e.* $V_{ij\mathcal{I}\mathcal{J}}\bar{V}^{ij\mathcal{I}\mathcal{J}}$, according to formula (3.4), one obtains that $(D^1)^2(\bar{D}_8)^2V_{ij\mathcal{I}\mathcal{J}}V^{ij\mathcal{I}\mathcal{J}}$ is quartic in fields in the linearised approximation, and the corresponding terms can be identified with $(D^1)^2(\bar{D}_8)^2(W\bar{W})^2$ in this approximation. We conclude therefore that the existence of the $B_{\alpha\dot{\alpha}}$ term in the normal-coordinate expansion of the supervielbein Berezinian has the effect that a superfield L satisfying the quadratic constraint (4.42) in the linearised approximation, without being a total derivative at the non-linear level, is effectively equivalent to the operator $(W\bar{W})L$ in the linearised approximation. In the case of the nonlinear integrand $M\chi^2$ in (4.37), this has the result that this integral is effectively equal to the superspace integral of $(W\bar{W})(M\chi^2+\bar{M}\bar{\chi}^2)$ in the linearised approximation, which is precisely the operator defining the (parity-even) five-point invariant discussed in [25].

5 Conclusion

In this paper we have seen, perhaps surprisingly, that the volume of four-dimensional $\mathcal{N}$-extended superspace vanishes on-shell. This means that the leading fully supersymmetric and duality invariant candidate counterterms for the first ultraviolet divergences of $\mathcal{N}\geq 4$ supergravity cannot after all be written as full superspace integrals.

On the other hand, in section 4.1 we have exhibited a fully supersymmetric and duality invariant expression for the $(\mathcal{N}-1)$-loop $\mathcal{N}$-extended supergravity counterterm of structure $\nabla^{2(\mathcal{N}-4)}R^4$ in the form of an integral over the $(\mathcal{N},1,1)$ harmonic superspace measure. This measure exists [40] at the non-linear level as opposed to the cases of harmonic measures $(\mathcal{N},p,q)$ with either $p>1$ or $q>1$ (for $\mathcal{N}\geq 5$). These invariants cannot be rewritten as full superspace integrals at the nonlinear level. For the $\mathcal{N}=8$ case, the purely gravitational component of this invariant is of the general form

$$I^8\sim\kappa^{12}\int d^4x\,e\left((\nabla^4R^2)^2+\text{s.s.c.}\right)\,. \tag{5.1}$$

It was shown in [47] that the absence of a superdiffeomorphism anomaly implies that there exists a duality-invariant form for the associated corrected action $S=S_{\text{class}}+I^8+\ldots$ in the Henneaux–Teitelboim formalism [48], which is equivalent to the existence of an action satisfying the Gaillard–Zumino constraint in the standard formulation. Duality invariance therefore poses no obstacle to the occurrence of a 7-loop logarithmic divergence, as opposed to what was claimed in [49].

There is no known requirement that the counterterm to an ultraviolet divergence be given by a full superspace integral with respect to the full on-shell supersymmetry. The situation is similar for counterterms to the ultraviolet divergences of maximal supergravity in higher dimensions, where BPS counterterms, written as subsurface integrals with respect to the full on-shell superspace (at least at the linearised level [7]), are known to occur in many cases. For example, the one-loop counterterm in eight dimensions is the R^4 invariant expressed as an on-shell half-superspace integral, the two-loop ∇^4R^4 counterterm in seven dimensions is an on-shell quarter-superspace integral, and the three-loop $\nabla^6 R^4$ counterterm in six dimensions is an on-shell eighth-superspace integral. Off-shell supersymmetry or algebraic renormalisation methods or superstring limiting methods [6, 7, 19, 20, 22, 25] can rule out certain BPS structures with respect to the full supersymmetry, but none of these methods are known to apply to the $D = 4$ seven-loop counterterm (4.1) for the $\mathcal{N} = 8$ theory, or to the same structure at corresponding loop orders for lesser $\mathcal{N}$-extended supergravities.

Nonetheless, the fact that the invariants (4.1) and (4.2) are not associated to full-superspace integrals might give one pause about their ultimate acceptability as counterterms. One can conceive of further non-renormalisation restrictions that might follow from nonstandard methods. And a full nonlinear analysis of their cocycle structure in the ectoplasm formalism has not yet been carried out.

There are possible analogues of further non-renormalisation restrictions in super Yang–Mills theories. An example concerns the absence of the three-loop double-trace divergence in six-dimensional $\mathcal{N} = 2$ super Yang–Mills theory [50]. In that case, the double-trace invariant $(\partial\, \mathrm{tr} F^2)^2$ descends from a 1/4 BPS primary operator. The cocycle structure of this invariant is moreover identical to that of the classical action, so that one does not at present have a non-renormalisation theorem for it within the framework of algebraic renormalisation. Since the 7-loop maximal supergravity divergence candidate turns out to be the superspace integral of a G-analytic superfield, it might have similar properties. Arguments using the pure spinor formalism in string theory and field theory [51–53] show the super Yang–Mills invariant to be protected beyond the two-loop order, but these arguments do not, however, carry over straightforwardly to the gravitational case.

In spacetime dimensions $D > 4$, it seems most likely that the full on-shell superspace volumes do not vanish. The volume of superspace is only pertinent for higher dimensional logarithmic divergences in the case of $\mathcal{N} = 1$ (half maximal) supergravity in 8 dimensions at one loop, and for $\mathcal{N} = 2$ (maximal) supergravity in nine dimensions at two loops. For example, the two-loop, four-graviton amplitude for maximal $D = 9$ supergravity is ultraviolet divergent with a $\nabla^8 R^4$ counterterm [54]. The duality-invariant supersymmetric counterterm of this dimension will be either the full superspace volume for $D = 9$ maximal supergravity or a partial superspace integral along the lines of Section 4.1 of this paper. If it turns out to be the superspace volume, this would not be in contradiction with the vanishing of the $D = 4$ superspace volumes that we have found, however. If a superspace volume is non-vanishing in a dimension $D > 4$, its reduction to $D = 4$ would lead to a non-duality-invariant $D = 4$ full-superspace integral of some function of the dilatonic scalars arising from the dimensional reduction, and not to one of the duality-invariant counterterms that we have constructed in Section 4.1.

For maximal supergravity in $D = 5$ the volume is not a possible counterterm. The first possible counterterms that are duality invariant and fully supersymmetric occur at the 6-loop order and are schematically of the form $\nabla^{12} R^4$. These can be expressed as full superspace integrals of dimension 4 superspace integrands constructed from the superspace tensors but with no explicit factors of the scalars.

The duality-invariance properties of a counterterm can be classified by the Laplace equation satisfied by the scalar-field prefactor of the purely gravitational part of the invariant [22, 55]. In perturbative supergravity field theory, where one requires invariance under continuous duality transformations, the scalar prefactor of a duality-invariant counterterm [19, 21, 22, 25, 56] as constructed in Section 4.1 must be an eigenfunction of the duality-invariant Laplace operator with zero eigenvalue.

In contrast, at the nonperturbative string-theory level, maximally supersymmetric string-theory considerations indicate [57] that the scalar prefactors of effective-action contributions such as the dimension-16 $\nabla^8 R^4$ operator will be sums of automorphic forms under the corresponding discrete duality group, arising from solutions to the corresponding Laplace equation with various eigenvalues. In the field-theory limit, such contributions nonetheless reduce to continuously duality-invariant expressions. For example, it was shown in [57] that the 2-loop $D = 9$ maximal supergravity divergence is contained in the zero-eigenvalue $SL(2,\mathbb{Z})$ invariant automorphic contribution to the $\nabla^8 R^4$ operator.

Of course, should duality symmetries be broken by anomalies, they cannot be used to constrain ultraviolet counterterms. This caveat applies in particular to the case of $\mathcal{N} = 4$ supergravity, where quantum corrections break the corresponding global $SU(1,1)$ symmetry, so that one can consider a full-superspace integral of any function $F(W\bar{W})$ of that theory's complex scalar field W parametrising $U(1)\backslash SU(1,1)$,

$$I_F^4 = \kappa^4 \int d^4x\, d^{16}\theta\, E(x,\theta)\, F(W\bar{W}) \;; \tag{5.2}$$

such integrals are in general non-vanishing and will contribute in the linearised approximation to couplings of the form $F^{(2)}(\phi\bar{\phi})\, R^4$ plus supersymmetric completions. So one should keep in mind that the strong limitations on the forms of ultraviolet counterterms that we have considered in this paper follow both from supersymmetry and from the requirement of continuous duality invariance where applicable.

Acknowledgments

We would like to thank Niklas Beisert, Nathan Berkovits, Henriette Elvang, Daniel Freedman, Michael Kiermaier, Emery Sokatchev and Boris Zupnik for useful discussions and comments on this work. P.V. and K.S.S. would like to thank the Kavli Institute for Theoretical Physics for hospitality during the course of this work, and for support in part by the National Science Foundation under Grant No. NSF PHY05-51164. K.S.S. would also like to thank the TEO Department of CBPF for hospitality during the course of the work, in a visit supported by a PCI-BEV grant. The work of K.S.S. was supported in part by the STFC under rolling grant ST/G000743/1. The work of G.B. was supported by the ITN programme PITN-GA-2009-237920, the ERC Advanced Grant 226371, the IFCPAR CEFIPRA programme 4104-2 and the ANR programme blanc NT09-573739.

A On-shell extended Superspace

In this appendix, we review the main properties of $\mathcal{N}$-extended superspace in four dimensions needed for the computation in the main text. We follow the conventions and notation of [28].

At the nonlinear level, the solutions to the Bianchi identities are expressed in terms of the spin 1/2 fermions χ_α^{ijk} and $\chi_{\alpha\, ijklm}$ and their complex conjugates:

$$\begin{aligned} R^i{}_j &= -\frac{1}{3}\,\bar{P}^{iklm} \wedge P_{jklm} \\ P^i_{\alpha jklm} &= 2\delta^i_{[j}\,\chi_{\alpha\, klm]}\ , \quad P_{\dot\alpha i\, jklm} = \bar\chi_{\dot\alpha\, ijklm} \\ D^i_\alpha \bar\chi_{\dot\beta\, jklmn} &= 5i\delta^i_{[j} P_{\alpha\dot\beta klmn]}\ , \quad \bar D_{\dot\alpha\, i}\chi_{\beta\, jkl} = 2iP_{\beta\dot\alpha\, ijkl} \\ D^i_\alpha \chi_\beta^{jklmn} &= M^{ijklmn}_{(\alpha\beta)} - \frac{5}{2}\varepsilon_{\alpha\beta}\,\bar\chi^{i[jk}_{\dot\alpha}\,\bar\chi^{\dot\alpha\, lmn]} \\ D^i_\alpha\chi_{\beta\, jkl} &= 3\delta^i_{[j} M_{\alpha\beta\, kl]} + \varepsilon_{\alpha\beta}\Big(\frac{2}{\mathcal{N}-4}\,\bar\chi_{\dot\alpha\, ijkmn}\bar\chi^{\dot\alpha\, lmn} \\ &\quad -\frac{3}{(\mathcal{N}-3)(\mathcal{N}-4)}\,\delta^l_{[i}\bar\chi_{\dot\alpha\, jk]mnr}\bar\chi^{\dot\alpha\, mnr}\Big)\ . \end{aligned} \tag{A.1}$$

All $i, j, \ldots$ indices are $(S)U(\mathcal{N})$ indices.

For $\mathcal{N}=8$, we have also $\bar P^{ijkl} = \frac{1}{24}\varepsilon^{ijklmnpq}P_{mnpq}$.

It was shown in [28] that the fermions χ_α^{ijk} and $\chi_{\alpha\, ijklm}$ arise from the fermionic part of the off-diagonal components of the superspace Maurer–Cartan form for the scalar potential $\mathcal{V}$ parametrising the coset space $K\backslash G$ given by $U(4)\backslash\big(SU(1,1)\times SU(4)\big) \cong U(1)\backslash SU(1,1)$ for $\mathcal{N}=4$, $U(5)\backslash SU(5,1)$ for $\mathcal{N}=5$, $U(6)\backslash SO^*(12)$ for $\mathcal{N}=6$ and $(SU(8)/\mathbb{Z}_2)\backslash E_{7(7)}$ for $\mathcal{N}=8$. For $\mathcal{N}=8$

$$d\mathcal{V}\cdot\mathcal{V}^{-1} = \begin{pmatrix} \frac{2}{3}\delta^{[i}_{[k}\Omega^{j]}{}_{l]} & P_{ijkl} \\ \bar P^{ijkl} & -\frac{2}{3}\delta^{[k}_{[i}\Omega^{l]}{}_{j]} \end{pmatrix}\ . \tag{A.2}$$

For further reference, we define the quantities

$$J_{\alpha\dot\beta kl}{}^{ij} = \bar\chi^{ijm}_{\dot\beta}\chi_{\alpha\, klm}\ , \qquad K_{\alpha\dot\beta kl}{}^{ij} = \chi_\alpha^{ijmnp}\bar\chi_{\dot\beta\, klmnp} \tag{A.3}$$

and

$$H_{\alpha\dot\beta j}{}^i = \begin{cases} \frac{1}{2}\,J_{\alpha\dot\beta jk}{}^{ik} - \frac{1}{16}\delta^i_j\,J_{\alpha\dot\beta mn}{}^{mn} & \text{for } \mathcal{N}=4,8 \\ \frac{1}{2}\,J_{\alpha\dot\beta jk}{}^{ik} - \frac{1}{16}\delta^i_j\,J_{\alpha\dot\beta mn}{}^{mn} + \frac{1}{6}\,K_{\alpha\dot\beta jk}{}^{ik} - \frac{1}{80}\delta^i_j\,K_{\alpha\dot\beta mn}{}^{mn} & \text{for } \mathcal{N}=5,6 \end{cases} \tag{A.4}$$

$$G_{\alpha\dot\beta} = \begin{cases} -\frac{1}{48}\,J_{\alpha\dot\beta mn}{}^{mn} & \text{for } \mathcal{N}=4,8 \\ -\frac{1}{48}\,J_{\alpha\dot\beta mn}{}^{mn} + \frac{7}{240}\,K_{\alpha\dot\beta mn}{}^{mn} & \text{for } \mathcal{N}=5,6 \end{cases} \tag{A.5}$$

and

$$N^{ij}_{\alpha\beta} = \begin{cases} 0 & \text{for } \mathcal{N}=4 \\ \frac{1}{3}\,\chi^{ijklm}_{(\alpha}\chi_{\beta)klm} & \text{for } \mathcal{N}=5,6 \\ -\frac{1}{72}\,\varepsilon^{ijklmnpq}\chi_{\alpha\, klm}\chi_{\beta npq} & \text{for } \mathcal{N}=8\,. \end{cases} \tag{A.6}$$

A.1 G-analyticity conditions in $\mathcal{N}=4$ superspace

We can check that $J_{\alpha\dot\beta 4j}^{\;\;1i}$ is G-analytic because $D^k_\alpha \chi^\alpha_{ijk}=0$ in [28, eq. (5.5)]:

$$D^1_\alpha J_{\alpha\dot\beta 4j}^{\;\;1i} = 0\ , \qquad \bar D_{\dot\alpha\,4} J_{\alpha\dot\beta 4j}^{\;\;1i} = 0 \tag{A.7}$$

so $B_{\alpha\dot\beta} = J_{\alpha\dot\beta 4i}^{\;\;1i}$ is G-analytic, as well as

$$C_4 = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\, J_{\alpha\dot\beta 4j}^{\;\;1i} J_{\beta\dot\alpha 4i}^{\;\;1j}\ . \tag{A.8}$$

However, in $SU(4)$ this expression for C_4 turns out to be proportional to $\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}B_{\alpha\dot\beta}B_{\beta\dot\alpha}$. Therefore for $\mathcal{N}=4$, $B_{\alpha\dot\beta}$ and all its powers are G-analytic.

A.2 G-analyticity conditions in $\mathcal{N}=5$ superspace

For $\mathcal{N}=5$, we have that

$$K_{\alpha\dot\beta 5\,j}^{\;\;1i} = -6\delta^i_5\delta^1_j\, \chi^{12345}_\alpha \bar\chi_{\dot\beta\, 12345}\ . \tag{A.9}$$

This implies that $K_{\alpha\dot\beta 5\,i}^{\;\;1i}=0$. Acting with the fermionic derivatives leads to

$$\begin{aligned} D^1_\gamma J_{\alpha\dot\beta 5j}^{\;\;1i} &= \frac{1}{6}\,\delta^i_5\delta^1_j\,\varepsilon_{\gamma\alpha}\bar\chi^{15p}_{\dot\beta}\,\bar\chi_{\dot\delta\,1pqr5}\bar\chi^{\dot\delta\,1qr} \\ D^1_\gamma K_{\alpha\dot\beta 5j}^{\;\;1i} &= \delta^i_5\delta^1_j\left(-\frac{3}{2}\varepsilon_{\gamma\alpha}\,\bar\chi^{15p}_{\dot\alpha}\bar\chi^{\dot\alpha\,qr1}\bar\chi_{\dot\beta\,1pqr5} - i6\chi^{12345}_\alpha P_{\gamma\dot\beta\,2345}\right)\ , \end{aligned} \tag{A.10}$$

with equivalent expressions for the action of $\bar D_{\dot\gamma\mathcal{N}}$. These equation imply that $D^1_\gamma J_{\alpha\dot\beta 5i}^{\;\;1i} = \bar D_{\dot\gamma\,5} J_{\alpha\dot\beta 5i}^{\;\;1i} = 0$, so $B_{\alpha\dot\beta} = J_{\alpha\dot\beta 5i}^{\;\;1i}$ is G-analytic.

Since $J_{\alpha\dot\beta 55}^{\;\;11} = K_{\alpha\dot\beta 55}^{\;\;11} = 0$, we find that $J_{\alpha\dot\beta 5j}^{\;\;1i}K_{\alpha\dot\beta 5i}^{\;\;1j} = 0$ and $K_{\alpha\dot\beta 5j}^{\;\;1i}K_{\alpha\dot\beta 5i}^{\;\;1j} = 0$, so the only term to analyse at quartic order is

$$C_5 = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\, J_{\alpha\dot\beta 5j}^{\;\;1i}\, J_{\beta\dot\alpha 5i}^{\;\;1j}\ , \tag{A.11}$$

but $C_5 \propto \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}B_{\alpha\dot\beta}B_{\beta\dot\alpha}$. Therefore for $\mathcal{N}=5$, $B_{\alpha\dot\beta}$ and all its powers are G-analytic.

A.3 G-analyticity conditions in $\mathcal{N}=6$ superspace

For $\mathcal{N}=6$, the J and K fermion bilinears are non-vanishing and are independent.

The variation of these bilinears is given by

$$\begin{aligned} D^1_\gamma J_{\alpha\dot\beta 6j}^{\;\;1i} &= \varepsilon_{\gamma\alpha}\bar\chi^{1im}_{\dot\beta}\left(\bar\chi_{\dot\delta\,6jmrs}\bar\chi^{\dot\delta\,1rs} + \frac{1}{6}\delta^1_j\bar\chi_{\dot\delta\,6mpqr}\bar\chi^{\dot\delta\,pqr}\right) \\ D^1_\gamma K_{\alpha\dot\beta 6j}^{\;\;1i} &= \left(-\frac{3}{2}\varepsilon_{\gamma\alpha}\,\bar\chi^{1[ip}_{\dot\alpha}\bar\chi^{\dot\alpha\,qr]1}\bar\chi_{\dot\beta\,jpqr6} - 5i\delta^1_j\,\chi^{1ipqr}_\alpha P_{\gamma\dot\beta\,pqr6}\right)\ , \end{aligned} \tag{A.12}$$

with equivalent equations for the action of $\bar D_{\dot\gamma 6}$.

These equations and the Fierz identity $\theta_\alpha\psi_\beta\psi^\beta = -2\theta_\beta\psi^\beta\,\psi_\alpha$ imply that

$$B_{\alpha\dot\beta} = J_{\alpha\dot\beta 6i}^{\;\;1i} + \frac{1}{3}K_{\alpha\dot\beta 6i}^{\;\;1i} \tag{A.13}$$

$$C_6 = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\left(J_{\alpha\dot\beta 6i}^{\;\;1i}J_{\beta\dot\alpha 6\,j}^{\;\;1j} + \frac{4}{3}\,J_{\alpha\dot\beta 6\,j}^{\;\;1i}J_{\beta\dot\alpha 6\,i}^{\;\;1j}\right)\ . \tag{A.14}$$

Therefore $B_{\alpha\dot\beta}$ and all its powers and C_6 are G-analytic.

A.4 G-analyticity conditions in $\mathcal{N}=8$ superspace

In $\mathcal{N}=8$, because we have the relations

$$\bar{\chi}_{\dot\alpha}^{ijklm} = \frac{1}{12}\varepsilon^{ijklmnpq}\chi_{\alpha\, npq} \; , \qquad \bar{\chi}_{\dot\alpha\, ijklm} = \frac{1}{12}\varepsilon_{ijklmnpq}\bar{\chi}_{\dot\alpha}^{npq} \; , \tag{A.15}$$

We find that the G-analyticity conditions lead to

$$\begin{aligned} D^1_\gamma J_{\alpha\dot\alpha}{}^{1i}_{8j} &= -\frac{1}{48}\,\varepsilon_{\gamma\alpha}\varepsilon^{\dot\gamma\dot\delta}\,\varepsilon_{18ikmnpq}\,\bar{\chi}_{\dot\alpha}^{1jk}\bar{\chi}_{\dot\gamma}^{1mn}\bar{\chi}_{\dot\delta}^{1pq} \; , \\ D^1_\gamma J_{\alpha\dot\alpha}{}^{1i}_{8i} &= 0 \; , \end{aligned} \tag{A.16}$$

and similarly for the complex conjugate. Therefore $B_{\alpha\dot\beta}$ and all its powers are G-analytic.

B Kinematic structure

Supersymmetry Ward identities imply that the four-graviton amplitude kinematic structure is always of the form

$$P(s,tu)\, C^{(1)}_{\alpha\beta\gamma\delta}C^{(2)\,\alpha\beta\gamma\delta}\bar{C}^{(3)}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\bar{C}^{(4)\,\dot\alpha\dot\beta\dot\gamma\dot\delta} + \text{c.c.} + \text{perms}\,(2,3,4) \tag{B.1}$$

where

$$C^{(n)}_{\alpha\beta\gamma\delta} = \sigma^{ab}_{(\alpha\beta}\sigma^{cd}_{\gamma\delta)}\, k^{(n)}_a k^{(n)}_c \epsilon^{(n)}_{bd}\,(k^{(n)}) \tag{B.2}$$

is the Weyl tensor associated to the $n^{\rm th}$ graviton of momentum $k^{(n)}$ and polarisation $\epsilon^{(n)}$, and perm $(2,3,4)$ denotes the sum over the permutations of the labels of the external particles while $s=(k^{(1)}+k^{(2)})^2$, $t=(k^{(1)}+k^{(4)})^2$ and $u=(k^{(1)}+k^{(3)})^2$ since we are working with the signature $(+---)$. For the contribution of order $\nabla^{2k}R^4$, $P_k(s,tu)$ is a polynomial of degree k in s,t,u:

$$P_k(s,tu) = \sum_{i=0}^{\lfloor k/2\rfloor} c^i_k\, s^{k-2i}\,(tu)^i \; . \tag{B.3}$$

One sees immediately that there are $\lfloor k/2\rfloor+1$ independent monomials at each order.

In the case of $\mathcal{N}=8$ supergravity, $C_{\alpha\beta\gamma\delta}$ and $\bar{C}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}$ occur in the same linearised supersymmetry multiplet, and the supersymmetry Ward identities therefore imply that the dependence on the polarisations factorises the four-graviton amplitude such that $P_k(s,tu)$ is a symmetric function in s,t,u. $P(s,tu)$ is then expressed as a polynomial in the invariants $\sigma_2 = s^2+t^2+u^2 = 2(s^2-tu)$ and $\sigma_3 = s^3+t^3+u^3 = 3stu$ as shown in [58]. The kinematic structure $\nabla^{2k}R^4$ has degeneracy $\lfloor(k+2)/2\rfloor - \lfloor(k+2)/3\rfloor$, and is unique for $k=0$, $2\le k\le 5$ and $k=7$.

References

[1] S. Deser, J. H. Kay and K. S. Stelle, "Renormalizability properties of supergravity," Phys. Rev. Lett. **38** (1977) 527.

[2] S. Deser and J. H. Kay, "Three loop counterterms for extended supergravity," Phys. Lett. B **76** (1978) 400.

[3] R. E. Kallosh, "Counterterms in extended supergravities," Phys. Lett. B **99** (1981) 122.

[4] P. S. Howe, K. S. Stelle and P. K. Townsend, "Superactions," Nucl. Phys. B **191** (1981) 445.

[5] P. S. Howe, K. S. Stelle and P. K. Townsend, "Miraculous ultraviolet cancellations in supersymmetry made manifest," Nucl. Phys. B **236** (1984) 125.

[6] P. S. Howe and K. S. Stelle, "Supersymmetry counterterms revisited," Phys. Lett. B **554** (2003) 190 [arXiv: `hep-th/0211279`].

[7] G. Bossard, P. S. Howe and K. S. Stelle, "The ultra-violet question in maximally supersymmetric field theories," Gen. Rel. Grav. **41** (2009) 919, [arXiv: `0901.4661 [hep-th]`].

[8] P. S. Howe and U. Lindstrom, "Higher order invariants in extended supergravity," Nucl. Phys. B **181** (1981) 487.

[9] J. Wess and B. Zumino, "Superfield Lagrangian for supergravity," Phys. Lett. B **74** (1978) 51.

[10] P. S. Howe, K. S. Stelle and P. K. Townsend, "The vanishing volume of $\mathcal{N}=1$ superspace," Phys. Lett. B **107** (1981) 420.

[11] M. Sohnius and P. C. West, "The tensor calculus and matter coupling of the alternative minimal auxiliary field formulation of $\mathcal{N}=1$ supergravity," Nucl. Phys. B **198** (1982) 493.

[12] E. Sokatchev, "A superspace action for $\mathcal{N}=2$ supergravity," Phys. Lett. B **100** (1981) 466.

[13] M. T. Grisaru, "Two loop renormalizability of supergravity," Phys. Lett. B **66** (1977) 75.

[14] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Amplitudes and ultraviolet behavior of $\mathcal{N}=8$ supergravity," [arXiv: `1103.1848 [hep-th]`].

[15] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, "Three-loop superfiniteness of $\mathcal{N}=8$ supergravity," Phys. Rev. Lett. **98** (2007) 161303, [arXiv: `hep-th/0702112`].

[16] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Manifest ultraviolet behavior for the three-loop four-point amplitude of $\mathcal{N}=8$ supergravity," Phys. Rev. D **78**, 105019 (2008) [arXiv: `0808.4112 [hep-th]`].

[17] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "The ultraviolet behavior of $\mathcal{N}=8$ supergravity at four loops," Phys. Rev. Lett. **103** (2009) 081301, [arXiv: `0905.2326 [hep-th]`].

[18] J. M. Drummond, P. J. Heslop, P. S. Howe and S. F. Kerstan, "Integral invariants in $\mathcal{N}=4$ SYM and the effective action for coincident D-branes," JHEP **0308** (2003) 016 [arXiv: `hep-th/0305202`].

[19] H. Elvang and M. Kiermaier, "Stringy KLT relations, global symmetries, and $E_{7(7)}$ violation," JHEP **1010** (2010) 108 [arXiv: `1007.4813 [hep-th]`].

[20] M. B. Green, J. G. Russo and P. Vanhove, "String theory dualities and supergravity divergences," JHEP **1006** (2010) 075 [arXiv: `1002.3805 [hep-th]`].

[21] M. B. Green, S. D. Miller, J. G. Russo and P. Vanhove, "Eisenstein series for higher-rank groups and string theory amplitudes," [arXiv: `1004.0163 [hep-th]`].

[22] G. Bossard, P. S. Howe and K. S. Stelle, "On duality symmetries of supergravity invariants,"

JHEP **1101** (2011) 020 [arXiv: `1009.0743 [hep-th]`].

[23] H. Elvang, D. Z. Freedman and M. Kiermaier, "A simple approach to counterterms in $\mathcal{N} = 8$ supergravity," JHEP **1011** (2010) 016 [arXiv: `1003.5018 [hep-th]`].

[24] J. M. Drummond, P. J. Heslop and P. S. Howe, "A note on $\mathcal{N} = 8$ counterterms," [arXiv: `1008.4939 [hep-th]`].

[25] N. Beisert, H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales and S. Stieberger, "$E_{7(7)}$ constraints on counterterms in $\mathcal{N} = 8$ supergravity," Phys. Lett. B **694** (2010) 265 [arXiv: `1009.1643 [hep-th]`].

[26] G. Bossard, C. Hillmann and H. Nicolai, "$E_{7(7)}$ symmetry in perturbatively quantised $\mathcal{N} = 8$ supergravity," JHEP **1012** (2010) 052 [arXiv: `1007.5472 [hep-th]`].

[27] S. J. J. Gates, K. S. Stelle and P. C. West, "Algebraic origins of superspace constraints in supergravity," Nucl. Phys. B **169** (1980) 347.

[28] P. S. Howe, "Supergravity in superspace," Nucl. Phys. B **199** (1982) 309.

[29] L. Brink and P. S. Howe, "The $\mathcal{N} = 8$ supergravity in superspace," Phys. Lett. B **88** (1979) 268.

[30] A. A. Rosly, "Super Yang–Mills constraints as integrability conditions," in *Proceedings of the International Seminar on Group Theoretical Methods in Physics*," (Zvenigorod, USSR, 1982), M. A. Markov (Ed.), Nauka, Moscow, 1983, Vol. 1, p. 263 (in Russian); English translation: in *Group Theoretical Methods in Physics*," M. A. Markov, V. I. Man'ko and A. E. Shabad (Eds.), Harwood Academic Publishers, London, Vol. 3, 1987, p. 587

[31] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, "Unconstrained $\mathcal{N} = 2$ matter, Yang–Mills and supergravity theories in harmonic superspace," Class. Quant. Grav. **1** (1984) 469.

[32] A. Karlhede, U. Lindstrom and M. Rocek, "Selfinteracting tensor multiplets In $\mathcal{N} = 2$ superspace," Phys. Lett. B **147** (1984) 297.

[33] P. S. Howe and G. G. Hartwell, "A superspace survey," Class. Quant. Grav. **12** (1995) 1823.

[34] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, "Harmonic superspace," (*Cambridge, UK: University Press (2001) 306 p*).

[35] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. Sokatchev, "$\mathcal{N} = 2$ supergravity in superspace: different versions and matter couplings," Class. Quant. Grav. **4** (1987) 1255.

[36] A. S. Galperin, N. A. Ky and E. Sokatchev, "$\mathcal{N} = 2$ supergravity in superspace: solution to the constraints," Class. Quant. Grav. **4** (1987) 1235.

[37] S. M. Kuzenko, U. Lindstrom, M. Rocek and G. Tartaglino-Mazzucchelli, "4D $\mathcal{N} = 2$ supergravity and projective superspace," JHEP **0809** (2008) 051 [arXiv: `0805.4683 [hep-th]`].

[38] S. M. Kuzenko, U. Lindstrom, M. Rocek and G. Tartaglino-Mazzucchelli, "On conformal supergravity and projective superspace," JHEP **0908** (2009) 023 [arXiv: `0905.0063 [hep-th]`].

[39] E. Sokatchev and B. Zupnik, unpublished.

[40] G. G. Hartwell and P. S. Howe, "(N, p, q) harmonic superspace," Int. J. Mod. Phys. A **10** (1995) 3901 [arXiv: `hep-th/9412147`].

[41] I. N. McArthur, "Superspace normal coordinates," Class. Quant. Grav. **1**, 233 (1984).

[42] S. J. Gates, M. T. Grisaru, M. E. Knutt-Wehlau and W. Siegel, "Component actions from curved superspace: Normal coordinates and ectoplasm," Phys. Lett. B **421** (1998) 203, [arXiv: `hep-th/9711151`].

[43] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, "Different representations for the action principle in $4D$ $\mathcal{N}=2$ supergravity," JHEP **0904** (2009) 007 [arXiv: `0812.3464 [hep-th]`].

[44] V. K. Dobrev and V. B. Petkova, "All positive energy unitary irreducible representations of extended conformal supersymmetry," Phys. Lett. B **162** (1985) 127.

[45] D. Z. Freedman and E. Tonni, "The $D^{2k}R^4$ invariants of $\mathcal{N}=8$ supergravity," JHEP **1104** (2011) 006 [arXiv: `1101.1672 [hep-th]`].

[46] B. de Wit and H. Nicolai, "$\mathcal{N}=8$ supergravity," Nucl. Phys. B **208** (1982) 323.

[47] G. Bossard and H. Nicolai, "Counterterms vs. dualities," [arXiv: `1105.1273 [hep-th]`].

[48] M. Henneaux and C. Teitelboim, "Dynamics of chiral (selfdual) p-forms," Phys. Lett. B **206** (1988) 650.

[49] R. Kallosh, "$E_{7(7)}$ symmetry and finiteness of $\mathcal{N}=8$ supergravity," [arXiv: `1103.4115 [hep-th]`].

[50] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "The complete four-loop four-point amplitude in $\mathcal{N}=4$ Super-Yang–Mills theory," Phys. Rev. D **82** (2010) 125040 [arXiv: `1008.3327 [hep-th]`].

[51] N. Berkovits, M. B. Green, J. G. Russo and P. Vanhove, "Non-renormalization conditions for four-gluon scattering in supersymmetric string and field theory," JHEP **0911** (2009) 063 [arXiv: `0908.1923 [hep-th]`].

[52] J. Bjornsson and M. B. Green, "5 loops in 24/5 dimensions," JHEP **1008** (2010) 132 [arXiv: `1004.2692 [hep-th]`].

[53] J. Bjornsson, "Multi-loop amplitudes in maximally supersymmetric pure spinor field theory," JHEP **1101** (2011) 002 [arXiv: `1009.5906 [hep-th]`].

[54] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, "On the relationship between Yang–Mills theory and gravity and its implication for ultraviolet divergences," Nucl. Phys. B **530** (1998) 401 [arXiv: `hep-th/9802162`].

[55] M. B. Green and P. Vanhove, "Duality and higher derivative terms in M theory," JHEP **0601**, 093 (2006) [arXiv: `hep-th/0510027`].

[56] M. B. Green, J. G. Russo and P. Vanhove, "Automorphic properties of low energy string amplitudes in various dimensions," Phys. Rev. D **81** (2010) 086008 [arXiv: `1001.2535 [hep-th]`].

[57] M. B. Green, J. G. Russo and P. Vanhove, "Modular properties of two-loop maximal supergravity and connections with string theory," JHEP **0807** (2008) 126 [arXiv: `0807.0389 [hep-th]`].

[58] M. B. Green and P. Vanhove, "The low energy expansion of the one-loop type II superstring amplitude," Phys. Rev. D **61** (2000) 104011 [arXiv: `hep-th/9910056`].

NOTAS DE FÍSICA é uma pré-publicação de trabalho original em Física.
Pedido de cópias desta publicação deve ser enviado aos autores ou ao:

Centro Brasileiro de Pesquisas Físicas
Área de Publicações
Rua Dr. Xavier Sigaud, 150 – 4º andar
22290-180 – Rio de Janeiro, RJ
Brasil
E-mail: socorro@cbpf.br/valeria@cbpf.br
http://www.biblioteca.cbpf.br/index_2.html

NOTAS DE FÍSICA is a preprint of original unpublished works in Physics.
Request for copies of this report should be addressed to:

Centro Brasileiro de Pesquisas Físicas
Área de Publicações
Rua Dr. Xavier Sigaud, 150 – 4º andar
22290-180 – Rio de Janeiro, RJ
Brazil
E-mail: socorro@cbpf.br/valeria@cbpf.br
http://www.biblioteca.cbpf.br/index_2.html