RANDOM MAGNETISM

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The "ingredients" which control a phase transition in well defined systems as well as in random ones (e.g. random magnetic systems) are listed and discussed within a somehow unifying perspective. Among these "ingredients" we find the couplings and elements responsible for the cooperative phenomenon, the topological connectivity as well as possible topological incompatibilities, the influence of new degrees of freedom, the order parameter dimensionality, the ground state degeneracy and finally the "quanticity" of the system. The general trends, though illustrated in magnetic systems, essentially hold for all phase transitions, and give a basis for connection of this area with Field theory, Theory of dynamical systems, etc.

I - INTRODUCTION

Since the early 70's a very rapidly increasing (theoretical as well as experimental) effort has been dedicated to study partially defined systems (glasses, polymers, random magnets, etc) in the sense that the *definition of the system* itself involves random aspects (e.g. the nature of the elements of the systems, their interactions, their mean positions in condensed matter, etc). In particular, in random magnetism (spin-glasses, bond- or site- mixed or dilute magnetic systems, etc) converge several problematics, related to the kind of long range magnetic order (commensurate or not), to the cross-over between different universality classes, to the underlying geometrical structure in what concerns its topology (regular lattices, Bethe's trees, cacti, partially removed loop structures, amorphous systems, etc) or its occupancy (independent or correlated site and/or bond percolation), to the method of preparation (annealed or quenched) of the system, etc. By now only a few of these points are clearly understood, some of them are only partially enlightened by preliminar theoretical and/or experimental work (therefore quite a number of controversies exist on these grounds), and finally a great number of aspects are still to be attacked. Within the theoretical approach a new non trivial difficulty must be handled: configurational averages besides the traditional thermal ones.

In the present work we have no intention of exhaustive review of the area of random magnetism, not even of all of its outstanding features(we shall indicate here and there the appropriate References): we shall restrict ourselves to list down the main "ingredients" which control phase transitions in well defined systems (Section II) and how they emerge in random systems (Section III). Though the general trends we intend to

point out pratically hold for any phase transition, we have voluntarily chosen most of our illustrations related to magnetic systems. Let us also stress that the list (certainly incomplete) has been constructed on simplicity or familiarity (for searchers in the area) grounds rather than within a strict logical classification; therefore some of the "ingredients" are closely related among them, and could of course have been listed together.

For clearness reasons let us introduce some precision in the current words we shall use. A *critical phenomenon* is characterized by the fact that *small variations of the causes lead to big variations of the effects*(for a mathematical formulation of this concept see R. Thom's Catastrophe Theory; for example Ref. (1)). The critical phenomenon will be called *cooperative* (or *non cooperative*) according to whether it is (or it is not) driven by the mutual enhancement of a particular tendency of the elements of the system (the fall of a ruler at the edge of a desk can clearly be considered as a non cooperative critical phenomenon).Each element might "know" about the others through energetic couplings (in the Hamiltonian or Lagrangian), through quantum effects such as symmetrization-antisymmetrization of the wave functions, through particular bounds, etc. Among the cooperative critical phenomena we find the *phase transitions* and the *regime changes*. The former connect two or more phases (collective states* of thermodynamical equilibrium) of a macroscopic** system (e.g. ferromagnetism or gravitational condensation of a gas of stars to form a galaxy); the latter connect two or more regimes (collective states of non equilibrium) of a (not necessarily macroscopic) system (e.g., the trigger of the laser effect, the laminar or turbulent water flow through a tap).

The theoretical study of a phase transition typically (but not exclusively) leads to the analysis of the thermodynamic free energy $F=U-TS$ (U,T and S being respectively the internal energy, the temperature and the entropy). In the limit $T \to 0$, U dominates and the cooperation (hence the order) is favoured; in the limit $T \to \infty$, S dominates and the cooperation (hence the order) is depressed. Anything which "helps" U enhances the phase transition, i.e. the critical temperature T_c increases (more generally the less disordered phase *expands* in the intensive parameter space where appears the phase diagram) and/or the eventual long range order becomes stronger and/or several correlation functions decay more slowly with distance. We shall next list the various "ingredients" which enter on these grounds.

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* Two phases might be different even if there is no breakdown of the symmetry (i.e. both are invariant with respect to the same group of symmetry), in other words if both have the same type of *macroscopic* order or disorder (see for example Refs.(2-5)).

** Only in the thermodynamical limit (N→∞) appear the various singularities which characterize a phase transition.

II - WELL DEFINED SYSTEMS

II.1 - Cooperative couplings

Obviously the phase transition is enhanced if the coefficients of the couplings responsible for the cooperation become stronger. Ferromagnetism exhibits this fact as typically $T_c \propto J$, where J is the exchange integral which couples neighboring spins (see for example Ref.(6.a)).Another example might be superconductivity where, in the limit $u\rho_F \to 0$ of the BCS approximation, it is (see, for example, Ref. (7)) $T_c \sim 1.14 T_D e^{-1/u\rho_F}$ (T_D, u and ρ_F are respectively the Debye temperature, the electron-lattice interaction coefficient and the density of electronic states at the Fermi level), hence T_c monotonically increases with u. Finally in gauge theories on lattices where a gauge invariant interaction appears characterized by a coupling constant λ (see, for example, Ref. (8)), the correlation function $< \exp i \oslash \phi_j >$ is assymptotically proportional to e^{-P} if λ is big enough and to e^{-A} if λ small enough ($<\cdots>$ denotes the canonical mean value, ϕ_j an angle associated to the j-th site, $\oslash$ a sum over a closed path (Wilson loop), and P and A respectively the perimeter and the area of the Wilson loop). In other words, the less disordered phase corresponds to big values of λ.

II.2 - Cooperative elements

The phase transition will be depressed if the concentration of cooperating elements decreases. This is self-evident and clearly related to the previous Subsection, as if the cooperation comes from let us say a two-body interaction, anything which weakens either the two elements or their interaction will obviously depress the critical phenomenon. To illustrate this let us imagine a regular linear chain where exist first- and second- neighbour ferromagnetic interactions. If we call M(N) a certain magnetic (non magnetic) atom, we will clearly have lower magnetic disorder in a crystal ... MMM ... than in a crystal ...MNMNM...

II.3 - Topological connectivity

If the *connectivity* between the cooperating elements of the system is increased, the stabilizing "messages" which a given element receives from the others will also increase, therefore the phase transition will be enhanced. This fact explains why for *any* particular model (e.g. the Ising model) T_c monotonically increases with the space dimensionality d. By the way, this is the reason why the Mean Field Approximation (MFA), which *underestimates* the fluctuations, systematically *overestimates* T_c; furthermore it is not surprising that at sufficiently high dimensionality the MFA becomes exact, as there the high connectivity makes the fluctuations to become neglectable indeed. From Ref.(9.a)we can obtain an illustration of the preceding statements, namely for the simplest ferromagnetic Ising model (n=1) we have that T_c/T_c^{MFA} (where T_c and T_c^{MFA} are respectively the actual and the MFA critical temperatures) equals 0 for d=1 (linear chain), 0.57 for d=2 (square lattice), 0.75 for d=3 (simple cubic lattice)and

1 for d → ∞ (simple hypercubic lattice).

A second illustration (still for n=1) of the influence of the connectivity might be taken from Ref. (10) where, for different two-dimensional lattices, we have that $k_B T_c/qJ$ (where q and J are respectively the coordination number and the exchange integral) equals 0.51 for the honeycomb (q=3), 0.57 for the square lattice (q=4) and 0.61 for the triangular lattice (q=6).

A last illustration can be taken from the range of the cooperative interactions. If we have a ferromagnetic exchange integral J_{ij} which decays with the distance R_{ij} (between the sites i and j) as $e^{-R_{ij}/\Lambda}$ (or as $1/R_{ij}^x$) then T_c must be a monotonically increasing (or decreasing) function of Λ (or of x); furthermore the MFA becomes exact in the limit $\Lambda \to \infty$ (or x → 0) (see, for example, Ref. (11)).

II.4 - Topological incompatibility

There exist situations where topological constraints prevent, even at vanishing temperature, the full and simultaneous "satisfaction" of every single few-body microscopic interaction. In such cases, on one hand the total internal energy is not as low as it should be without those topological constraints, on the other hand the entropy tends to be favoured by a high degeneracy of the ground state. As a consequence of these facts, the eventual long range order that might appear is not a simple one (like ferromagnetism or antiferromagnetism), it tends to be incommensurate* with the underlying (crystalline) structure (as it is frequently the case of helimagnetism; see,for example, Ref. (6.b)), and its stability is quite vulnerable.

Such situation appears in the triangular lattice with first-neighbour antiferromagnetic Ising interactions. This case is a fully frustrated[13] one in the sense that if we consider an elementary triangular plaquette, there is no up-down configuration of the three spins (at the vertex of the triangle) which *simultaneously* satisfies the three bonds.

Another illustration of the influence of a topological incompatibility is given by the R-S model[14]. This magnetic model essentially consists in a set of parallel layers: within a layer the spins are strongly coupled through a ferromagnetic interaction, and between first- and second- neighbouring layers exist exchange interactions respectively characterized by $J_1 > 0$ and $J_2 \gtrless 0$. At low temperatures the system presents, for sufficiently high (low) values of $\alpha \equiv J_2/J_1$, a ferromagnetic (helical) long range order. At high temperatures the system is paramagnetic for any value of α. The three phases coexist on a very special point, the *Lifshitz point* (noted L on the plane p=1 of Fig. 1); another very interesting point (noted M on the same figure) is the *multiphase point*[15], which separates at vanishing temperature the two ordered phases. Now, all the interesting features of this model come

* Through this fact appear analogies with the Theory of Fractals (see, for example, Ref. (12)).

from the topological incompatibility which appears for $\alpha < 0$, namely if we consider three consecutive layers of the system (each of them being ferromagnetically ordered), there is no up-down configuration of the spins which can *simultaneously* satisfy both first- and second- neighbour interactions. For such a system T_c should monotonically increase with α.

Fig. 1 - Possible critical surface of a site dilute R-S model: its "outside" corresponds to the paramagnetic phase (P); the "right" ("left") part (with respect to the first order transition surface LMN) of its "inside" corresponds to the ferromagnetic phase (F) (helimagnetic phases (H)); LN is a Lifshitz line and MN a multiphase line; the T=0 line connecting (P) with (F) and (H) is oversimplified in the figure as it overlooks the consequences of the fact that the $\alpha = 0$ percolation cannot be via 2nd neighbouring layers.
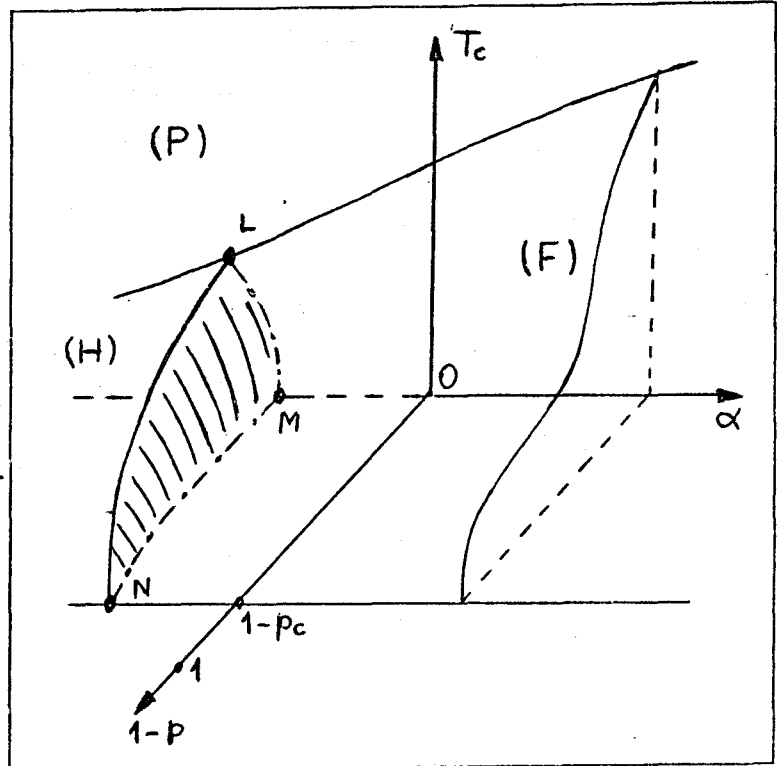
II.5 - New degrees of freedom

A new degree of freedom, besides those responsible for the phase transition, presents two faces: on one side it "helps" U because it introduces more possibilities for exploiting the cooperating couplings, but on the other side it "helps" S because the number of fluctuating variables is increased. So a new degree of freedom might be "useful" to the phase transition if it "helps" more U than S, or "destructive" in the opposite case. Let us illustrate the latter situation. An assembly of localized non interacting (electronic) spins leads, in the presence of an external magnetic field, to the *Brillouin paramagnetism*, characterized by a divergence, at vanishing temperature, of the isothermal susceptibility χ_T (see Fig. 2). If we release the electronic translational degree of freedom, the overlap of the wave functions will increase and, through Pauli principle, the spins will be prevented to be parallel, therefore the function $\chi_T(T)$ will be depressed and the singularity will desappear (*Pauli paramagnetism*). Of course there is no phase transition in the present case, but if there was the depressing effect would be the same.

Fig. 2 - The isothermal magnetic susceptibility for vanishing external field for an ideal assembly of localized (Brillouin paramagnetism) or free (Pauli paramagnetism) electrons as a function of temperature.
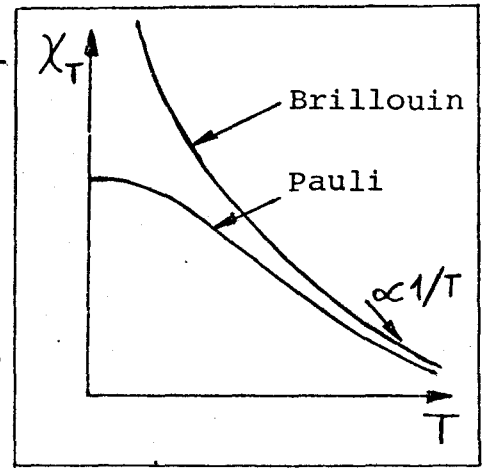
Let us now give an example of an "useful" degree of freedom. In KH_2PO_4 (KDP)-type ferroelectricity the protons move in double-wells and their "left-right" ordering (isomorphic to the up-down configurations of Ising pseudo-spins[16]) leads to the phase transition. The tunneling through the central barrier of each double-well creates a dynamics which can be seen as pseudo-magnons (see Fig. 3). When the critical temperature T_c is approached by above a soft mode appears (in a quasi harmonic framework) for vanishing wave vector: when its frequency vanishes, the crystal becomes structurally unstable and the ferroelectric phase appears. Now it happens in the real system that, through dipolar interactions, the pseudo-magnons are bilinearly coupled to a new degree of freedom, namely a particular type of collective oscillations (transverse optic phonons) of the heavy ions (see Fig. 3). The diagonalization of the Hamiltonian which describes the whole system leads[17] to two new types of elementary excitations, which might be called quasi-phonons and quasi-pseudo-magnons. The interesting feature is that the instability is now related to a new soft mode whose frequency is slightly lower than before, hence it will vanish "earlier" (at higher temperature) than before. Therefore we see that the essential effect of the new degree of freedom is to increase T_c.

Fig. 3 - The frequencies (as functions of the wave vector q) of the elementary excitations before (pseudo-magnons (m) and phonons (p)) and after (quasi-pseudo-magnons (qm) and quasi-phonons (qp)) taking into account the bilinear interaction between protons and heavy ions. The empty (full) dot marks the original (final) soft mode; q_B denotes the frontier of the first Brillouin zone.
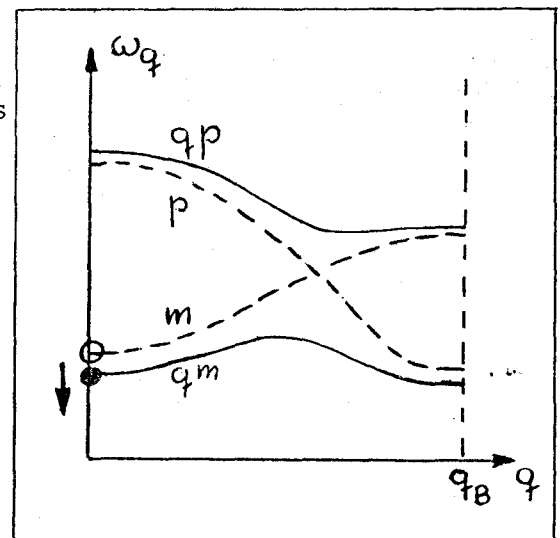
II.6 - Ground state degeneracy

The entropy effects at low temperatures are clearly determined by the degeneracy of the collective fundamental level of the system, and a strong degeneracy will provoke a weak phase transition. The dimensionality n of the order parameter (or of the field in Field theory) provides a direct manner for modifying the ground state degeneracy.

T_c is a monotonically *decreasing* function of n, as n gives the number of fluctuating variables per element of the system; in other words, if n increases, the probability for a given element to be in the "right" direction   (the one that has been favoured by the eventual breakdown of the symmetry) decreases. See Table I for d=3 and Fig. 4 for d=2. Within this context it is worth while to recall that Onsager's   Reaction Field Approximation (RFA) always *underestimates* T_c because it *overestimates* fluctuations: this is why RFA becomes exact in the limit n $\rightarrow$ ∞ (spherical model).

Model	$\mathcal{H}/J$	n	order parameter	$\dfrac{T_c(n)}{T_c(1)}$
Ising	$\displaystyle\sum_{i,j} S_i^z S_j^z$	1	scalar	1
XY	$\displaystyle\sum_{i,j} (S_i^x S_j^x + S_i^y S_j^y)$	2	complex number	0.99
Heisenberg	$\displaystyle\sum_{i,j} \vec{S}_i \cdot \vec{S}_j$	3	vector	0.97
Spherical	$\displaystyle\sum_{i,j} \sum_{\alpha=1}^{\infty} S_i^\alpha S_j^\alpha$	∞	tensor	0.91

Table I - Various three-dimensional models ($\mathcal{H}$ and J being respectively the Hamiltonian and the first-neighbour ferromagnetic exchange integral) and their approximative critical temperatures (Ref. (9.b)) for the fcc lattice.

Besides n there are many other ways of increasing the ground state degeneracy.   For example the q-state Potts model[18] attributes to each site of a given lattice a variable σ_i (for the i-th site) which might take the values $\sigma_i = i, 2, \ldots, q$, the Hamiltonian being, for example,

$$\mathcal{H} = -J \sum_{i,j} \delta_{\sigma_i, \sigma_j}$$

where J is a first-neighbour positive coupling constant and $\delta_{\sigma_i, \sigma_j}$ is the Kronecker delta (q=1 leads to bond percolation and q=2 to the $\frac{1}{2}$ - spin Ising model). The elementary two-body interaction of the present model has a ground state whose degeneracy clearly equals q. We verify[19] that T_c monotonically decreases with q.

II.7 - "Quanticity"

The present "ingredient" is directly related to the previous one: more a system is classical, the bigger is the number of states within a given energy interval, therefore we can say that in a loose sense, its degeneracy increases (remember that in Classical Statistical Mechanics the Third Principle of Thermodynamics is violated because $\lim_{T \to 0} S(T) = -\infty$), hence the phase transition is depressed. A trivial illustration of this fact can be the Einstein condensation of a three-dimensional ideal gas of bosons of mass m and concentration ρ : the critical temperature is given (see, for example, Ref. (20)) by

$$T_c = 0.084 \ h^2 \ \rho^{2/3}/m \ k_B$$
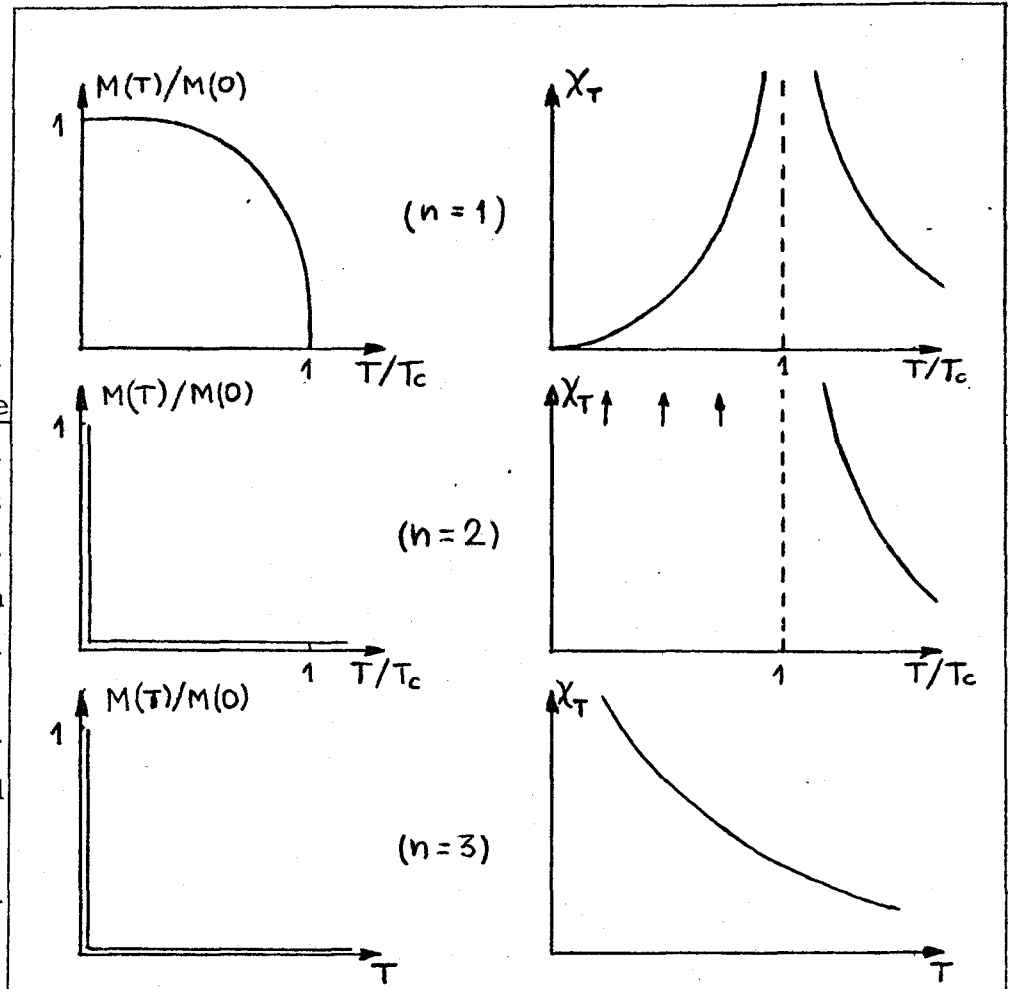
hence the Planck constant $h \to 0$ leads to $T_c \to 0$.

Fig. 4 - Spontaneous magnetization and isothermal susceptibility for two-dimensional ferromagnetic Ising (n=1), XY (n=2) and Heisenberg (n=3) models. For n=2 the susceptibility diverges for $T < T_c$ and the correlation function decays for $T < T_c$ ($T > T_c$) as a power (exponential) law with distance (see Refs. (2-5)).

Another illustration can be given by the MFA results for simple ferromagnetism associated to spins with length S: the spontaneous magnetization (as it is presented in Fig. 5; see, for example, Ref. (6.c)) monotonically decreases with S.

III - RANDOM SYSTEMS

III.1 - Introduction

Until now we have discussed well defined systems, particularly pure magnetism, where the space dimensionality d, the order parameter dimensionality n, the spin size S, among others, are perfectly defined, and where the site concentration p_s and bond concentration p_b equal unity. The statistical treatment of such systems involves only

thermal averages. We intend now to discuss random systems, and more precisely random magnetism, where one or more of the "ingredients" mentioned in Section II become random variables, so that the system is partially undefined (typically $p_s < 1$ and/or $p_b < 1$). Its statistical treatment obliges us to simultaneously deal with *thermal and configurational* averages: a further difficulty appears. There is however a particular case, the *pure percolation*, where the situation is greatly simplified because it corresponds to the limit of vanishing temperature,* therefore there is no energetic undefinition, and the statistics reduces to configurational averages only.

The *systems of interest* can be classified into two main categories: the *insulating* (where no
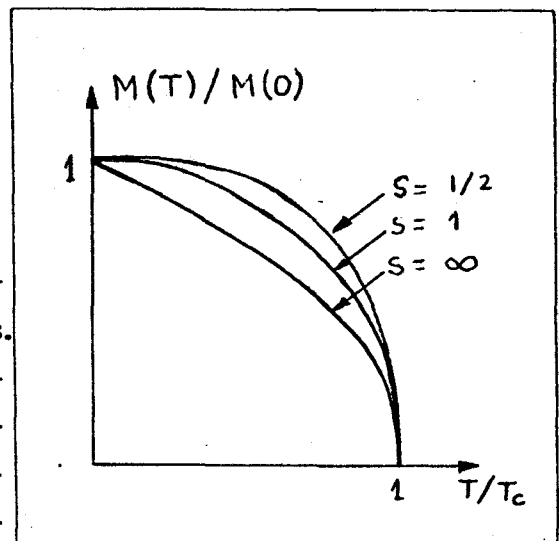

Fig. 5 - MFA spontaneous magnetization as a function of temperature. The spin length S → ∞ corresponds to the classical limit.

electronic translation is allowed) and the *conducting* (where the electronic translation is released, appearing therefore phenomena like hopping, valence fluctuation , etc) magnetic systems. We shall be concerned here with the former, where the critical aspects appear in a more pure manner.

The *regions of interest* in the external parameters space of dimensionality D (e.g. , T-p space, where p denotes a concentration) can be classified as follows (see Fig.6) :

a)   the *critical frontier* (or phase diagram) of dimensionality (D-1) (e.g., a line in the T-p space), i.e. *where* the various singularities appear; in the T-p space the limits p → 1 (pure magnetism) and T → 0 (pure percolation) are typically less difficult to discuss; the point p=1 (or T=0) itself might be approached by let us say high and low temperature (or concentration) series;

b)   the *neighbourhood* of the critical frontier, in other words *how* and *what* singularities appear; this is where the *critical exponents* come in, and also eventual *crossover* from one critical set of exponents to another;

c)   the *rest*, which from a certain standpoint is less interesting, as there are no singularities (with possible exceptions at T=0 and/or p=1); this is a very hard region in what concerns exact results, excepting the point (p,T) = (1,0) (where there are no averages at all to be done) and the limits T → ∞ and p → 0 which are usually trivial.

Finally the physical *quantities of interest* can be classified   into   two   (related)

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* There are situations[21] where the pure uncorrelated percolation appears in   the limit T → ∞ .

groups: *macroscopic* and *microscopic* quantities. Among the former we find the *equili-brium* thermodynamical quantities (extensive-type functions of intensive-type varia-bles, e.g., equations of states, specific heat, isothermal susceptibility, compress-ibility, piezomagnetic and magnetostrictive coefficients, etc) and the *non-equilibrium* thermodynamical quantities (transport properties, such as viscosity, heat and electric conductivities, etc). Among the microscopic quantities we find *static* ones (e.g., spin correlation functions such as $< S_i^z \ S_j^z >$ which might be proportional to $\exp - \frac{R_{ij}}{\xi}$, where R_{ij} is the distance between the i-th and j-th sites and ξ the correlation length which typically diverges at the critical frontier; or eventually the space Fourier transform $S(\vec{k})$ of the correlation function, where $\vec{k}$ is a wave vector) as well as *dynamic* quantities, which might generalize the static ones (e.g., the correlation function $< S_i^z(0) \ S_j^z(t) >$ or its space and time Fourier transform $S(\vec{k},\omega)$ where ω is a frequence ; in this function appear the central peak related to solitons , etc). Let us next see how the various "ingredients" we mentioned in Section II can become random variables, and their consequences.

III.2 - Cooperative couplings

On these grounds, the typical discussion concerns the so called *bond problem*. Let us consider, as an example, the following first-neighbour Ising Hamiltonian:

$$\mathcal{H} = - \sum_{i,j} J_{ij} \ S_i^z \ S_j^z$$

where J_{ij} is an independent random variable whose distribution law is given by

$$P (J_{ij}) = (1-p) \ \delta \ (J_{ij}-J_1) + p\delta \ (J_{ij}-J_1) \qquad \forall(i,j) \qquad (J_1 \neq J_2)$$

As the law is the same for each bond, the present problem is a *quenched* one. The particular case $J_1=0$ is called the *bond-dilute* problem; the general case $J_1 \neq J_2$ is called the *bond-mixed* problem. If J_1 and J_2 are both of the same sign the problem is normally less complicate than when they have different signs (as here spin-glass orders might appear).

The simple dilute problem has received attention from several authors (Refs. (22-28) among others). The critical frontier for the case $J_2 > 0$ is indicated in Fig. 6. In the pure bond percolation point (A in the Fig.) the slope is infinite because the Ising model has no dynamics, therefore in the magnons spectrum appears a gap, hence in the limit of low temperatures one should assymptotically expect $e^{-cte/T}c \propto p - p_c$. In the pure Ising limit $p \to 1$ (point C of of the Fig.) one should expect a linear be-haviour because it is essentially a one hole (or absent bond) problem; however there is no reason[(29)] for expect-ing the frontier to be analytic in the point p=1, because the next assymptotic correction concerns a two-hole problem; and the two holes "see" each other critically through the correlation length ξ , i.e. we expect
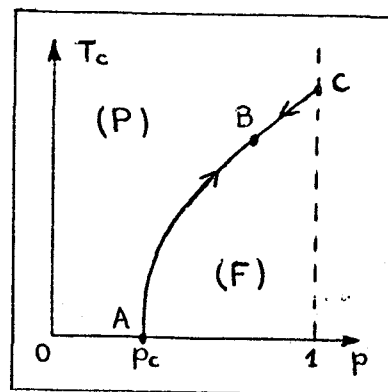
Fig.6 - Critical tempera-ture as a function of bond concentration of the ferro-magnetic Ising model (d>1); P (F) denotes the paramagnet ic (ferromagnetic) phase. A and C respectively corres pond to the pure percolat-ion and pure Ising fixed points; B corresponds to the possible random fixed point; the arrows denote the possible flow senses.

in the limit p → 1,

$$T_c - T_c(1) \sim \text{cte}(1-p) + \text{cte}'(1-p)^{2+x} \qquad (d \geqslant 1)$$

with x eventually different from zero. Another inter-
esting aspect concerns the set of critical exponents,
and now we shall restrict ourselves to d=2. For $p = p_c$
there is a percolation critical set which is differ-
ent from the pure Ising critical set (p=1). What
about $p_c < p < 1$? It is commonly believed that the
eventual *random* critical set is different from the
percolation one (in Renormalization Group (RG) terms,
the pure percolation fixed point is expected to be
unstable, with a crossover to another fixed point).
Is it the same as the pure Ising one? (in RG terms,
is the pure Ising fixed point a stable one?). The
Harris criterium[22] says "yes" if the specific heat
critical exponent $\alpha_t < 0$, says "no" if $\alpha_t > 0$, and
gives no answer in the marginal case $\alpha_t = 0$, (which
is precisely the d=2 case !). Nevertheless the spec-
ific heat logarithmic divergence of the pure Ising
model can be represented by $\alpha_t = + 0$, therefore
suggesting that the unknown correct answer could be
"no", i.e., a *third* critical set could exist (with a
"random" fixed point, like point B in Fig. 6), valid
for the whole critical line, excepting its two ends;
if so one could expect $\alpha_r < 0$ (r stands for "random"),
because it is rather intuitive that the weak loga-
rithmic singularity characterized by $\alpha_t = + 0$ should
be smeared under dilution. More precisely, a possi-
ble answer could be[30] that presented in Fig. 7. We
see that the divergence might become a cusp, which
should of course disappear at $p = p_c$. Furthermore a
Schottky-type anomaly, due to the ensemble of finite
clusters which coexist with the infinite one for
$p > p_c$, could be present. The total area is expected
to decrease with decreasing p. Let us finally remark
that the smearing effect makes experiences as well
as Monte Carlo (or other computer) treatments quite
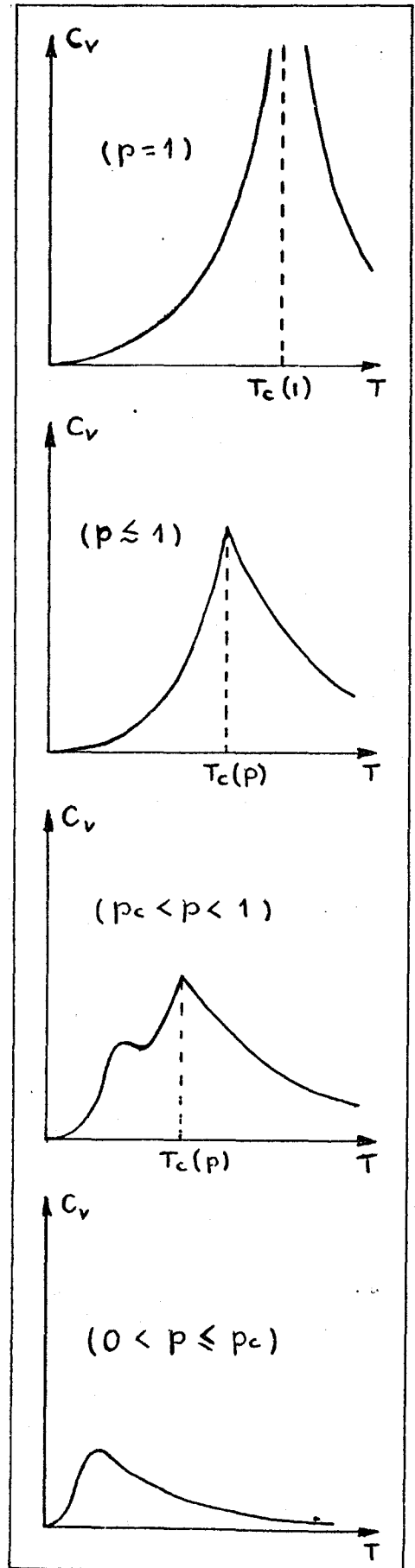difficult in the neighbourhood of p_c.

Fig. 7 - Possible specific heat
vs. temperature for two-dimen-
sional bond-dilute Ising models
(p denotes the bond concentra-
tion).

In what concerns the mixed problem[31,32] for the case $0 < J_1 < J_2$ (with $\alpha \equiv J_1/J_2$), typical results are presented in Fig. 8 (the different sign mixed problem will be mentioned in Subsection III.5). An experimental realization of the bond-mixed model can be obtained in systems such as[33] $Co(S_pSe_{1-p})_2$ where the Co atoms interact between them essentially via superexchange through the S atoms or the Se atoms (and this of course simulates J_1 and J_2).

III.3 - Cooperative elements

The *site problem* is very analogous to the bond one, the dilution concerning the cooperative elements of the system instead of their interactions. A typical problem is the first-neighbour ferromagnetic *site-dilute* Ising model; the Hamiltonian and the independent occupancy probability law are respectively given by

$$\mathcal{H} = - J \sum_{i,j} c_i c_j \, S_i^z S_j^z \qquad (J > 0)$$

and $P(c_i) = (1-p)\delta(c_i) + p\delta(c_i-1) \qquad \forall i$

Because of the independent occupancy the problem is said to be *quenched*. In Fig. 9 we compare site and bond critical frontiers for the *same* lattice and interactions. We verify that
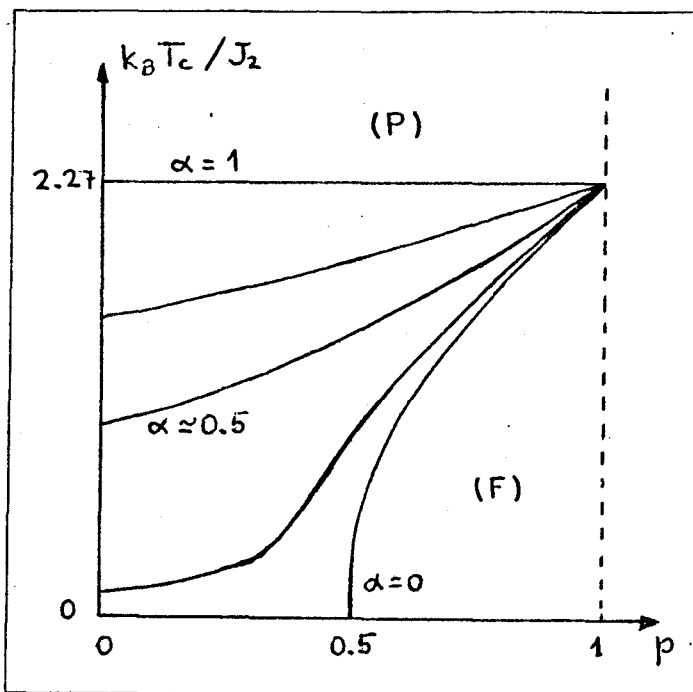

Fig. 8 - Critical line[32] separating, for different values of $\alpha \equiv J_1/J_2$, the ferromagnetic (F) from the paramagnetic (P) phase in the square lattice first-neighbour $\frac{1}{2}$ - spin ferromagnetic bond-mixed Ising model.

T_c(site) $\leq T_c$(bond) for all concentrations. It is easy to understand why; let us think on the square lattice: in order to make a perfect covering we must associate two bonds to each site. If one bond is absent we loose one cooperative interaction, if one site is absent we loose four cooperative interactions, i.e. *twice* the topological proportion, hence site dilution deteriorates the cooperative phenomenon more than bond dilution. The theoretical treatment of the site problem is more complicate than that of the bond problem: the absence of a bond does not perturbate the contributions of the neighbouring bonds, whereas the impact of a site absence depends on the absence or presence of the neighbouring sites. On the other hand, the site problem is easier than the bond one in what concerns the experimental realizations, and various systems have been studied: $Mn_pZn_{1-p}F_2$[36], $Rb_2Mn_pMg_{1-p}F_4$[37,38], $Fe_pMg_{1-p}Cl_2$[39], $Rb_2Co_pMg_{1-p}F_4$[40], $K_2Mn_pFe_{1-p}F_4$[41], $Fe_pCo_{1-p}Cl_2\cdot2H_2O$[42], $Fe_pCo_{1-p}Cl_2$[43], $Cd_{1-p}Mn_pTe$[44], among others. As an illustration we present in Fig. 10 the EPR re-

sults[44] obtained with a single crystal of Cd_{1-p} $Mn_p Te$ (antiferromagnet with the zinc blende struc-ture). Furthermore, in the same work[44], the exponent x defined by $\Delta H \propto |T-T_c(p)|^{-x}$ (where ΔH is the EPR line width) has been measured; the results, for (p,x), are $(0.25, 1.00)$, $(0.40, 0.96)$, $(0.50, 0.91)$ and $(0.60, 0.93)$, clearly exhibiting universality along the critical line T_c vs. p.

The site-dilute problem might be generalized into the site-mixed one, where atoms A (B) might independently occupy a given site with probability p $(1-p)$. Therefore the exchange integral J_{AA} will appear with probability p^2, J_{BB} with $(1-p)^2$ and J_{AB} with $2p(1-p)$. We see that this problem implies 3 different coupling constants, instead of only 2 for its bond analogous.

Another possible generalization might be the inclusion of further-neighbour interactions[45,46]: if we respectively call $J_1 > 0$ and $J_2 > 0$ the first-and second-neighbour interactions in a given lattice, we have represented in Fig. 11 a typical phase diagram parametrized by $\alpha \equiv J_2/J_1$. It is easy to understand, through topological connectivity arguments, why $T_c(p)$ monotonically increases when α varies from 0 to 1.

Let us finally mention another generalization : the uncorrelated quenched site-bond-dilute problem. The Hamiltonian and probability laws are respectively given by

$$\mathcal{H} = -\sum_{i,j} J_{ij}\, c_i c_j\, s_i^z s_j^z \qquad ,$$

$$P(J_{ij}) = (1-p^B)\delta(J_{ij}) + p^B\, \delta(J_{ij}-J) \quad \forall(i,j) \qquad (J > 0),$$

$$Q(c_i) = (1-p^S)\delta(c_i) + p^S\, \delta(c_i-1) \qquad \forall\, i.$$

Fig. 9 - Typical critical frontiers for site-and bond-dilute ferromagnetic Ising models. It is always (see, for example, Refs. (34,35)) $p_c^B \leq p_c^S$ (the equality holds for structures with no loops, like Bethe trees).

Fig. 10 - Critical temperature vs. concentration obtained[44] through EPR measurement on $Cd_{1-p}Mn_p Te$ (the arrow points the percolation threshold).

The expected critical surface is indicated in Fig. 12. If M (N) is a magnetic (non magnetic) atom, and V (U) is (is not) an atom which allows for superexchange interaction of the M atoms between them (which otherwise do not interact), an experimental realization of the above model can be imagined with a substance of the type
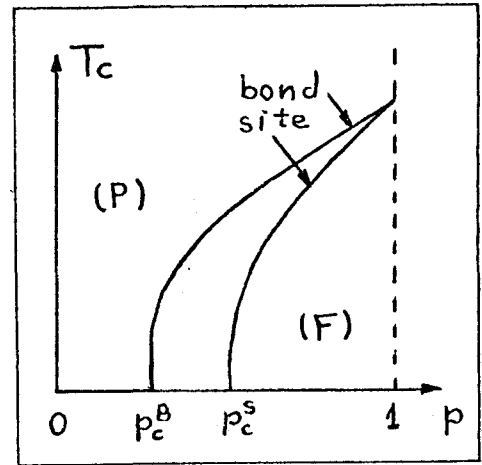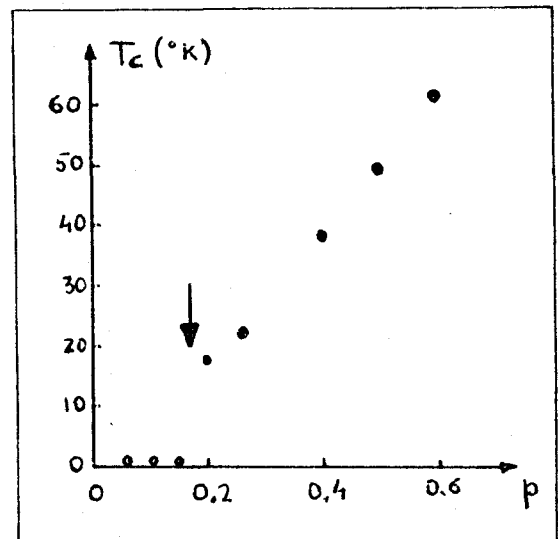
$$M_p s \ N_{1-p} s \ V_p B \ U_{1-p} B \ .$$

III.4 - Topological connectivity

In order to illustrate the effects of random
ness on the connectivity of the system, let
us imagine a layered Ising ferromagnet ($J > 0$
and $I > 0$ respectively being the intra-layer
and inter-layer exchange integrals). If we
call $\alpha \equiv I/J$, we have that the dimensional-
ity d of the problem is respectively 2,3 and
1 for $\alpha = 0$, $\alpha \simeq 1$ and $\alpha \rightarrow \infty$. Furthermore, if
we remember that the site occupancy critical
probability p_c(d) monotonically increases
with decreasing d (and that $p_c(1) = 1$), it
is natural to expect for the critical front
iers the intrincated tailing behaviour in-
dicated in Fig. 13. The problem of the
complex crossovers between different di-
mensionalities critical sets of exponents has
not yet been attacked, to the best of our
knowledge.

Fig. 11 - Ferro-para critical line of
a site-dilute first-and second-neighbour
interaction (J_1 and J_2 respectively)
Ising model ($\alpha \equiv J_2/J_1$); $p_c(1)$ and p_c
(1+2) respectively denote the first and
and first-and-second neighbour critical
probability for site percolation.

III.5 - Topological incompatibility

In Fig. 1 we have represented a possible crit-
ical surface for a dilute R-S model[14]. Pract-
ically nothing is known about such a model
where no doubt very interesting features must
exist: one of the most fascinating certainly
is the point N of the Fig., as it is simulta -
neously a Lifshitz and a multiphase point.

Another interesting illustration of random to-
pological incompatibilities certainly is the
different sign bond-mixed problem, where a
spin-glass phase might appear (Refs. (13, 48-

Fig. 12 - Critical surface for the
uncorrelated quenched site-bond-di-
lute first-neighbour Ising model
(the T=0 critical line has recently
been studied[47]).

50) among others). To fix ideas let us imagine a random distribution of first-neigh-
bour Ising interactions $J_1 < 0$ (with probability (1-p)) and $J_2 > 0$ (with probabili -
ty p)on a square lattice of $\frac{1}{2}$ - spins. If we consider an elementary square plaquette,
we see that, for any bond distribution which contains an even number of each exchange
constant, two spin configurations exist which simultaneously satisfy all the bonds,
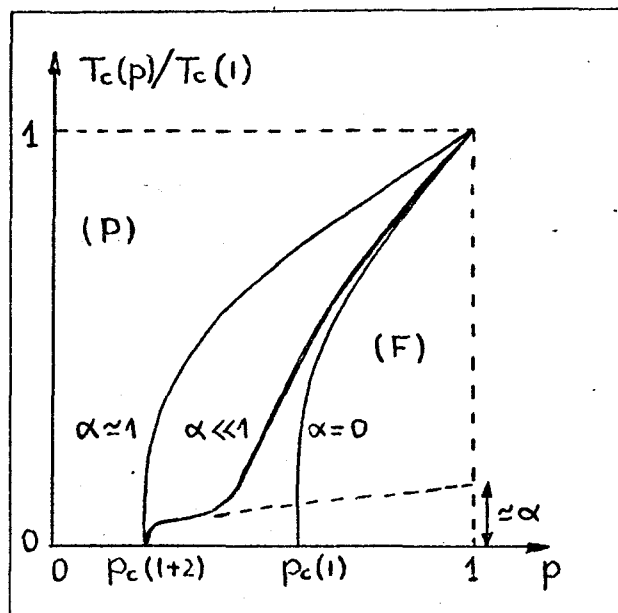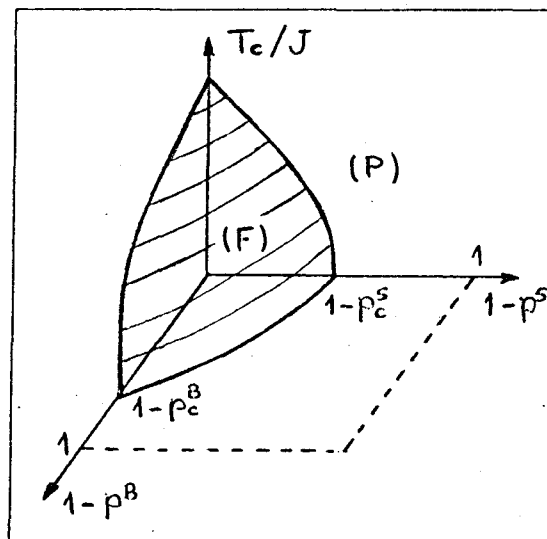whereas for any bond distribution which contains an odd number of each exchange constant,

there is no configuration which simultaneous_ly satisfy all the bonds: the plaquette is said to be *frustrated*[13]; if this phenome - non occurs frequently in the system, a phase (spin-glass) which is neither ferro- or anti_ferromagnetic might be the most stable (the higher the dimensionality of the system, the easier for the spin glass phase to appear); see Fig. 14.

III.6 - New degrees of freedom

Let us illustrate this "ingredient" by comp-aring the *quenched* models (where migration of the elements or of the bonds is forbidden) and the *annealed* ones (where that migration is allowed). Though an annealed crystal is frequently easier to prepare, its theoretic-al treatment presents usually greater diffi-culty (with respect to its quenched version), precisely because of the additional degree of freedom.

Let us fix our ideas on site dilution on a square lattice ferromagnetic model. If we have only one absent site, both quenched and annealed problems are equivalent to a one hole problem, hence their slopes in the li-mit $p \rightarrow 1$ should be identical. The situation is quite different if we have two absent sites. In the quenched problem the two ho-les will probably be far away one from the other, therefore 8 cooperative interactions will be lost; in the annealed version, the two holes will probably migrate in order to become first neighbours, therefore only 7 cooperative interactions will be lost. As a consequence of these facts we expect T_c (annealed) $\geq T_c$ (quenched) $\forall (n,d)$. An example [50] is presented in Fig. 15.
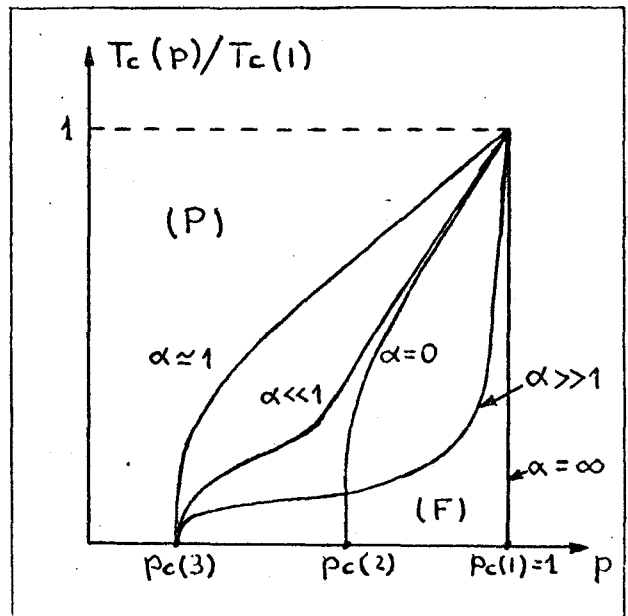

Fig. 13 - Ferro-para critical lines for a layered site-dilute Ising ferromagnet ($\alpha \equiv I/J$ where I and J are respect-ively the inter-and intra-layer exchan-ge constants).
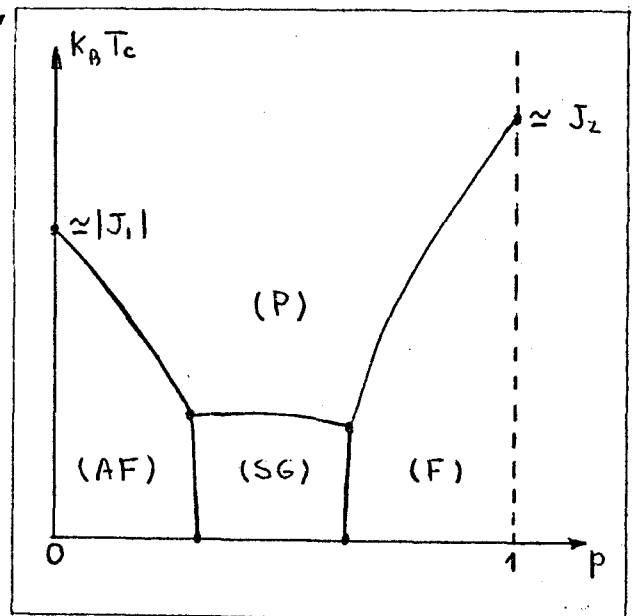

Fig. 14 - A possible phase diagram for a bond-mixed Ising model with $J_1 < 0 < J_2$ (P,F,AF and SG respectively denote the para-, ferro, antiferromagnetic and spin-glass phases).

III.7 - Ground state degeneracy

Until now we have basically restricted ourselves
to the Ising model (n = 1). Not very much is
known about random magnets with n > 1. If we call
p*(n) the probability where the critical temper-
ature of a dilute n- model vanishes, and p_c the
corresponding percolation critical probability ,
it is obvious that $p_c(n) \geq p_c$ $\forall (n,d)$, because
the existance of an infinite cluster is *necessa-
ry* for the development of a long range order.
The equality will hold if it is also *sufficient*.
That seems to be the case for the Ising model,
i.e. $p_c(1) = p_c$ $\forall d$, but the question is contro- (51)
versial for n > 1 (for example, Barma et al
found p*(3) > p_c for a particular model; see
Fig. 16). In any case it is clear that p*(n)
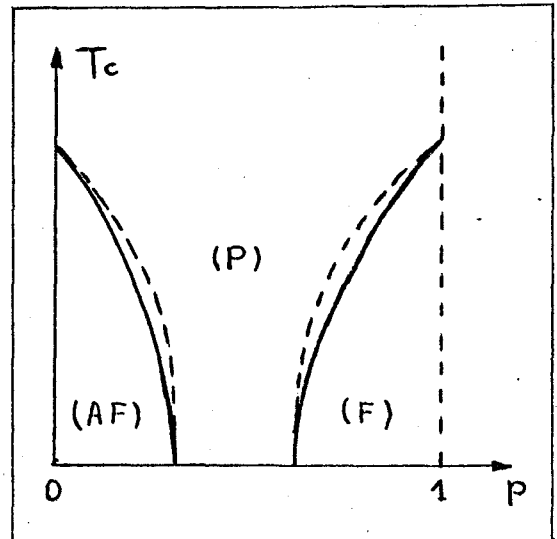cannot decrease with increasing n, as the phase

Fig. 15 - Approximate phase dia-
gram(50) of a planar quenched (full
lines) and annealed (dashed lines)
bond-mixed Ising model (with
$-J_1 = J_2$); P, F and AF respectively
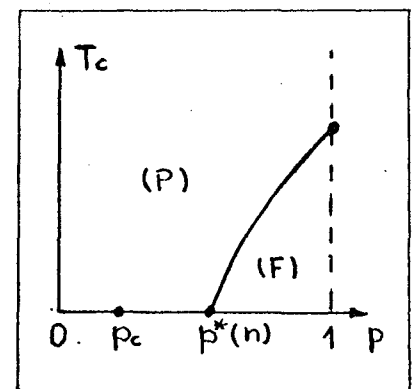denote the para-, ferro-, and anti-
ferromagnetic phases.

transition weakens in this situation. The critical frontier is expected, for the same
reasons as for n = 1, to be linear in the limit p → 1. In the simple n- models, T_c
is expected to follow a power-law ($T_c \propto (p-p^*)^x$, with x > 0) in the limit p → p*, be-
cause of the absence of gap in the spectrum of elementary magnetic excitations (mag-
nons); the absence of a gap is of course due to the fact that, for n > 1, the Hamil-
tonian of the system is invariant with respect to a continuous group of symmetry.

A very rich (and, to the best of our knowledge, complete
ly unexplored) discussion is certainly related to a let
us say three-dimensional first-neighbour model like the
following one:

$$\mathcal{H} = - J \sum_{i,j} c_i c_j \{ s_i^z s_j^z + \alpha(s_i^x s_j^x + s_i^y s_j^y) \} \qquad (J > 0)$$

$$P(c_i) = (1-p)\delta(c_i) + p\delta(c_i - 1) \qquad \forall i$$

We see that α = 0, 1, ∞ respectively correspond to n = 1,
3, 2, therefore quite a number of unknown possibilities
(in what concerns the critical lines and sets of expo-
nents) must be involved for arbitrary values of α. In
Fig. 17 we have indicated two possible results (and have
omitted the eventual complicate tailing effects for arbi-
trary values of α).

Fig. 16 - Critical line
for bond- or site-dilute
ferromagnetic n- model
(d ≥ 2 and n > 1); P and
F respectively denote the
para- and ferromagnetic
phases (it is expected
p*(n) ≥ p_c ≡ percolation
threshold).

Another illustration of randomness related to the ground state degeneracy might be

the quenched bond-dilute q-state ferromagnetic first-neighbour Potts model, namely

$$\mathcal{H} = - \sum_{i,j} J_{ij}\, \delta_{\sigma_i,\sigma_j}$$

$$P(J_{ij}) = (1-p)\,\delta(J_{ij}) + p\,\delta(J_{ij}-J) \quad \forall (i,j) \qquad (J > 0)$$

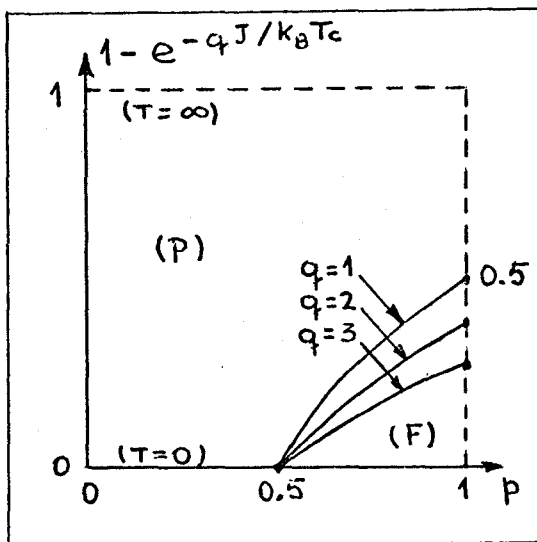In Fig. 18 we have reproduced the results obtained for this model by Southern and Thorpe[52].

Fig. 18 - Ferro (F) -para (P) critical lines for the quenched bond-dilute q-state ferromagnetic first neighbour Potts model[52] in square lattice.
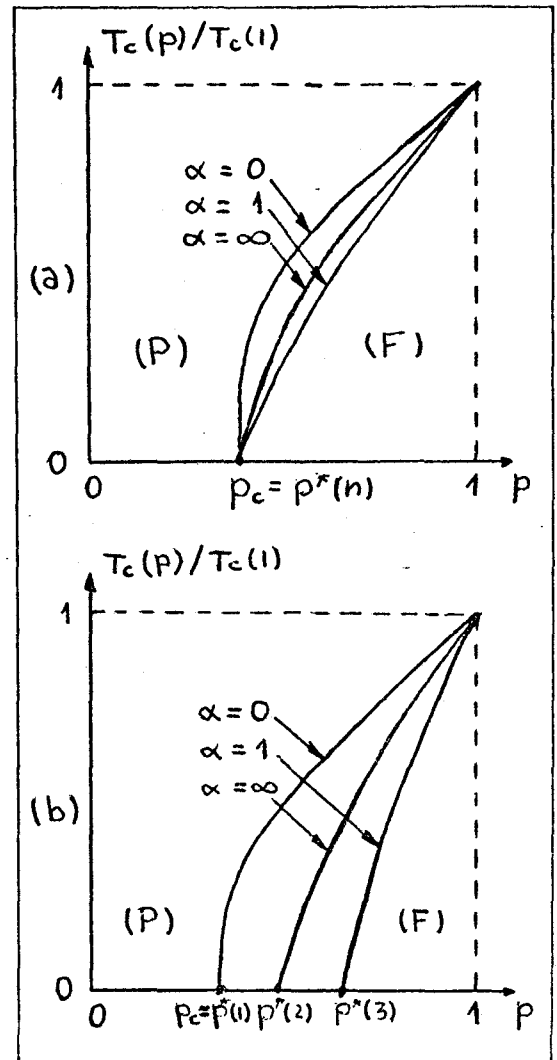
Fig. 17 - Possible ferro (F) -para (P) critical lines for let us say three-dimensional site-dilute mixed model ($\alpha=0,1,\infty$ respectively correspond to n=1,3,2).

(a) $p^*(3)=p^*(2)=p^*(1)=p_c$;

(b) $p^*(3)>p^*(2)>p^*(1)=p_c$.

III.8 - "Quanticity"

To illustrate this "ingredient" we shall restrict ourselves to a single simple example[53]. Let us have a quenched site-mixed first-neighbour ferromagnetic Ising model in a z-coordinated lattice. The family A is made of spins with size 1 (and occupancy probability p) and the family B is made of spins with size $\frac{1}{2}$ (and occupancy probability(1-p)). For simplicity let it be $J_{AA} = J_{BB} = J_{AB} \equiv J > 0$. Then the critical temperature is given[53], within the MFA, by

$$T_c = \frac{z\,J}{4\,k_B}\,(2-p)$$

Therefore T_c decreases when the weigh of the larger spin increases (i.e. when it becomes less quantical).

IV - CONCLUSION

Random magnetism is a good "bridge" between geometrical and thermal physics. Though we have been, in the present quick survey, extremely restrictive (we have not gone very much beyond the phase diagrams), an enormous number of almost completely unexplored possibilities has become evident. This fact (as well as the various non uniform convergences that have appeared along this work) is typical of this area where very much is still to be done, on theoretical as well as experimental grounds. Through the general trends, it has been exhibited the relevance of the topological concepts in random magnetism, and through them (as well as for other reasons) the Theory of Critical Phenomena becomes related to the mathematical Catastrophe Theory and Theory of Fractals as well as to other branches of Physics such as Field Theory or Cosmology. We might be a little bit closer to the "fundamental unity" of Physics.

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