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ABSTRACT

We analyse and extend the real-space renormalization group proposed by Reynolds, Klein and Stanley 1977 to treat the site percolation. The best among 3 possible definitions of "percolating" configurations and among 5 possible methods to weigh these configurations, are established for percolation in square lattices. The use of n x n square clusters leads, for n = 2 (RKS), n = 3 and n = 4, to $\nu_p \simeq 1.635$, $\nu_p \simeq 1.533$ and $\nu_p \simeq 1.498$, and also to $p_c \simeq 0.382$, $p_c \simeq 0.388$ and $p_c \simeq 0.398$, exhibiting in this way the correct (but slow) tendency towards the best up to date values.

RÉSUMÉ

Le groupe de rénormalisation dans l'espace réel proposé par Reynolds, Klein et Stanley 1977 pour traiter la percolation de sites, est analysé et étendu. La meilleure (parmi 3 possibilités étudiées) definition de configuration "percolatrice" et le meilleur poids (parmi 5 possibilités étudiées) sont établis pour la percolation dans le réseau carré. L'utilisation d'amas carrés n x n amène, pour n = 2 (RKS), n = 3 et n = 4, à $\nu_p \simeq 1{,}635$, $\nu_p \simeq 1{,}533$ et $\nu_p \simeq 1{,}498$, et à $p_c \simeq 0{,}382$, $p_c \simeq 0{,}388$ et $p_c \simeq 0{,}398$, manifestant ainsi la tendance correcte (quoieque lente) vers les meilleurs valeurs actuellement connues.

I - INTRODUCTION

Bond and site percolation problems have received great attention during last years because of their applications in many fields, as well in Physics as in other branches of knowledge (see the reviews by Shante and Kirkpatrick(1971) and by Essam (1972)). Different theoretical approaches have been attempted, particularly by using the ideas of the reciprocal space (see Wilson and Kogut (1974), Toulouse and Pfeuty (1975) , Ma (1976)) and real space (see Niemeyer and Van Leeuwen (1974, 1976) and Harris and Lubensky (1974)) renormalization group (RG). A great number of interesting related topics are analysed in the collection of reviews edited by Domb and Green (1972 - 1976).

Young and Stinchcombe (1975, 1976), Stinchcombe and Watson (1976), and Kirkpatrick (1977) directly transformed the probability in the bond problem. Harris et al (1975,1976) and Dasgupta (1976) made an RG expansion near d = 6. Stanley et al (1976) discussed two-dimensional phase transitions near the percolation threshold. Plischke and Zobin (1977) used RG arguments to discuss the two-dimensional dilute Ising and Ising spin-glass models. Finally Reynolds, Klein and Stanley (1977, RKS) made a real space RG proposal for treating the site and bond percolation problems in regular lattices.

This last work (RKS) is the one we mainly deal with in the present paper, where we analyse and extend the RKS ideas for site percolation. By a simple and interesting way, RKS introduced a RG which allows for the calculation of the critical

probability p_c and the exponent ν_p*. Their method essentially consists in making a partition of the lattice into cells or clusters (which will become the renormalized sites) that completely covers it (all sites must be considered one and only one time) and simultaneously preserves its original symmetry. Once the cluster has been chosen, they consider it as a renormalized site (with probability p' of being independently occupied) of an expanded lattice (the expansion factor is noted b). The next step is to relate the original and renormalized probabilities by a non linear transformation $p' = R(p)$, which immediately leads to p_c and ν_p; more precisely, p_c corresponds to the non trivial (p_c different from 0 and 1) fixed point of the transformation (in other words $p_c = R(p_c)$) and ν_p is given by (RKS) $\nu_p = \ln b / \ln \lambda_p$, where $\lambda_p \equiv (dR(p)/dp)_{p_c}$ is the eigenvalue of the linearized transformation $R^L(p-p_c) = \lambda_p (p-p_c)$.

It is clear that the whole point is <u>how</u> to effectively write down the transformation $R(p)$, or in other words <u>what</u> cluster configurations are to be considered as "percolating" (and with what weight). This is the point we shall examine in the present work by introducing three different but "reasonable" definitions of percolating cluster configuration. We concretize this by studying the square lattice. Let us anticipate that all three definitions become, for simple clusters, precisely the one used by RKS. Furthermore, once the percolating configurations

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* We recall that the exponent ν_p is defined by $\xi_p \simeq |p-p_c|^{-\nu_p}$ in the limit $p \to p_c$, where ξ is the mean size of a cluster and p is the probability of a site being independently occupied.

have been determined, it is not obvious that they should not be weighted to obtain their contribution to $R(p)$. This point has been examined in the present paper, by introducing five different but "reasonable" ways of calculating the weight. In other words we have considered, for each choice of cluster, 15 different possibilities. The complete discussion has been performed only for a 3 x 3 square cluster (besides the simple case of the RKS 2 x 2 square cluster). For the best among the 15 possibilities, we have also performed the calculations for the 4 x 4 square cluster. The discussions relative to the definition and weight of a percolating configuration are presented in Sections II and III respectively.

II - DEFINITION OF PERCOLATING CLUSTER

Let us fix our ideas on the square lattice with only first-neighbor interactions. We choose a particular cell which completely covers the lattice, for example a n x n square. This cell will usually present a great number of configurations (2^{n^2} in our example) if we remember that each site may be empty or occupied (see Fig. 1). Some of these configurations will contain "paths" (defined by first-neighborhood), and among them will be the "percolating" ones, which necessarily begin and end on two different sites of the periphery of the chosen cell. A configuration will be considered as "percolating" (and will contribute to $R(p)$) if and only if it contains at least one percolating path. It is clear that this definition allowds for configurations where appear two or more disconnected percolating

regions in the cluster. Now the central point is what paths are we going to consider as percolating for our n x n square cluster. Let us now state three possible definitions for this kind of path:

1st definition: it must contain at least 2 sites;

2nd definition: it must contain at least n sites, including the possible "hanged" ones;

3rd definition: it must contain at least n sites, after elimination of the possible "hanged" ones.

In Fig. 1, configuration (a) does not percolate, configuration (b) percolates only within the 1st definition, configuration (c) percolates within both 1st and 2nd definitions, and finally configurations (d), (e) and (f) percolate within all three definitions. The aim pursued with the 1st definition is to mantain without changements the definition used by RKS which essentially demands the possibility of transfert of the information. In the 2nd and 3rd definitions we essentially demand the possibility of getting across the cell in the most direct manner (straight line). We have not succeeded in finding out a clear a priori argument in favour or against the inclusion of the possible hanged points, because on one hand they clearly do not contribute in the percolation process, but on the other they do affect the mean size ξ of the infinite percolating cluster, particularly if $p > p_c$. Note that for the 2 x 2 square cluster used by RKS, all three definitions reduce to only one.

III - PERCOLATIVITY

Let us now turn our attention on to the problem of the weight atributed to each percolating configuration in what concerns its contribution to $R(p)$. For example, configurations (e) and (f) of Fig. 1 equally percolate ? Should not we rather consider that configuration (e) percolates "more" than the (f) one ? This question brings to us the necessity of introducing the concept of percolativity P ($0 \leq P \leq 1$) of a given configuration. In order to be relatively complete, we introduce 5 different methods for evaluating the percolativity of a given configuration:

1^{st} method: P = 1 for every percolating configuration;

2^{nd} method: P = (number of peripheric occupied sites which belong to a percolating path)/(total number of peripheric occupied sites);

3^{rd} method: P = (number of external peripheric paths leading to occupied sites which belong to a percolating path)/(total number of external peripheric paths leading to occupied sites);

4^{th} method: P = (number of peripheric occupied sites which belong to a percolating path)/4(n-1) ;

5^{th} method: P = (number of external peripheric paths leading to occupied sites which belong to a percolating path)/4n.

For example, for the configuration (e) of Fig. 1 we have P = 1, 7/8, 9/11, 7/12 and 9/16 for the 1^{st}, 2^{nd}, 3^{rd}, 4^{th} and 5^{th} methods respectivelly. Notice that the unitary

weight is the one adopted by RKS, and that, for the 2 x 2 square cluster, the 1^{st}, 2^{nd} and 3^{rd} methods coincide, like the 4^{th} and 5^{th} ones between them. Remark also that the 3^{rd} and 5^{th} methods atribute double weight to the 4 corner sites (as they are twice more accessible from the outside), and that the 4^{th} and 5^{th} methods lead to P = 1 only when <u>all</u> the 4(n-1) peripheric sites are occupied. On the other hand the 2^{nd} and 3^{rd} methods will lead to P = 1 only when <u>all</u> the peripheric occupied sites belong to percolating paths. Let us finally add that, for all 5 methods, we atribute the value P = 0 for every non percolating configuration.

Before writting down the expression of $R(p)$ let us define the "multiplicity" m of a given cluster configuration as the number of configurations which are topologically equivalent to a given one. For example m = 4 for the configurations (b) and (f) of Fig. 1, and m = 8 for the configurations (a), (c), (d) and (e).

Let us now explicitely write the transformation which relates the original probability p of independent site occupation to the renormalized probability p' associated to the n x n square cluster considered as a site of the expanded lattice:

$$p' = R(p) = \sum_{r=o}^{n^2-2} A_r \, p^{n^2-r} \, (1-p)^r \qquad (1)$$

with $A_o = 1$ and $A_r = \sum_i m_i^{(r)} \, P_i^{(r)}$, where the index i refers to all topologically non equivalent percolating configurations which have (n^2-r) occupied and r empty sites (their number is usually a few times smaller than $(n^2)!/(n^2-r)!\,r!$).

IV - RESULTS

Let us first recall the RKS results for the 2 x 2 square cluster* (hence n = b = 2), which correspond to our 1^{st}, 2^{nd} and 3^{rd} methods:

$$p' = p^4+4p^3(1-p) + 4p^2(1-p)^2 \qquad \text{(Fig.2(a))}$$

hence $p_c \simeq 0.382$, $\lambda_p \simeq 1.528$ and $\nu_p \simeq 1.635$.

The 4^{th} and 5^{th} methods lead to

$$p' = p^4+3p^3(1-p) + 2p^2(1-p)^2 \qquad \text{(Fig.2(b))}$$

hence there is no non trivial fixed point.

Let us now present the results for the 3 x 3 square cluster (hence n = b = 3): relation (1) stands

$$p' = \sum_{r=o}^{7} A_r \, p^{9-r}(1-p)^r$$

In the limit $p \to 0$, we obtain, for all five methods, $p' \sim A_7 p^2$ within the 1^{st} definition and $p' \sim A_6 p^3$ within the 2^{nd} and 3^{rd} definitions. On the other hand in the limit $p \to 1$, we obtain, for all three definitions, $p' \sim 1 - (126-A_4)(1-p)^4$ for the 1^{st} method, $p' \sim 1 - (36-A_2)(1-p)^2$ for the 2^{nd} and 3^{rd} methods, and $p' \sim 1 - (1-p)-(28-A_2)(1-p)^2$ for the 4^{th} and 5^{th} methods. In general terms, we may say that, for all three definitions, the 1^{st}, 2^{nd} and 3^{rd} methods lead to a behaviour

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* Remember that in this case all three definitions are equivalent.

like that indicated in Fig. 2(a), and the 4^{th} and 5^{th} methods like that in Fig. 2(b) (in other words, no fixed point other than the trivial ones p = 0 and p = 1).

The results for the case n = 3 are indicated in Table 1 and Table 2. It is interesting to remark in Table 2 the completely monotonic "horizontal" and "vertical" behaviours in what concerns the values of p_c and ν_p. These results are to be compared with the most up to date known values (Shante and Kirkpatrick (1971), Essam (1972), Dunn et al (1975), and Kirkpatrick (1976)) $p_c \simeq 0.590 \pm 0.010$ and $\nu_p \simeq 1.34 \pm 0.02$. In order to decide what definition and method are preferable, we have used as criterium the value of ν_p rather than that of p_c (essentially because the central goal of the RG theory is the knowledge of the critical exponents rather than the critical probabilities or temperatures). So it is clear that our best trial corresponds to the 3^{rd} definition and the 1^{st} method. The 4 x 4 square cluster (n = b = 4) has been studied (with computer) only for this trial, which leads, for the relation (1), to

$$\begin{aligned} p' = {} & p^{16}+16p^{15}(1-p)+120p^{14}(1-p)^2+560p^{13}(1-p)^3 \\ & + 1820p^{12}(1-p)^4+4364p^{11}(1-p)^5+7890p^{10}(1-p)^6 \\ & + 10440p^{9}(1-p)^7+9514p^{8}(1-p)^8+5664p^{7}(1-p)^9 \\ & + 2096p^{6}(1-p)^{10}+440p^{5}(1-p)^{11}+40p^{4}(1-p)^{12} \end{aligned}$$

This expression leads to the results indicated in Table 3, together with those corresponding to n = 2 and n = 3 for the same trial (3^{rd} definition and 1^{st} method). An analysis of this Table shows that, within this trial, the numerical values of p_c

and ν_p exhibit, for increasing n, the correct tendency towards $p_c \simeq 0,590$ and $\nu_p \simeq 1,34$ respectively. However, because of the relatively slow convergence, it is impossible to make, at this stage, a good extrapolation for achieving the limit values when $n \to \infty$, or in other words, to say if this RG supports the numerical results obtained by other methods.

V - CONCLUSION

In order to conclude let us say that the real space renormalization group proposal of Reynolds, Klein and Stanley (1977) for treating the site percolation, can be considered as essentially satisfactory. In particular, the procedure they used for the weight of the percolating configurations (namely to take the percolavity equal one) proved to be the best among the five possibilities we considered here. On the other hand, it was necessary, for complex clusters (n x n squares in square lattice), to extend the RKS definition of percolating configuration, and we found that the best among the three definitions we considered, is the following: a cluster configuration can be considered as percolating for the renormalization group needs, if and only if it contains at least one percolating path wich begins and ends in two different peripheric occupied sites and contains at least n occupied sites after elimination of the possible "hanged" ones. Let us finally recall our central result: the sequence $n = 2$(RKS), $n = 3$ and $n = 4$ leads to $\nu_p \simeq 1.635$, 1.533 and 1.498 respectively. These values exhibit the correct tendency towards the best known value 1.34 ± 0.02 (Dunn et al(1975)).

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CAPTION FOR FIGURES AND TABLES

Fig. 1 - Six different configurations (among the 2^{16}) of a 4 x 4 square cell.

Fig. 2 - The renormalization transformation $R(p)$. The (a)-type corresponds to the 1^{st}, 2^{nd} and 3^{rd} methods, and the (b)-type to the 4^{th} and 5^{th} methods.

Table 1 - The set of $\{A_r\}$ for all five weighting methods for a 3 x 3 square cluster: (a) 1^{st} definition; (b) 2^{nd} definition; (c) 3^{rd} definition.

Table 2 - 3 x 3 square cluster: the critical probability p_c (top number) and the critical exponent ν_p (buttom number) for all three definitions and the first three weighting methods (the 4^{th} and 5^{th} methods lead to no non trivial fixed point).

Table 3 - The values of the critical probability p_c and the critical exponent ν_p obtained with a n x n square cluster within the 3^{rd} definition and the 1^{st} method; in the last column the most up to date values (Shante and Kirkpatrick (1971) Essam (1972), Dunn et al (1975), Kirkpatrick (1976)).

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
a) 1st Meth.	9	36	84	125	116	54	8
a) 2nd Meth.	9	35,33333	78,66667	108	91,33333	43,33333	8
a) 3rd Meth.	9	35,2	77,83333	106,31746	89,94286	43,6	8
a) 4th Meth.	8	27,5	52,5	60	40,5	14,5	2
a) 5th Meth.	8	27,33333	51,66667	58,33333	39	14	2
b) 1st Meth.	9	36	84	117	80	22	0
b) 2nd Meth.	9	35,33333	76	95,2	67,33333	22	0
b) 3rd Meth.	9	35,2	75,25556	93,46032	65,71429	22	0
b) 4th Meth.	8	27,5	50,5	52	28,5	6,5	0
b) 5th Meth.	8	27,33333	49,66667	49,66667	27	6	0
c) 1st Meth.	9	36	84	109	64	14	0
c) 2nd Meth.	9	35,33333	76	91,2	56,66667	14	0
c) 3rd Meth.	9	35,2	75,25556	90,03175	56,11429	14	0
c) 4th Meth.	8	27,5	50,5	50	24,5	4,5	0
c) 5th Meth.	8	27,33333	49,66667	48,33333	23	4	0

TABLE 1

Definition \ Method	1st	2nd	3rd
1st	0.141 1.851	0.1687 2.140	0.1689 2.152
2nd	0.328 1.540	0.378 1.826	0.385 1.865
3rd	0.388 1.533	0.438 1.760	0.443 1.785

TABLE 2

n	2	3	4	almost exact
p_c	0.382	0.388	0.398	0,590
ν_p	1.635	1.533	1.498	1,34

TABLE 3

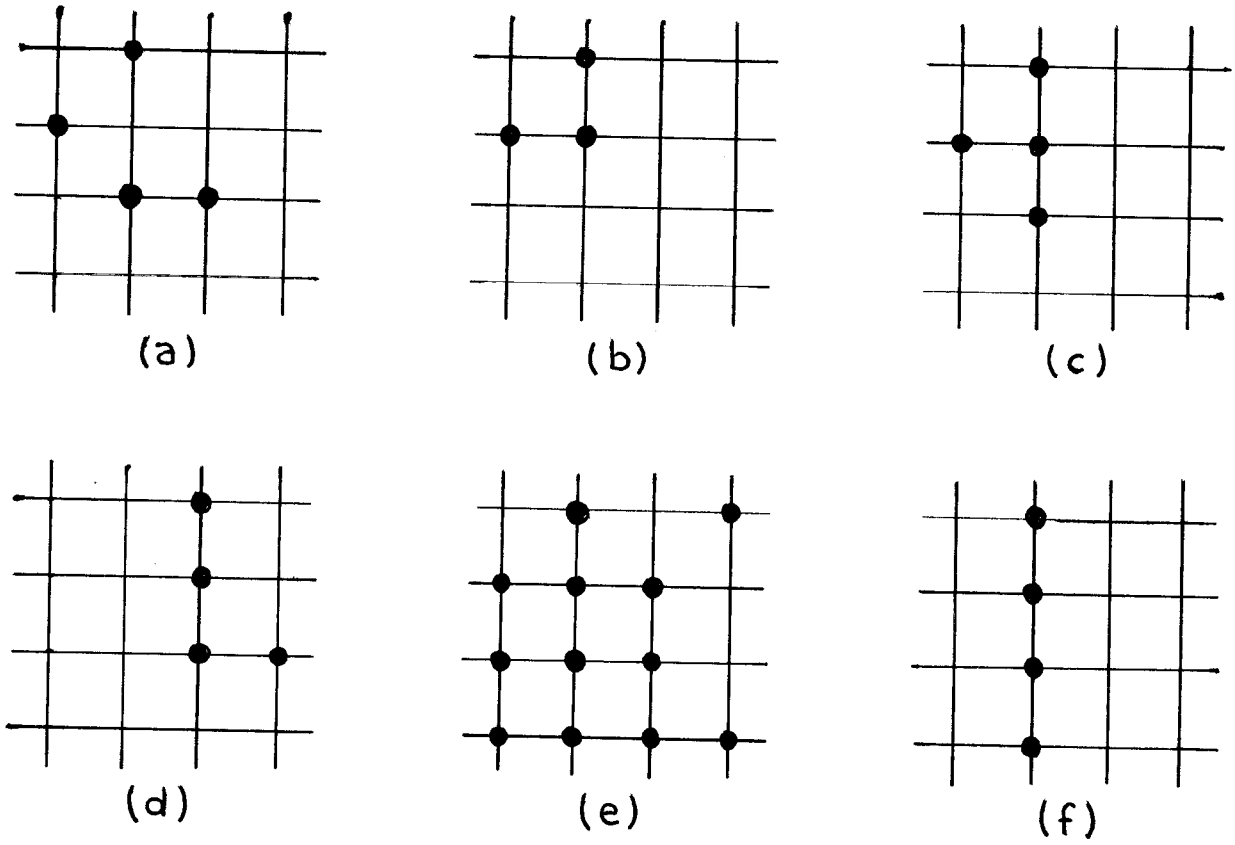

Fig. 1

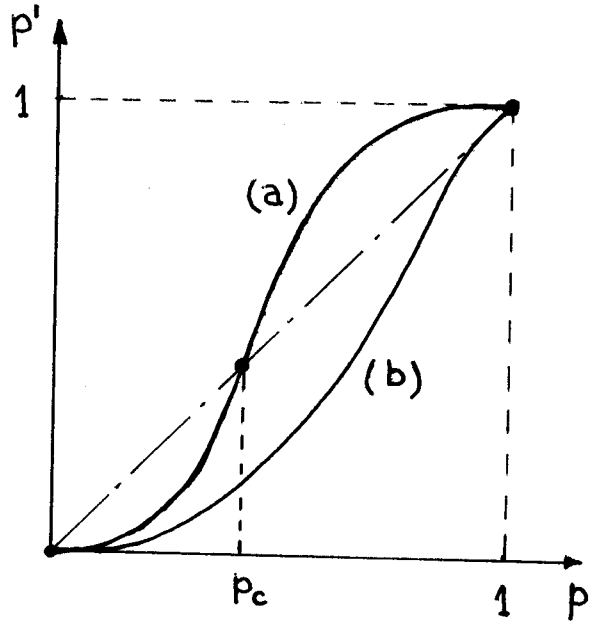

Fig. 2