SPHERICAL DIFFUSE YUKAWA SOURCES IN RELATIVITY

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ABSTRACT

A system only containing two diffuse massless (bearing a uniform ratio) of two different short range scalar fields is studied, according to Einstein gravitational theory. One field is attractive, the other is repulsive. The distribution is in static equilibrium with spherical symmetry. A class of solutions of the field equations is obtained. The solutions are nonsingular and have simple physical interpretation. A Schwarzschild-type gravitation is found at infinity, with mass parameter solely arising from the scalar fields; these rapidly disappear with a Yukawa-type behaviour. The stability of the system is briefly stated, and the applicability of the model to large or small physical systems is pointed out.

1. INTRODUCTION

One finds in literature (Kurşunoğlu 1976) an ever growing belief that general relativity must play an important role in the description of elementary physical systems. Nonsingular solutions of Einstein's equations are particularly desirable, corresponding to systems with specified values for some physical parameters such as mass, electric and nuclear charge, angular momentum, etc.

In an earlier study Duan' - I - Shi (1956) obtained a class of solutions corresponding to the fields of a point nuclear charge; however, his solutions contain a number of functions whose forms are not explicitly known. Stephenson (1962) enlarged that study by introducing electric charge, and obtained an explicit but approximate solution which ressembles the classical fields of a proton; however, his solution is singular in the origin.

More recently Teixeira et al. (1975) considered a system of dust whose constituents are sources of gravitation and of a repulsive short range scalar field; though non-singular, their solution does not allow inclusion of electrostatic charge to a physically desirable value without destroying the equilibrium.

Quantum effects are certainly important for microscopic objects, where the uncertainty relations play a fundamental role; nevertheless it seems worthwhile to investigate from the purely classical viewpoint the formation and stability of elementary systems, under the combined effects of short range fields and space curvature.

Very recently Souza et al. (1977) studied a system not containing matter explicitly; the constituents of their system were sources of a long range attractive scalar field and of a short range repulsive scalar field. The asymptotic curvature of spacetime produced by such a distribution is analogous to that produced by a usual "material" source. The system is under static equilibrium and does not show any singularity. In their approximate solution, however, the asymptotic attractive effects of the long range scalar field exceed the gravitational effects, what seems not to be a desirable result.

In the present paper we consider a physical system which does not present that inconvenient. We here study a spherically symmetric static distribution of two different diffuse sources of short range scalar fields. The ratio between these two sources is taken constant. The attractive character of one of the fields prevents an indefinite expansion of the system, while the repulsive character of the other field (with a shorter range) prevents a collapse. From the solution of the linearized field equations one finds that the sole interaction present in regions far from the center of symmetry is the gravitational one, which has a Schwarzschild-type behaviour; both scalar fields rapidly disappear with a Yukawa-type behaviour.

2. BASIC EQUATIONS

We derive our field equations from a Lagrangean density

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_A + \mathcal{L}_B \quad , \tag{2.1}$$

$$\kappa \mathcal{L}_G = -\frac{1}{2} (-g)^{1/2} R \quad , \qquad \kappa = 8\pi G/c^4 \quad , \tag{2.2}$$

$$\kappa \mathcal{L}_A = -(-g)^{1/2} (A_{,\mu} A_{,\nu}\, g^{\mu\nu} - \alpha^2 A^2) + 2\, A a^* \quad , \tag{2.3}$$

$$\kappa \mathcal{L}_B = (-g)^{1/2} (B_{,\mu} B_{,\nu} g^{\mu\nu} - \beta^2 B^2) + 2Bb^* \quad ; \tag{2.4}$$

in these expressions g is the determinant of the gravitational potentials $g_{\mu\nu}$, R is the scalar curvature (Anderson 1967), A and B are attractive and repulsive (Teixeira et al. 1976) scalar fields of finite inverse ranges α and $\beta > \alpha$. Subscripted commas mean ordinary derivative. The quantities a^* and b^* are scalar densities of weight + 1, and represent the diffuse sources of the fields A and B.

Einstein's equations are obtained upon variations of the gravitational potentials $g_{\mu\nu}$,

$$R^\mu_\nu = - 2A^{,\mu}A_{,\nu} + 2B^{,\mu}B_{,\nu} + (\alpha^2 A^2 - \beta^2 B^2)\delta^\mu_\nu \quad , \tag{2.5}$$

while the variations of the fields A and B give respectively

$$A^{;\nu}_{;\nu} + \alpha^2 A = - a \quad , \tag{2.6}$$

$$B^{;\nu}_{;\nu} + \beta^2 B = b \quad ; \tag{2.7}$$

a semicolon means covariant derivative, and we introduced the scalar quantities of weight zero (Souza et al. 1977)

$$a = a^* (-g)^{-1/2} \quad , \quad b = b^* (-g)^{-1/2} \quad . \tag{2.8}$$

A very useful expression is provided by the contracted Bianchi identities, which here take the simple form

$$aA_{,\nu} + bB_{,\nu} = 0 \quad . \tag{2.9}$$

Spherically symmetric static systems are conveniently described by the metric element

$$ds^2 = e^{2\eta}(dx^0)^2 - e^{2\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \quad , \tag{2.10}$$

with the potentials η and λ , the fields A and B, and the densities a and b all functions of r alone. We then obtain from the preceding equations

$$(\eta_{11} + 2\eta_1/r + \eta_1^2 - \eta_1\lambda_1)e^{-2\lambda} = -\alpha^2A^2 + \beta^2B^2 \quad , \tag{2.11}$$

$$(\eta_{11} - 2\lambda_1/r + \eta_1^2 - \eta_1\lambda_1)e^{-2\lambda} = -\alpha^2A^2 + \beta^2B^2 - 2(A_1^2 - B_1^2)e^{-2\lambda}, \tag{2.12}$$

$$(\eta_1/r - \lambda_1/r + 1/r^2)e^{-2\lambda} - 1/r^2 = -\alpha^2A^2 + \beta^2B^2 \quad , \tag{2.13}$$

$$r^{-2}(r^2\, e^{\eta-\alpha}A_1)_1\, e^{-\eta-\lambda} - \alpha^2A = a \quad , \tag{2.14}$$

$$r^{-2}(r^2\, e^{\eta-\alpha}\, B_1)_1\, e^{-\eta-\lambda} - \beta^2B = -b \quad , \tag{2.15}$$

where a subscript 1 means d/dr; these equations satisfy the identity

$$aA_1 + bB_1 = 0 \quad . \tag{2.16}$$

In the five independent equations (2.11) to (2.15) we have six functions (η, λ, A, B, a, b); one constraint is then necessary if one wants explicit solutions. We assume a proportionality between the sources,

$$a = f\,b \quad , \quad f^2 = \text{const} < 1 \quad . \tag{2.17}$$

In view of difficulty in finding exact solutions we try an approximate method: expansion is made of the four fields η , λ, A, B and of the two densities a, b in integral powers of some dimensionless parameter ε , to be identified later. In the lowest approximation we have taken A, B, a, b proportional to ε , and we have taken η, λ proportional to ε^2; the equations (2.11) to (2.17) then simplify to

$$\eta_{11} + 2\eta_1/r = - \alpha^2 A^2 + \beta^2 B^2 \quad , \tag{2.18}$$

$$\lambda = r\eta_1 - \frac{1}{2} r^2 (A_1^2 - B_1^2 - \alpha^2 A^2 + \beta^2 B^2) \quad , \tag{2.19}$$

$$A_{11} + 2A_1/r - \alpha^2 A = f\, b \quad , \tag{2.20}$$

$$B_{11} + 2B_1/r - \beta^2 B = - b \quad , \tag{2.21}$$

$$(fA_1 + B_1)b = 0 \quad , \tag{2.22}$$

$$a = f\, b \quad . \tag{2.23}$$

One finds that the four last equations determine the four functions A, B, a, b; the potential η is next obtained by integration of (2.18), and finally λ is directly given by (2.19).

3. THE SCALAR FIELDS

In regions where the sources a and b are nonzero we get from (2.20) to (2.23)

$$A_i = - \gamma\left[(\mu r)^{-1} \sin \mu r + \delta\right] \quad , \tag{3.1}$$

$$B_i = \gamma f\left[(\mu r)^{-1} \sin \mu r + \delta(\alpha/f\beta)^2\right] \quad , \tag{3.2}$$

$$a = \gamma\left[(\mu^2 + \alpha^2)(\mu r)^{-1} \sin \mu r + \delta\alpha^2\right] \quad , \tag{3.3}$$

$$b = a/f \quad , \tag{3.4}$$

where the subscript i means internal, γ and δ are dimensionless constants of integration, and where we defined the constant

$$\mu^2 = (f^2\beta^2 - \alpha^2)/(1 - f^2) > 0 \quad . \tag{3.5}$$

In writing these results we have discarded solutions presenting singularities in the origin. We also eliminated the possibilities $f^2\beta^2 \leq \alpha^2$, since the corresponding interior solutions do not satisfy the continuity conditions at the boundary.

In regions where the sources a = b = 0 we easily obtain from (2.20) and (2.21)

$$A_e = -\gamma_A \rho r^{-1} \exp(-\alpha r) , \qquad (3.6)$$

$$B_e = \gamma_B \rho r^{-1} \exp(-\beta r) , \qquad (3.7)$$

where the subscript e means external, and where $\gamma_A \rho$ and $\gamma_B \rho$ are constants of integration; solutions diverging at infinity were eliminated.

We now impose the continuity of the fields A and B, and of their first radial derivatives on the boundary of the sphere. These four boundary conditions will fix the three constants γ_A, γ_B, δ and the radius ρ of the sphere in terms of the constants α, β, γ and f. From the continuity of A and B we obtain

$$\gamma_A = \gamma\left[(\mu\rho)^{-1} \sin \mu\rho + \delta\right] \exp(\alpha\rho) , \qquad (3.8)$$

$$\gamma_B = \gamma f\left[(\mu\rho)^{-1} \sin\mu\rho + \delta(\alpha/f\beta)^2\right] \exp(\beta\rho) , \qquad (3.9)$$

while the continuity of the radial derivative of A gives

$$\delta = -(1+\alpha\rho)^{-1} (\cos\mu\rho + \alpha\mu^{-1} \sin\mu\rho) ; \qquad (3.10)$$

finally the continuity of the radial derivative of B imposes a constraint on the radius ρ,

$$\left[f^2\beta^3(1+\alpha\rho)-\alpha^3(1+\beta\rho)\right] \tan\mu\rho = -\mu\left[f^2\beta^2(1+\alpha\rho)-\alpha^2(1+\beta\rho)\right] . \qquad (3.11)$$

4. THE GRAVITATIONAL FIELD

We now integrate (2.18) to obtain $\eta(r)$ in the interior and exterior regions. We impose that $\eta(0)$ be finite, and that $\eta(\infty) = 0$; we also impose that $\eta(r)$ and its radial derivative both be continuous through the surface of the sphere. Under these four boundary conditions we obtain for the interior region $(r < \rho)$

$$\eta_i(r) = \eta(0) + \frac{1}{2}\gamma^2 (1 - f^2)\Big[C + \log 2\mu r - \text{ci}\,(2\mu r) - \frac{1}{3}(\zeta\mu r)^2 + (2\mu r)^{-1} \sin 2\mu r - 1\Big] , \tag{4.1}$$

where, for short, we used the constants

$$\zeta = \delta\alpha/f\beta , \tag{4.2}$$

$$\eta(0) = -\frac{1}{2}\gamma^2 (1-f^2) \Big[C + \log 2\mu\rho - \text{ci}\,(2\mu\rho) - (\zeta\mu\rho)^2\Big] - (\gamma_A\alpha\rho)^2\, \text{Ei}(-2\alpha\rho) + (\gamma_B\beta\rho)^2\, \text{Ei}(-2\beta\rho) ; \tag{4.3}$$

in these expressions C is the Euler constant 0.577... and the cosine and exponential integrals are defined by

$$\text{ci}(x) = -\int_x^\infty t^{-1} \cos t\, dt , \quad \text{Ei}(-x) = -\int_x^\infty t^{-1} e^{-t}\, dt, \quad x > 0 . \tag{4.4}$$

For the exterior region $(r > \rho)$ we obtain

$$\eta_e(r) = -(Gm/c^2) r^{-1} - \frac{1}{2}\alpha r A^2(r) + \frac{1}{2}\beta r B^2(r) - (\gamma_A\alpha\rho)^2\, \text{Ei}(-2\alpha r) + (\gamma_B\beta\rho)^2\, \text{Ei}(-2\beta r) , \tag{4.5}$$

where we used the mass parameter m given by

$$Gm/(c^2\rho) = \frac{1}{2}\gamma^2 (1 - f^2)\left[1 - (2\mu\rho)^{-1} \sin 2\mu\rho - \frac{2}{3}(\zeta\mu\rho)^2\right]$$

$$- \frac{1}{2}\gamma_A^2 \alpha\rho \exp(-2\alpha\rho) + \frac{1}{2}\gamma_B^2 \beta\rho \exp(-2\beta\rho) \quad ; \qquad (4.6)$$

one finds in (4.5) the usual Schwarzschild asymptotic behaviour $-Gm/c^2r$ for $\eta_e(r)$, since the fields $A(r)$, $B(r)$ and the exponential integrals all rapidly disappear with increasing r.

We finally use (2.19) to obtain the metric potential $\lambda(r)$: for the interior region ($r < \rho$) we get

$$\lambda_i(r) = - \frac{1}{2}\gamma^2 (1-f^2)\left[(1-\mu r \cot \mu r)(\mu r)^{-2} \sin^2\mu r - \frac{1}{3}(\zeta\mu r)^2\right] \quad , \qquad (4.7)$$

while for $r > \rho$ we obtain

$$\lambda_e(r) = (Gm/c^2)r^{-1} - \frac{1}{2}(1+\alpha r)A^2(r) + \frac{1}{2}(1+\beta r)B^2(r) \quad . \qquad (4.8)$$

One finds from (4.7) that $\lambda(r)$ vanishes on the origin, and from (4.8) one finds the Schwarzschild asymptotic behaviour Gm/c^2r for $\lambda_e(r)$. The continuity of $\lambda(r)$ through the boundary of the sphere is obvious from (2.19).

5. DISCUSSION

A special case of the present work is already known (Souza et al. 1977), in which the attractive scalar field is of long range ($\alpha = 0$).

Though being unaware of any exact solution of Einstein's equations involving short range fields, we still regret our inability to exactly solve the system of equations (2.11) to (2.17). Our linearized equations (2.18) to (2.22) present however an important convenience, namely that their corresponding weak field solutions have a dynamical behaviour which can be described in terms of usual nonrelativistic concepts; this will save us some deal of labor when studying the stability of our system.

For example, a short reflection shows that in the weak fields limit the attractive field A must have a range larger than that of the repulsive field B; this is a necessary condition for avoiding the escape of scalar sources. We then only considered systems satisfying the condition $\beta > \alpha$. We also have only considered systems with repulsive source stronger than attractive source, on physical grounds; this condition, expressed by $f^2 < 1$, is necessary for preventing the collapse of the system.

A third condition $f^2\beta^2 > \alpha^2$ for the equilibrium of the diffuse system was later found, in (3.5); this condition results from local cummulative effects of the diffuse sources, and seems to have no counterpart in systems made of point scalar sources.

The Bianchi identity (2.16) ressembles the equilibrium equation found in the interior Schwarzschild solution, where the gravitational attraction of the matter is balanced by the gradient of pressure; we now have a static equilibrium between the radial forces produced by the attractive and repulsive scalar fields upon the respective densities of source.

The dimensionless parameter γ can be seen to represent the intensity of the distribution in our first order approximation; it is then identified with the necessarily small dimensionless parameter ε in terms of which we made our series expansion. One finds from (4.6) that the condition $\gamma^2 << 1$ implies $Gm/c^2\rho << 1$, a condition usually met both in the very small as well as in the very large physical systems.

Differently from the Schwarzschild interior-exterior solution we have in our system not only η, λ and the radial derivative η_1 continuous through the boundary $r = \rho$, but also η_{11}, η_{111}, λ_1 and λ_{11} are continuous; this can be seen either from direct calculation or most easily from the differential equations (2.18) to (2.22).

It is well known that a rigorous demonstration of the stability of a dynamical system is in general a considerably hard task; however, we may content ourselves with the following simple reasoning, similar to that already used by Teixeira et al. (1975) and by Souza et al. (1977), to state the stability of our weak field system. If some small disturbance produces a local compression of the diffuse sources, the additional repulsive forces of short range (produced by, and acting on the compressed sources) exceed the fainter additional attractive forces of a longer range; a tendency to restore the equilibrium configuration thus manifests itself. The same final tendency is observed in the reverse situation of a local small expansion of the diffuse sources.

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