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# ON SPINOR SPACE REPRESENTATION OF LORENTZ GROUP

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ABSTRACT

The spinor space representation of Homogeneous Lorentz group offered by Clifford numbers in Minskowski space is reviewed. Two spinor calculus naturally follows when spinor matrix representation for these numbers is used. Representations of the improper four group are also discussed.

* * *

## 1. INTRODUCTION.

The purpose of the present article is to discuss the four component and two component spinor analysis, starting from the representation of Lorentz group in terms of Clifford numbers. The results are not new[1], however, a good deal of clarity in discussion is achieved. With the extensive use of spinors in Riemanian space [2] this is perhaps desirable.

In sections 2 and 3 we review the four spinor representation of the restricted homogeneous Lorentz group offered by Clifford algebra in Minskowski space. Section 4 is devoted to 2-spinor calculus which naturally follows when we express the matrices $\gamma^{\mu}$ in the spinor representation and the SL(2, C) group structure is made transparent. In section 5 we discuss how a spin frame in two dimensional spinor space can be defined in terms two legs like the four-legs or tetrads of vectors frequently used in Minskowski space. A set of null tetrad of vectors $\sigma^{\mu(A)(\dot{B})}$ is also constructed. Finally, in sections 5 and 6 we discuss in detail the representations of the improper Four group in spinor space together with the transformations of bilinear invariants.

## 2. NOTATION. REPRESENTATION OF LORENTZ GROUP BY CLIFFORD NUMBERS. SPINOR SPACE.

Homogeneous Lorentz group (H. L. G.) may be defined as the group of 4x4 real matrices $\{\Lambda\}$ which satisfy

$$\Lambda^{T} G \Lambda = G \qquad (1)$$

where [3] $\Lambda = (\Lambda^{\mu}{}_{\nu})$, $G = (g_{\mu\nu}) = (g_{\nu\mu})$, $(\Lambda^{T})^{\mu}{}_{\nu} = \Lambda^{\nu}{}_{\mu}$ with $\mu,\nu = 0,1,2,3$

and $g_{oo} = 1$, $g_{kk} = -1$, $k = 1,2,3$ $g_{\mu\nu} = 0$ for $\mu \neq \nu$. We will be mostly concerned [4] here with restricted H.L.G. - indicated simply as Lorentz group, for which

$$\Lambda^{o}{}_{o} > 1 \qquad \text{and} \quad \det \Lambda = +1 \tag{2}$$

The equation (2.1) written explicitly reads

$$(\Lambda^{T})^{\mu}{}_{\alpha} \; g_{\alpha\beta} \, \Lambda^{\beta}{}_{\nu} = g_{\alpha\beta} \, \Lambda^{\alpha}{}_{\mu} \, \Lambda^{\beta}{}_{\nu} = g_{\mu\nu} \tag{3}$$

Here the summation on repeated indices is understood.

The matrix group can be represented by the group of linear transformations on a four dimensional real linear vector space, called Minskowski space, with basis vectors $\underline{e}_{\mu}$ which transform as:

$$\underline{e}'_{\mu} \equiv L(\Lambda)\, \underline{e}_{\mu} = \Lambda^{\nu}{}_{\mu}\, \underline{e}_{\nu} \tag{4}$$

The contravariant components of a (real) vector $\underline{A}$ w.r.t. the basis $\{\underline{e}_{\mu}\}$, indicated by real components $A^{\mu}$,e.g., $\underline{A} = A^{\mu}\, \underline{e}_{\mu}$, transform as

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu}\, A^{\nu} \tag{5}$$

since

$$\underline{A}' \equiv L(\Lambda) A^{\nu}\, \underline{e}_{\nu} = A^{\nu}\, \Lambda^{\mu}{}_{\nu}\, \underline{e}_{\mu} \equiv A'^{\mu}\, \underline{e}_{\mu} \tag{6}$$

The group of contragradient matrices $\{\Lambda^{-1\,T}\}$ is isomorphic to the matrix group $\{\Lambda\}$. Indicating the basis vectors in the corresponding representation space by $\underline{e}^{\mu}$ it can be realized as a group of linear transformations defined by

$$\underline{e}'^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}\, \underline{e}^{\nu} \tag{7}$$

The covariant components $A_{\mu}$ of vector $\underline{A}$ w.r.t. this basis ($\underline{A} \equiv A_{\mu}\, \underline{e}^{\mu}$) transform as

$$A'_\mu = (\Lambda^{-1})^\nu{}_\mu A_\nu = (\Lambda^{-1T})^\mu{}_\nu A_\nu \tag{8}$$

Equation (2.1) implies

$$\Lambda^{T^{-1}} = G \Lambda G^{-1} \tag{9}$$

so that the contragradient representation is equivalent to the representation $\{\Lambda\}$. We note also from equations (2.5) and (2.8) that the Kronecker delta $\delta^\mu_\nu$ is an invariant tensor. From the fact that $\Lambda^{-1}$ is also a Lorentz transformation equation (2.3) implies $g_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu g_{\alpha\beta}$ which states that the indices $\mu$ and $\nu$ are covariant tensor indices and that $g_{\mu\nu}$ is an invariant tensor.

It is clear that $(g_{\mu\nu} \underline{e}^\nu)$ transforms like $\underline{e}_\mu$ for

$$(g_{\mu\nu} \underline{e}^\nu)' = g_{\mu\nu} \underline{e}'^\nu = g_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu (\Lambda^{-1})^\nu{}_\rho \underline{e}^\rho$$

$$= \Lambda^\alpha{}_\mu (g_{\alpha\rho} \underline{e}^\rho)$$

Thus we may define

$$\underline{e}_\mu = g_{\mu\nu} \underline{e}^\nu \tag{10}$$

which is an alternative statement of the equivalence expressed by equation (2.9). This leads to

$$A_\mu = g_{\nu\mu} A^\mu \tag{11}$$

In other words while the components $(A^0, A^1, A^2, A^3)$ transform by matrix $\Lambda$, the components $(A^0, -A^1, -A^2, -A^3)$ transform according to the matrix $(\Lambda^{-1T})$. The two representations are equivalent since the former can be obtained from the latter by a change of basis in the representation space according to equation (2.10).

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tion of the Lorentz group. The Clifford algebra in Minskowski space is defined by a set of four hypercomplex numbers $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ which satisfy the anti-commutation relation [7]

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu} \quad I \tag{18}$$

Any product of $\gamma$'s can be reduced to, using equation (2.18), to one of the 16 elements $I, \gamma^\mu, (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \gamma_5 \gamma^\mu, \gamma_5$ where $\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ Representing the $\gamma$'s by $(r \times r)$ matrices we can show that the 16 elements are linearly independent so that $r$ must be $\geqslant 4$. It also follows that the representation of the algebra by 4 x 4 matrices is irreducible. In the following $\gamma$'s will be regarded as (4x4) (irreducible) matrices.

It may be easily shown that the 6 elements $\Sigma^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ satisfy the commutation relations of the Lie algebra of the generators $M^{\mu\nu}$ of the homogeneous Lorentz group, viz,

$$\left[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\right] = i(g^{\mu\sigma}\, \Sigma^{\nu\rho} - g^{\mu\rho}\, \Sigma^{\nu\sigma} - g^{\nu\sigma}\, \Sigma^{\mu\rho} + g^{\nu\rho}\, \Sigma^{\mu\sigma}) \tag{19}$$

Thus we can obtain a representation by (4x4) complex matrices [9] $\{S(\Lambda)\}$ of Lorentz group in terms of Clifford numbers with

$$S(\Lambda) = e^{-i/2\, \omega_{\mu\nu}\, \Sigma^{\mu\nu}} \tag{20}$$

where [10] $\omega_{\mu\nu} = \Lambda_{\mu\nu} - g_{\mu\nu}$. The corresponding representation space is 4-dimensional complex vector space called <u>Spinor Space</u>. Equations (2.18) to (2.20) lead to

$$S^{-1}(\Lambda)\gamma^\mu\, S(\Lambda) = \Lambda^\mu{}_\nu\, \gamma^\nu \tag{21}$$

Denoting the basis vectors of the Spinor space by $\underline{Z}_a$ (a = 1,2,3,4), and the components of a vector $\underline{\xi}$ w.r.t. this basis by $\xi^a$ e.g., $\underline{\xi} = \xi^a\, \underline{Z}_a$, corresponding to a Lorentz transformation $\Lambda$ in Minskowski space the transformation in spinor space is defined to be the linear transformation given by

$$\underline{Z}'_a \equiv U(\Lambda)\underline{Z}_a = S(\Lambda)^b{}_a \, \underline{Z}_b \tag{22}$$

Here $S^a{}_b$ are the matrix elements of the matrix S. Group property of the transformations (or the operators $U(\Lambda)$ defined on spinor space) may be easily verified. The components $\xi^a$ are seen to transform as

$$\xi'^a \equiv (U(\Lambda)\underline{\xi})^a = S(\Lambda)^a{}_b \, \xi^b \tag{23}$$

The contragradient representation constituted by the group of matrices [11] $\left\{S^{-1T}(\Lambda)\right\}$ is realized on a representation space, whose basis vectors will be indicated by $\underline{Z}^a$. The group of linear operators acts according to

$$\underline{Z}'^a = (S^{-1}(\Lambda))^a{}_b \, \underline{Z}^b \tag{24}$$

and the components $\xi_a$ of a vector $\underline{\xi} = \xi_a Z^a$ transform as

$$\xi'_a = (S^{-1}(\Lambda))^b{}_a \, \xi_b = (S^{-1T}(\Lambda))^a{}_b \, \xi_b \tag{25}$$

We observe that $\xi^a \eta_a$ is an invariant under homogeneous Lorentz transformations.

The conjugate representation by the group of matrices $\{S^*(\Lambda)\}$ is realized on a space with basis vectors indicated by $\underline{Z}_{\dot{a}}$ with

$$\underline{Z}'_{\dot{a}} = S(\Lambda)^{\dot{b}}{}_{\dot{a}} \, \underline{Z}_{\dot{b}} \; ; \qquad S^{\dot{b}}{}_{\dot{a}} = (S^b{}_a)^* \tag{26}$$

and

$$\xi'^{\dot{a}} = S(\Lambda)^{\dot{a}}{}_{\dot{b}} \, \xi^{\dot{b}} \tag{27}$$

where $\xi^{\dot{a}}$ are components of $\underline{\xi}$ w.r.t. the basis $\underline{Z}_{\dot{a}}$.

The representation contragradient to the conjugate one is realized on a vector space with basis vectors denoted by $\underline{Z}^{\dot{a}}$ with

$$\underline{Z}'^{\dot{a}} = (S^{-1}(\Lambda))^{\dot{a}}{}_{\dot{b}}\, \underline{Z}^{\dot{b}} \tag{28}$$

and

$$\xi'_{\dot{a}} = (S^{-1}(\Lambda))^{\dot{b}}{}_{\dot{a}}\, \xi_{\dot{b}} \tag{29}$$

3. INVARIANT TENSORS

It will be shown below that all these representation, in the present case are equivalent to each other and there is essentially one irreducible representation. However, it is convenient to work with upper, lower, dotted and undotted indices (just as in the case of Minskowski space).

The equation (2.20) can be written explicitly as $((\gamma^{\mu})^{a}{}_{b} \equiv \gamma^{\mu a}{}_{b})$:

$$\gamma^{\mu a}{}_{b} = \Lambda^{\mu}{}_{\nu}\, S(\Lambda)^{a}{}_{c}\, (S^{-1}(\Lambda))^{d}{}_{b}\, \gamma^{\nu\, c}{}_{d} \tag{1}$$

The "mixed quantities" $\gamma^{\mu a}{}_{b}$ thus are invariant or held fixed under the Lorentz transformation of the indices defined above and the tacit assumption made that the index 'μ' in $\gamma^{\mu}$ is a space time contravariant index is consistently assigned. Since $\Lambda^{-1}$ is a Lorentz transformation we also have:

$$\gamma^{\mu a}{}_{b} = (\Lambda^{-1})^{\mu}{}_{\nu}\, (S^{-1}(\Lambda))^{a}{}_{c} (S(\Lambda))^{b}{}_{d}\, \gamma^{\nu c}{}_{d} \tag{2}$$

Taking the complex conjugate of equation (3.1)

$$\gamma^{\mu \dot{a}}{}_{\dot{b}} = \Lambda^{\mu}{}_{\nu}\, S(\Lambda)^{\dot{a}}{}_{\dot{c}}\, (S^{-1}(\Lambda))^{\dot{d}}{}_{\dot{b}}\, \gamma^{\nu \dot{c}}{}_{\dot{d}} \tag{3}$$

It may be remarked that the Kronecker deltas $\delta^a{}_b$, $\delta^{\dot a}{}_{\dot b}$ are also invariant tensors.

The equivalence of the representations indicated above follow from

$$\{\gamma^{\mu T}, \gamma^{\nu T}\}_+ = \{\gamma^{\mu *}, \gamma^{\nu *}\}_+ = \{\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\}_+ = 2\, g^{\mu\nu}\, I \tag{4}$$

which from the fundamental lemma [8] assures the existence of non-singular matrices A, B, C such that [12]

$$\begin{aligned} A\,\gamma^\mu A^{-1} &= \gamma^{\mu\dagger} \\ B\,\gamma^\mu B^{-1} &= \gamma^{\mu T} \\ C\,\gamma^\mu C^{-1} &= \gamma^{\mu *} \end{aligned} \tag{5}$$

One can show then

$$B\,S(\Lambda)\,B^{-1} = S^{-1T}(\Lambda) \quad \text{or} \quad B^{-1} = S(\Lambda)\,B^{-1}\,S^T(\Lambda) \tag{6}$$

$$A\,S(\Lambda)A^{-1} = S^{\dagger -1}(\Lambda) \tag{7}$$

$$C\,S(\Lambda)C^{-1} = S^*(\Lambda) \tag{8}$$

We can write equation (3.6) explicitly as:

$$(S^{-1T})^a{}_c\, B_{c\,d}\, (S^{-1})^d{}_b = (S^{-1})^c{}_a\, B_{cd}\, (S^{-1})^d{}_b = B_{ab}$$

or

$$\begin{aligned} B_{ab} &= (S^{-1}(\Lambda))^c{}_a\, (S^{-1}(\Lambda))^d{}_b\, B_{cd} \\ &= S(\Lambda)^c{}_a\, S(\Lambda)^d{}_b\, B_{cd} \end{aligned} \tag{9}$$

where the matrix $B \equiv (B_{ab})$. This relation shows that $B_{ab}$ is an invariant tensor with a and b transforming as covariant indices. B matrix plays the role of metric tensor in spinor space. Since $B_{ab}\underline{Z}^b$ transforms as $\underline{Z}_a$ we may define

$$\underline{Z}_a = B_{ab}\, Z^b \tag{10}$$

which implies

$$\xi_a = B_{ba}\, \xi^b \tag{11}$$

It may be shown that B may be chosen unitary and antisymmetric.[13] Metric tensor $B^{ab}$ can be introduced by

$$B_{ab} = B_{ca}\, B_{db}\, B^{cd} \tag{12}$$

so that ($B_{ab} = -B_{ba}$)

$$B_{db}\, B^{cd} = -\,\delta^c{}_b$$

$$B_{ca}\, B^{cd} = \delta^d{}_a \tag{13}$$

and $B^{ab} = -B^{ba}$ as expected. Also if $B \equiv (B_{ab})$ then $(B^{ab}) = -B^{-1}$. We may also choose a representation of $\gamma$ matrices such that B is a real matrix, then, $B^* = B = -B^T = -B^{-1}$ and $(B^{ab}) = B$. Using equation (3.13) we have

$$\xi^a = B^{ab}\, \xi_b \qquad \text{and} \qquad \underline{Z}^a = -\,B^{ab}\, \underline{Z}_b$$

against $\xi_a = -B_{ab}\, \xi^b$ and $\underline{Z}_a = B_{ab}\, \underline{Z}^b$.

We may define the inner product between two vectors $\underline{\xi}$ and $\underline{\eta}$ by

$$\underline{\xi}\cdot\underline{\eta} = \xi_a\, \eta^a = B_{ab}\, \xi^a \eta^b = B^{ab}\, \xi_a\, \eta_b \tag{15}$$

From $\underline{Z}_a = \delta^b{}_a\, \underline{Z}_b$ etc. it follows [14]

$$\underline{Z}_a\cdot\underline{Z}_b = B_{ab} \qquad \underline{Z}^a\cdot\underline{Z}^b = B^{ab}$$

$$\underline{Z}^a\cdot\underline{Z}_b = \delta^a{}_b = -\,\underline{Z}_a\cdot\underline{Z}^b \tag{16}$$

Other properties of inner product are

$$\underline{\xi}\cdot\underline{\eta} = -\,\underline{\eta}\cdot\underline{\xi}$$

$$(\alpha\underline{\xi})\cdot\underline{\eta} = \underline{\xi}\cdot(\alpha\underline{\eta}) = \alpha\,\underline{\xi}\cdot\underline{\eta}$$

$$(\underline{\xi}_1 + \underline{\xi}_2)\cdot\underline{\eta} = \underline{\xi}_1\cdot\underline{\eta} + \underline{\xi}_2\cdot\underline{\eta}\,, \qquad \underline{\xi}\cdot(\underline{\eta}_1+\underline{\eta}_2) = \underline{\xi}\cdot\underline{\eta}_1 + \underline{\xi}\cdot\underline{\eta}_2 \tag{17}$$

$$(\underline{\xi}\cdot\underline{\xi} \equiv \xi^a\xi_a = 0 \qquad \text{for all } \underline{\xi}\ )$$

and $\underline{\xi}\cdot\underline{\eta} = 0$ for all $\underline{\eta}$ implies $\underline{\xi} \equiv 0$. The representation space is called Symplectic space and the transformations $S(\Lambda)$ leave invariant the non-degenerate skew symmetric bilinear form given by equation (3.15).

An exactly similar discussion can be carried out for conjugate and its contragradient representations. Since $(S^{-1\,T})^* = B^*S^*B^{*-1}$ the invariant metric tensors are $B_{\dot a\dot b}$ and $B^{\dot a\dot b}$ where $(B_{\dot a\dot b}) = B^*$ and $(B^{\dot a\dot b}) = -B^{*-1}$ which for real matrix B are the same as $B_{ab}$ and $B^{ab}$. We observe that $\xi^{\dot a}\,\eta_{\dot a}$ is an invariant but $\xi^{\dot a}\,\eta_a$ is not so and that $\xi^{\dot a}$ transforms like $\xi^{a*}$ while $\xi_{\dot a} = B_{\dot a\dot b}\,\xi^{\dot b}$ transforms like $\xi^*_a$.

We consider now the equivalence relation of equation (3.7). It can be written explicitly as:

$$(S^{-1T})^{\dot a}{}_{\dot c}\, A_{\dot c d}(S^{-1})^d{}_b = (S^{-1})^{\dot c}{}_{\dot a}\, A_{\dot c d}(S^{-1})^d{}_b = A_{\dot a b}$$

or

$$A_{\dot a b} = (S^{-1}(\Lambda))^{\dot c}{}_{\dot a}\,(S^{-1}(\Lambda))^d{}_b\, A_{\dot c d} \tag{18}$$

where we write $A \equiv (A_{\dot a b})$. This relation shows that $A_{\dot a b}$ is an invariant tensor with one dotted and another undotted covariant index. Taking the complex conjugate we obtain invariant tensor $A_{a\dot{b}}$. Raising the indices by metric tensors we obtain invariant tensors [15] $A^{a\dot{b}}$, $A^{\dot{a}b}$, $A^{\dot{a}}{}_{b}$, $A^{a}{}_{\dot{b}}$. etc. It is clear that they are useful in constructing invariants of type $\xi^{\dot{a}}\ A_{\dot{a}b}\ \eta^{b}$ and of type $\xi^{a*} A_{\dot{a}b}\ \xi^{b}$ which may not vanish in contrast to $\xi^{a}\ \xi_{a} = 0$. We may choose [8] A to be hermitian e.g. $A_{a\dot{b}} = A_{\dot{b}a} = (A_{b\dot{a}})^{*}$ and $A^{2} = I$.

Other invariant hermitian tensors are [16] $\Gamma^{\mu} = (\Gamma^{\mu}_{\dot{a}b}) = (A_{\dot{a}c}\ \gamma^{\mu c}{}_{b})$, $A\ \Sigma^{\mu\nu} \equiv (\Gamma^{\mu\nu}_{\dot{a}b})$, $A\ \gamma_{5} \equiv (\Gamma_{5\dot{a}b})$, $(iA\ \gamma_{5}\gamma^{\mu}) = (\Gamma^{\mu}_{5\dot{a}b})$. For example, $\Gamma^{\mu}_{b\dot{a}} = (\Gamma^{\mu}_{\dot{b}a})^{*} = (A_{\dot{b}c}\ \gamma^{\mu c}{}_{a})^{*} = A_{b\dot{c}}\ \gamma^{\mu\dot{c}}{}_{\dot{a}} = \gamma^{\mu\dot{c}}{}_{\dot{a}}\ A_{\dot{c}b} = (\gamma^{\mu\dagger}\ A)_{\dot{a}b} = (A\ \gamma^{\mu})_{\dot{a}b} = \Gamma^{\mu}_{\dot{a}b}$ since from equation (3.5) $A\ \gamma^{\mu} = \gamma^{\mu\dagger}\ A$. Tensor quantities may be constructed from a quantity like $\eta^{\dot{a}b}$ e.g. scalar $A_{\dot{a}b}\ \eta^{\dot{a}b}$, pseudoscalar $\Gamma_{5\dot{a}b}\ \eta^{\dot{a}b}$, four-vector $\Gamma^{\mu}_{\dot{a}b}\ \eta^{\dot{a}b}$, pseudo four-vector $\Gamma^{\mu}_{5\dot{a}b}\ \eta^{\dot{a}b}$, antisymmetric tensor $\Gamma^{\mu\nu}_{\dot{a}b}\ \eta^{\dot{a}b}$. [17]

In particular, reminding that $\xi^{\dot{a}}$ transforms as $\xi^{a*}$ we have the well known bilinear covariants $\xi^{*a}\ A_{\dot{a}b}\ \xi^{b} \equiv \xi^{\dagger}\ A\xi$, $\xi^{\dagger}\ A\ \gamma^{\mu}\ \xi = \xi^{\dot{a}*}\ \Gamma^{\mu}_{\dot{a}b}\ \xi^{b}$, $\xi^{\dagger}\ A\ \gamma_{5}\ \gamma^{\mu}\ \xi$, $\xi^{\dagger}\ A\ \Sigma^{\mu\nu}\ \xi$ and $\xi^{\dagger}\ A\ \gamma_{5}\ \xi$ transforming as scalar, vector, pseudo-vector,antisymmetric tensor, and pseudo-scalar in Minskowski space.

## 4. SPINOR REPRESENTATION OF $\gamma$ MATRICES. TWO SPINORS

To bring out clearly the relationship of the 2-spinor calculus with the 4-spinor calculus discussed above we use a convenient matrix representation for traceless $\gamma$ matrices.

We take:

$$\gamma^{o\dagger} = \gamma^{o} \qquad \gamma^{k\dagger} = -\gamma^{k}, \qquad k = 1, 2, 3 \tag{1}$$

so that $\gamma_5 \equiv \gamma^o \gamma^1 \gamma^2 \gamma^3 = -\gamma_5^\dagger$; $\gamma_5^2 = -I$. Clearly $\gamma^o \gamma^\mu \gamma^o = \gamma^{\mu\dagger}$ so that we may identify $A \equiv (A_{a\dot{b}}) = \gamma^o = A^\dagger = A^{-1}$. Further we will take [18] $\gamma^\mu$ to be odd matrices so that $\Sigma^{\mu\nu}$, hence, $S(\Lambda)$. are even. A suitable representation is spinor representation defined by

$$\gamma^o = \begin{pmatrix} 0 & \sigma^o \\ \sigma^o & 0 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \qquad \gamma_5 = i \begin{pmatrix} \sigma^o & 0 \\ 0 & -\sigma^o \end{pmatrix} \tag{2}$$

where $\sigma^k$ are the Pauli 2x2 matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note also for this representation $\gamma^{oT} = \gamma^o$, $\gamma^{2T} = \gamma^2$, $\gamma^{1T} = -\gamma^1$, $\gamma^{3T} = -\gamma^3$ and $\gamma_5^T = \gamma_5$. A real matrix B satisfying equation ( 3.5 ) can be taken to be ($B = B^* = -B^T = -B^{-1}$):

$$B = -\gamma_5 \gamma^o \gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \tag{3}$$

so that

$$(B_{ab}) = (B^{ab}) = (B_{\dot{a}\dot{b}}) = (B^{\dot{a}\dot{b}}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4}$$

Also

$$\omega_{\rho\sigma}\,\Sigma^{\rho\sigma} = \begin{pmatrix} \vec{\sigma}(\vec{a} - i\vec{b}) & 0 \\ 0 & \vec{\sigma}\cdot(\vec{a} + ib) \end{pmatrix} \tag{5}$$

where $b^k = \frac{1}{2}(\omega_{ok} - \omega_{ko})$; $a^k = \frac{1}{2}(\omega_{\ell m} - \omega_{m\ell})$ k, ℓ, m cyclic and $\vec{\sigma}\cdot\vec{a} \equiv \equiv \sigma^1 a^1 + \sigma^2 a^2 + \sigma^3 a^3$. For $S(\Lambda)$ we find

$$S(\Lambda) = \begin{pmatrix} S_1(\Lambda) & 0 \\ 0 & S_1^{\dagger-1}(\Lambda) \end{pmatrix} \tag{6}$$

where

$$S_1(\Lambda) = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{a}-i\vec{b})} \tag{7}$$

The representation is unitary for space rotations but is in general non-unitary. In so far as only restricted Lorentz group is concerned $S(\Lambda)$ appears as direct sum of two-dimensional representations.

The (2x2) matrix groups $\{S_1(\Lambda)\}$ and $\{S_1^{\dagger-1}(\Lambda)\}$ themselves constitute [19] two inequivalents representations of the Lorentz group. Under parity transformation, we will see below, the two get interchanged so that representation is irreducible under full Lorentz group. We note det $(S_1(\Lambda)) = +1$ so that $\{S_1(\Lambda)\}$ and $\{S_1^{\dagger-1}(\Lambda)\}$ constitute the two inequivalent representations of SL(2,C) group. [20]

It is clear the two upper components $(\xi^1,\xi^2)$ of $\xi^a$ transform, under a Lorentz transformation, among themselves according to (2x2) matrix $S_1(\Lambda)$ while the lower components $(\xi^3, \xi^4)$ according to $S_1^{\dagger-1}(\Lambda)$. A change of notation is thus suggested:

$$u^1 \equiv \xi^1 \qquad u^2 \equiv \xi^2 \qquad v_{\dot{1}} \equiv \xi^3 \qquad v_{\dot{2}} \equiv \xi^4 \tag{8}$$

$$u'^A = S_1(\Lambda)^A{}_B \; u^B$$
$$v'_{\dot{A}} = (S_1^{-1}(\Lambda))^{\dot{B}}{}_{\dot{A}} \; v_{\dot{B}} \tag{9}$$

where $A, B = 1, 2$ and $\dot{A}, \dot{B} = \dot{1}, \dot{2}$. Also $\xi_1 = -\xi^2 = -u^2$, $\xi_2 = \xi^1 = u^1$, $\xi_3 = -\xi^4 = -v_{\dot{2}}$ $\xi_4 = \xi^3 = v_{\dot{1}}$ and $(\xi_1, \xi_2)$ transform according to matrix $S_1^{-1T}(\Lambda)$ while $(\xi_3, \xi_4)$ according to the matrix $S_1^*(\Lambda)$. We may thus introduce the notation [21].

$$u_1 \equiv \xi_1 \qquad u_2 \equiv \xi_2 \qquad -v^{\dot{1}} \equiv \xi_3 \qquad -v^{\dot{2}} \equiv \xi_4 \tag{10}$$

so that

$$u_A = -\,\varepsilon_{AB}\, u^B \,, \qquad v_{\dot{A}} = -\,\varepsilon_{\dot{A}\dot{B}}\, v^{\dot{B}} \tag{11}$$

where $\varepsilon_{AB}$, $\varepsilon_{\dot{A}\dot{B}}$ are Levi-oevitta symbols [22] and $u'_A = (S_1^{-1}(\Lambda))^B{}_A\, u_B$, $u'^{\dot{A}} = (S_1(\Lambda))^{\dot{A}}{}_{\dot{B}}\, u^{\dot{B}}$. We remark that the invariant tensor B is an even matrix in our representation

$$B = (B_{ab}) = \left(\begin{array}{c|c} (\varepsilon_{AB}) & 0 \\ \hline 0 & (\varepsilon^{\dot{A}\dot{B}}) \end{array}\right) \quad ; \quad (B^{ab}) = \left(\begin{array}{c|c} (\varepsilon^{AB}) & 0 \\ \hline 0 & (\varepsilon_{\dot{A}\dot{B}}) \end{array}\right) \tag{12}$$

where $(\varepsilon_{AB}) = (\varepsilon^{AB}) = (\varepsilon_{\dot{A}\dot{B}}) = (\varepsilon^{\dot{A}\dot{B}}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. From the identities like [23]

$$\varepsilon^{CD} S_{1\ C}^{A} S_{1\ D}^{B} = (\det S_1)\, \varepsilon^{AB} = \varepsilon^{AB} \qquad (13)$$

if det $S_1 = 1$ we see that $\varepsilon^{AB}$, $\varepsilon_{AB}$, $\varepsilon^{\dot{A}\dot{B}}$, $\varepsilon_{\dot{A}\dot{B}}$ are invariant tensors. Hence equation (4.11) expresses the equivalence of representation $S_1$ with $(S_1^{T-1})$ and $S_1^*$ with $S_1^{-1\dagger}$. Since $\xi^{\overset{*}{\dot{a}}}$ transforms like $\xi^{\dot{a}}$ we see that $u^{\overset{*}{\dot{A}}}$ transforms like $u^{\dot{A}}$ while $v_{\overset{*}{\dot{A}}}$ transforms like $v_A$. We will adopt the customary practice of identifying $u^{\dot{A}} = u^{*A}$ and $u_{\dot{A}} = u^{*}_{A}$. There is no invariant quantity (like $A_{\dot{a}b}$) which related the dotted and undotted components since the conjugate representation $(u^{\dot{A}})$ is not equivalent to the representation $(u^A)$. Likewise we define the basis vectors $\underline{h}$ by

$$\underline{h}_1 \equiv Z_1\,, \qquad \underline{h}_2 \equiv \underline{Z}_2\,, \qquad \underline{h}^{\dot{1}} \equiv \underline{Z}_3 \qquad \underline{h}^{\dot{2}} \equiv \underline{Z}_4 \qquad (14)$$

and

$$-\underline{h}^1 \equiv \underline{Z}^1 = -\underline{Z}_2 = -\underline{h}_2\,, \qquad -\underline{h}^2 \equiv \underline{Z}^2 = \underline{Z}_1 = \underline{h}_1\,,$$

$$\underline{h}_{\dot{1}} = \underline{Z}^3 = -\underline{Z}_4 = -\underline{h}^{\dot{2}}\,, \qquad \underline{h}_{\dot{2}} = \underline{Z}^4 = \underline{Z}_3 = \underline{h}^{\dot{1}} \qquad (15)$$

so that

$$\underline{\xi} = u^A \underline{h}_A + v_{\dot{A}} \underline{h}^{\dot{A}} = -u_A \underline{h}^A - v^{\dot{A}} \underline{h}_{\dot{A}} \qquad (16)$$

and

$$\underline{h}_A = -\varepsilon_{AB}\, \underline{h}^B\,, \qquad \underline{h}_{\dot{A}} = -\varepsilon_{\dot{A}\dot{B}}\, \underline{h}^{\dot{B}} \qquad (17)$$

Under a Lorentz transformation the basis vectors $\underline{h}$ transform as

$$\begin{aligned}
\underline{h}'_A &= S_1(\Lambda)^B{}_A\, \underline{h}_B \\
\underline{h}'^A &= (S_1^{-1}(\Lambda))^A{}_B\, \underline{h}^B \\
\underline{h}'_{\dot{A}} &= (S_1(\Lambda))^{\dot{B}}{}_{\dot{A}}\, \underline{h}_{\dot{B}} \\
\underline{h}'^{\dot{A}} &= (S_1^{-1}(\Lambda))^{\dot{A}}{}_{\dot{B}}\, \underline{h}^{\dot{B}}
\end{aligned} \qquad (18)$$

From

$$\varepsilon_{AB}\,\varepsilon^{BC} = -\,\delta^{C}{}_{A}\,, \qquad \varepsilon_{\dot{A}\dot{B}}\,\varepsilon^{\dot{B}\dot{C}} = -\,\delta^{\dot{C}}_{\dot{A}} \tag{19}$$

it follows

$$u^{A} = \varepsilon^{AB}\,u_{B}\,, \qquad \underline{h}^{A} = \varepsilon^{AB}\,\underline{h}_{B} \tag{20}$$

and similar expression for dotted indices.

From equation (3.16 ) we find the following inner products

$$\underline{h}_{A}\cdot\underline{h}^{B} = -\underline{h}^{A}\cdot\underline{h}_{B} = \delta^{A}_{B} = \underline{h}_{\dot{A}}\cdot\underline{h}^{\dot{B}} = -\underline{h}^{\dot{A}}\cdot\underline{h}_{\dot{B}}$$

$$\underline{h}_{A}\cdot\underline{h}^{\dot{B}} = \underline{h}_{\dot{A}}\cdot\underline{h}^{B} = \underline{h}^{A}\cdot\underline{h}_{\dot{B}} = \underline{h}^{\dot{A}}\cdot\underline{h}_{B} = 0 \tag{21}$$

and

$$\underline{h}_{A}\cdot\underline{h}_{B} = \varepsilon_{AB} = \varepsilon^{AB} = \underline{h}^{A}\cdot\underline{h}^{B}$$

$$\underline{h}_{\dot{A}}\cdot\underline{h}_{\dot{B}} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \underline{h}^{\dot{A}}\cdot\underline{h}^{\dot{B}} \tag{22}$$

The vector spaces generated by undotted and dotted basis vectors are orthogonal. The inner product of two vectors $\underline{u}$ and $\underline{\phi}$ in undotted space is

$$\underline{u}\cdot\underline{\phi} = u^{A}\underline{h}_{A}\cdot\phi^{B}\,\underline{h}_{B} = \varepsilon_{AB}\,u^{A}\,\phi^{B} = \varepsilon^{AB}\,u_{A}\,\phi_{B} = -\,u^{A}\,\phi_{A} = u_{A}\,\phi^{A} \tag{23}$$

and satisfies properties given in equation (3.17). The representation space is a symplectic space Sp(2) in two dimensions. The same goes for the dotted vector space and the linear independence of basis vectors follows from equation (4.22). We remark that $u^{A}\,\phi_{A} = -u_{A}\,\phi^{A}$ and $u^{A}\,u_{A} = 0$.

In the spinor representation the equation (2.21) leads to [24]

$$\sigma^{\mu} = \Lambda^{\mu}{}_{\nu}\,S_{1}(\Lambda)\,\sigma^{\nu}\,S_{1}{}^{\dagger}(\Lambda) \tag{24}$$

from which it follows that

$$\sigma^{\mu} \equiv (\sigma^{\mu\ A\dot{B}}) \tag{25}$$

and

$$\sigma^{\mu A\dot{B}} = \Lambda^{\mu}{}_{\nu}\ S_1(\Lambda)^{A}{}_{C}\ S_1(\Lambda)^{\dot{B}}{}_{\dot{D}}\ \sigma^{\nu C\dot{D}} \tag{26}$$

showing that $\sigma^{\mu A\dot{B}}$ defined by equation (4.25) is an invariant mixed quantity like $(\gamma^{\mu a}{}_{b})$. On lowering the indices by invariant metric tensor $\varepsilon_{AB}$ and using

$$\varepsilon_{AC}\ S_1(\Lambda)^{C}{}_{D}\ \varepsilon^{DB} = -\ (S_1^{-1}(\Lambda))^{B}{}_{A} = -S_1(\Lambda^{-1})^{B}{}_{A} \tag{27}$$

we show

$$\sigma^{\mu}_{A\dot{B}} = \Lambda^{\mu}{}_{\nu}\ (S_1^{-1}(\Lambda))^{C}{}_{A}\ (S_1^{-1}(\Lambda))^{\dot{D}}{}_{\dot{B}}\ \sigma^{\nu}{}_{C\dot{D}} \tag{28}$$

Both $(\sigma^{\mu A\dot{B}})$ and $(\sigma^{\mu}_{A\dot{B}})$ are hermitian matrices, that is $\sigma^{\mu\ A\dot{B}} = \sigma^{\mu\dot{B}A}$, $\sigma^{\mu}_{A\dot{B}} = \sigma^{\mu}_{\dot{B}A} \equiv (\sigma^{\mu}{}_{B\dot{A}})^{*}$. From $(\sigma^{\mu}{}_{A\dot{B}}) = (\varepsilon_{AC}\ \ \varepsilon_{\dot{B}\dot{D}}\ \sigma^{\mu C\dot{D}}) = -\ (\varepsilon\ \sigma^{\mu}\ \varepsilon)$ we see $(\sigma^{o}_{A\dot{B}}) = \sigma^{o}$, $(\sigma^{1}{}_{A\dot{B}}) = -\sigma^{1}$, $(\sigma^{2}{}_{A\dot{B}}) = \sigma^{2}$, $(\sigma^{3}{}_{A\dot{B}}) = -\ \sigma^{3}$ .so that

$$(\sigma^{o}{}_{A\dot{B}}) = \sigma^{o};\quad (\sigma^{k}{}_{A\dot{B}}) = (\sigma^{k}{}_{A\dot{B}})^{*} = -\ \sigma^{k}\ , \qquad k = 1,\ 2,\ 3 \tag{29}$$

The matrices $(\sigma^{\mu A}{}_{\dot{B}})$ are not all hermitian contrary to $(\sigma^{\mu A\dot{B}})$, $(\sigma^{\mu}_{A\dot{B}})$ and $(\sigma^{\mu}{}_{A\dot{B}})$. It can be shown easily

$$\sigma^{\mu A\dot{B}}\ \sigma^{\nu}{}_{\dot{B}C} + \sigma^{\nu A\dot{B}}\ \sigma^{\mu}{}_{\dot{B}C} = 2\ g^{\mu\nu}\ \ \delta^{A}_{C} \tag{30}$$

from which follows

$$\sigma^{\mu\ A\dot{B}}\ \sigma^{\nu}{}_{A\dot{B}} = 2\ g^{\mu\nu} \tag{31}$$

Then

$$\sigma_{\mu\ C\dot{D}}\ \ \sigma^{\mu A\dot{B}}\ \sigma^{\nu}{}_{A\dot{B}} = 2\ g^{\mu\nu}\ \sigma_{\mu C\dot{D}} = 2\ \sigma^{\nu}\ {}_{C\dot{D}}$$

which implies [25]

$$\sigma_{\mu C\dot{D}}\, \sigma^{\mu A\dot{B}} = 2\, \delta^{A}{}_{C}\, \delta^{\dot{B}}{}_{\dot{D}} \tag{32}$$

Other similar relations follow by raising a lowering the indices and taking complex conjugation. From equations (4.26) and (4.31) we have

$$2g^{\mu\nu} = \Lambda^{\mu}{}_{\rho}(S_1(\Lambda))^{A}{}_{C}\, (S_1(\Lambda))^{\dot{B}}{}_{\dot{D}}\, \sigma^{\rho C\dot{D}}\, \sigma^{\nu}{}_{A\dot{B}} \tag{33}$$

or

$$\Lambda^{\nu\alpha} \equiv g^{\mu\nu}(\Lambda^{-1})^{\alpha}{}_{\mu} = \frac{1}{2}\, S_1(\Lambda)^{A}{}_{C}\, S_1(\Lambda)^{\dot{B}}{}_{\dot{D}}\, \sigma^{\alpha C\dot{D}}\, \sigma^{\nu}{}_{A\dot{B}}$$

so that

$$\Lambda^{\mu}_{\nu} = \frac{1}{2}\, S_1(\Lambda)^{A}{}_{C}\, S_1(\Lambda)^{\dot{B}}{}_{\dot{D}}\, \sigma_{\nu}{}^{C\dot{D}}\, \sigma^{\mu}{}_{A\dot{B}} \tag{34}$$

$$= \frac{1}{2}\, g_{\alpha\nu}\, \mathrm{Tr}\, \{(\sigma^{\mu}{}_{\dot{B}A})\, S_1(\Lambda)\, \sigma^{\alpha}\, S_1^{\dagger}(\Lambda)\}$$

The following explicit form of $\gamma^{\mu}$ will be useful latter.

$$(\gamma^{\mu a}{}_{b}) = \left(\begin{array}{c|c} 0 & (\sigma^{\mu A\dot{B}}) \\ \hline (\sigma^{\mu}{}_{\dot{A}B}) & 0 \end{array}\right) \tag{35}$$

$$(A_{\dot{a}b}\, \gamma^{\mu b}{}_{c}) = \left(\begin{array}{c|c} (\sigma^{\mu}{}_{\dot{A}B}) & 0 \\ \hline 0 & (\sigma^{\mu A\dot{B}}) \end{array}\right) \tag{36}$$

The results obtained here are the main tools of 2-spinor calculus discussed in references 1 and 2.

Eqn. (4.26) shows that $\sigma^{\mu A\dot{B}} u_{A\dot{B}}$ transforms as a 4-vector in Minskowski space while, for a vector $U^{\mu}$, $(U_{\mu}\, \sigma^{\mu}{}_{A\dot{B}})$ transforms as $u_{A\dot{B}}$. Thus we may establish a correspondence between $u_{A\dot{B}}$ and a 4-vector $U^{\mu}$ by the relation [26]

$$U^{\mu} = \frac{1}{2}\, \sigma^{\mu A\dot{B}}\, u_{A\dot{B}}$$
$$u_{A\dot{B}} = \sigma_{\mu A\dot{B}}\, U^{\mu} \tag{37}$$

$U^{\mu}$ is real if $u_{A\dot{B}}$ is hermitian, it is a null vector if $u_{A\dot{B}} = \xi_A\, \eta_{\dot{B}}$ and a real null vector if $u_{A\dot{B}} = \pm\, \xi_A\, \xi_{\dot{B}}$ (e.g. a 2-spinor $\xi_A$ determines a real null vector). Since $u^{A\dot{B}}$ are component of a vector in the direct product spinor space spanned by $\{\underline{h}_A \otimes \underline{h}_{\dot{B}}\}$

$$\underline{u} \equiv u^{A\dot{B}}(\underline{h}_A \otimes \underline{h}_{\dot{B}}) = U_{\mu}\, \sigma^{\mu\, A\dot{B}}(\underline{h}_A \otimes \underline{h}_{\dot{B}}) \equiv U_{\mu}\, \underline{E}^{\mu} \tag{38}$$

where

$$\underline{E}^{\mu} = \sigma^{\mu A\dot{B}}\, (\underline{h}_A \otimes \underline{h}_{\dot{B}}) \tag{39}$$

constitute a basis for the representation of the Lorentz group. In fact

$$\underline{E}'^{\mu} = (\Lambda^{-1\mu}{}_{\nu}\, \sigma^{\nu C\dot{D}}\, S_1^{-1}(\Lambda)^{A}{}_{C}\, S_1^{-1}(\Lambda)^{\dot{B}}{}_{\dot{D}})\, S_1(\Lambda)^{E}{}_{A}\, S_1(\Lambda)^{\dot{F}}{}_{\dot{B}}\, (\underline{h}_E \otimes \underline{h}_{\dot{F}})$$
$$= \Lambda^{-1\mu}{}_{\nu}\, \sigma^{\nu C\dot{D}}\, \underline{h}_C \otimes \underline{h}_{\dot{D}} = (\Lambda^{-1})^{\mu}{}_{\nu}\, \underline{E}^{\nu} \tag{40}$$

thus establishing the correspondence $\underline{e}^{\mu} \longleftrightarrow \underline{E}^{\mu}$. Explicitly,

$$\underline{E}^{0} = (\underline{h}_1 \otimes \underline{h}_{\dot{1}} + \underline{h}_2 \otimes \underline{h}_{\dot{2}}), \qquad \underline{E}^{1} = (\underline{h}_2 \otimes \underline{h}_{\dot{1}} + \underline{h}_1 \otimes \underline{h}_{\dot{2}})$$
$$\underline{E}^{2} = -i(\underline{h}_1 \otimes \underline{h}_{\dot{2}} - \underline{h}_2 \otimes \underline{h}_{\dot{1}}), \qquad \underline{E}^{3} = (\underline{h}_1 \otimes \underline{h}_{\dot{1}} - \underline{h}_2 \otimes \underline{h}_{\dot{2}}) \tag{41}$$

The inner product is found to be

$$\underline{E}^{\mu} \cdot \underline{E}^{\nu} = \sigma^{\mu A\dot{B}}\, \sigma^{\nu C\dot{D}}\, (\underline{h}_A \cdot \underline{h}_C)(\underline{h}_{\dot{B}} \cdot \underline{h}_{\dot{D}})$$

$$= \sigma^{\mu A\dot{B}}\, \sigma^{\nu C\dot{D}}\, \varepsilon_{AC}\, \varepsilon_{\dot{B}\dot{D}} = \sigma^{\mu A\dot{B}}\, \sigma^{\nu}{}_{A\dot{B}} = 2g^{\mu\nu} \tag{42}$$

5. SPIN FRAME

The expression in equation 4.37 reminds us of tetrad formalism frequently used in general relativity. The formalism is useful for discussion in Riemannian space where the metric tensor $g^{\mu\nu}$ becomes function of space time coordinates while at the same time we introduce local cartesian frame of reference at each point in space-time. The tetrads or four legs then connect the world component $A^{\mu}$ with local components $A^{(\mu)}$. We will limit ourselves to the discussion in which the metric tensors remains constant independent of the coordinates. The discussion in 2-dimensional spinor space goes in close analogy to the case of 4-dimensional Minskowski space which we first review briefly.

Consider four vectors $\underline{n}_{(\alpha)}$, $(\alpha) = (0), (1), (2), (3)$ such that

$$\underline{n}_{(\alpha)} \cdot \underline{n}_{(\beta)} = g_{(\alpha)(\beta)} \tag{1}$$

where $g_{(\alpha)(\alpha)} = (1, -1, -1, -1)$, $g_{(\alpha)(\beta)} = 0$ for $(\alpha) \neq (\beta)$ e.g. $\underline{n}_0$ is time-like and $\underline{n}_{(1)}, \underline{n}_{(2)}, \underline{n}_{(3)}$ are space-like. They are clearly linearly independent and we may write

$$\underline{A} = A^{(\alpha)}\, \underline{n}_{(\alpha)} = A_{(\alpha)}\, \underline{n}^{(\alpha)} \tag{2}$$

where we define $g^{(\alpha)(\beta)} \equiv g_{(\alpha)(\beta)}$ and $A^{(\alpha)} = g^{(\alpha)(\beta)}\, A_{(\beta)}$. We expand

$\underline{n}_{(\alpha)}$ w.r.t. the basis $\{\underline{e}_\mu\}$

$$\underline{n}_{(\alpha)} = h^\mu_{(\alpha)}\,\underline{e}_\mu = h_{(\alpha)\mu}\,\underline{e}^\mu$$
$$\underline{n}^{(\alpha)} = g^{(\alpha)(\beta)}\,\underline{n}_{(\beta)} = h^{(\alpha)}_{\ \ \mu}\,\underline{e}^\mu = h^{(\alpha)\mu}\,\underline{e}_\mu \qquad (3)$$

whence it follows

$$\underline{A} = A^\mu\,\underline{e}_\mu = A^{(\alpha)}\,h^\mu_{(\alpha)}\,\underline{e}_\mu \qquad (4)$$

or

$$A^\mu = A^{(\alpha)}\,h^\mu_{(\alpha)} \qquad (5)$$

and similar relations obtained by raising and lowering the indices. The normalization conditions of $\underline{e}_\mu$ and $\underline{n}_{(\alpha)}$ gives [27]

$$g_{(\alpha)(\beta)}\,h^{(\alpha)}_{\ \ \mu}\,h^{(\beta)}_{\ \ \nu} = g_{\mu\nu}$$
$$g_{\mu\nu}\,h^\mu_{(\alpha)}\,h^\nu_{(\beta)} = g_{(\alpha)(\beta)} \qquad (6)$$

From the discussion in section 2 we find that under a Lorentz transformation

$$\underline{n}'_{(\alpha)} \equiv L(\Lambda)\,\underline{n}_{(\alpha)} = h^\mu_{(\alpha)}\,\Lambda^\nu_{\ \ \mu}\,\underline{e}_\nu \equiv h'^\mu_{(\alpha)}\,\underline{e}_\mu \qquad (7)$$

e.g. $h'^\mu_{(\alpha)} = \Lambda^\mu_{\ \ \nu}\,h^\nu_{(\alpha)}$ so that index $(\alpha)$ is un affected. From $A^{(\alpha)} = h^{(\alpha)}_{\ \ \mu}\,A^\mu$, shown easily, we see that $A^{(\alpha)}$ components are unchanged too. Thus tetrads of vectors $h^\mu_{(\alpha)}$ (or $\underline{n}_{(\alpha)}$) define a (fixed) frame of reference w.r.t. which any vector $A^\mu$ can be decomposed. The linear independence of $h^\mu_{(\alpha)}$ is easily demonstrated. We also note $\underline{A}\cdot\underline{B} = A^\mu B_\mu = A^{(\alpha)} B_{(\alpha)}$ and that the inner product remains invariant under a Lorentz transformation as well as under a rotation of the frame of reference, that is, when

$\underline{n}_{(\alpha)} \rightarrow \underline{N}_{(\alpha)}$ with $\underline{N}_{(\alpha)} \cdot \underline{N}_{(\beta)} = g_{(\alpha)(\beta)}$ or equivalently [28]
$h^{\mu}_{(\alpha)} \rightarrow R^{(\beta)}{}_{(\alpha)} \; h^{\mu}{}_{(\beta)}$ with $g_{(\sigma)(\rho)} \, R^{(\sigma)}{}_{(\alpha)} \, R^{(\rho)}{}_{(\beta)} = g_{(\alpha)(\beta)}$

For 2-spinor space a spin frame may be defined in terms of two vectors $\underline{n}_{(1)}$ and $\underline{n}_{(2)}$, in complex two dimensional space with basis vectors $\underline{h}_1$, and $\underline{h}_2$ which satisfy, like $\underline{h}_A$, the normalization condition

$$\underline{n}_{(A)} \cdot \underline{n}_{(B)} = \varepsilon_{(A)(B)} \tag{8}$$

where $\varepsilon_{(1)(2)} = -\varepsilon_{(2)(1)} = 1$, $\varepsilon_{(A)(B)} = 0$ for $(A) \neq (B)$. [29] The spin frame is completely specified in terms of the components $h^{B}{}_{(A)}$ of the vectors $\underline{n}_{(A)} = h^{B}_{(A)} \; \underline{h}_B = -h_{(A)B} \, \underline{h}^B$ just as $h^{\mu}{}_{(\alpha)}$ did so in earlier case. From equation (4.23) it follows

$$\varepsilon_{(A)(B)} = \varepsilon_{CD} \, h^{C}{}_{(A)} \, h^{D}{}_{(B)} = h_{D(A)} \, h^{D}{}_{(B)} = - h^{D}{}_{(A)} \, h_{(B)D} \tag{9}$$

This leads to [30]

$$\varepsilon^{AB} = \varepsilon^{(C)(D)} \, h^{A}{}_{(C)} \, h^{B}{}_{(D)} = h^{A}{}_{(C)} \, h^{(C)B} = - h^{(D)A} \, h^{B}{}_{(D)} \tag{10}$$

where $\varepsilon^{(A)(B)} = \varepsilon_{(A)(B)}$ and they are used to raise or lower the indices inside brackets in the fashion identical to that of $\varepsilon_{AB}$ and $\varepsilon^{AB}$, for example, $\underline{n}^{(A)} = \varepsilon^{(A)(B)} \, \underline{n}_{(B)}$ and $u_{(A)} = -\varepsilon_{(A)(B)} \, u^{(B)}$ etc. From

$$\underline{u} = u^{(A)} \, \underline{n}_{(A)} = - u_{(A)} \, \underline{n}^{(A)} = u^{B} \, \underline{h}_B = - u_B \, \underline{h}^B \tag{11}$$

we have the expansion

$$u^{A} = h^{A}{}_{(B)} \, u^{(B)} = - h^{(B)A} \, u_{(B)} \tag{12}$$

The inverse relations [31] are

$$u^{(A)} = -h_B^{(A)}\, u^B = h^{(A)B}\, u_B$$
$$\underline{h}_A = -h_A^{(C)}\, \underline{n}_{(C)} = h_{(C)A}\, \underline{n}^{(C)} \tag{13}$$

and others obtained by raising and lowering the indices. A Lorentz transformation $\Lambda$ induces, according to the discussion in section 4, the transformation:

$$\underline{n}'_{(A)} = h^B_{(A)}\, \underline{h}'_B = S_1(\Lambda)^C{}_B\, h^B_{(A)}\, \underline{h}_C \equiv h'^B_{(A)}\, \underline{h}_B \tag{14}$$

or

$$h'^B_{(A)} = S_1(\Lambda)^B{}_C\, h^C_{(A)} \tag{15}$$

Similarly, we have $h'_{(A)B} = (S_1^{-1}(\Lambda))^C{}_B\, h_{(A)C}$. Thus the components $u^{(A)}$ are unaltered. For the inner product we note $\underline{u}\cdot\underline{\phi} = u_A\, \phi^A = -u^A\, \phi_A = u_{(A)}\, \phi^{(A)} = -u^{(A)}\, \phi_{(A)}$. It thus remains invariant under Lorentz transformations as well as under the spin frame rotations. The latter constitute the transformations defined by

$$\underline{N}_{(A)} = S^{(B)}{}_{(A)}\, \underline{n}_{(B)} \equiv H^B_{(A)}\, \underline{h}_B \tag{16}$$

such that $\underline{N}_{(A)} \cdot \underline{N}_{(B)} = \varepsilon_{(A)(B)}$. It follows that $\varepsilon_{(A)(B)}\, S^{(A)}{}_{(C)}\, S^{(B)}{}_{(D)} = \varepsilon_{(C)(D)}$ so that the complex matrix $(S^{(A)}_{(B)})$ is unimodular and belongs to SL(2, C) group. Also $H^B_{(A)} = S^{(C)}_{(A)}\, h^B_{(C)}$ and $\underline{u} = u^{(A)}\, \underline{n}_{(A)} = U^{(A)}\, \underline{N}_{(A)}$ implies $U^{(A)} = (S^{-1})^{(A)}{}_{(B)}\, u^{(B)}$. We observe that while $u^{(A)}$ is unaltered under Lorentz transformations, $u^A$ is unaltered under spin frame rotations. An exactly analogous discussion goes for the complex 2-spinor space with dotted indices spanned by $\{\underline{h}_{\dot 1}, \underline{h}_{\dot 2}\}$.

An arbitrary spinor $u^{A\dot{B}}$, likewise, may be expanded w.r.t. the spin frame , $u^{A\dot{B}} = h^{A}_{(C)}\, h^{\dot{B}}_{(\dot{D})}\, u^{(C)(\dot{D})}$. For the case of $\sigma^{\mu A\dot{B}}$ we have

$$\sigma^{\mu A\dot{B}} = h^{A}_{(C)}\, h^{\dot{B}}_{(\dot{D})}\, \sigma^{\mu(C)(\dot{D})} \tag{17}$$

or

$$\sigma^{\mu(A)(\dot{B})} = h^{(A)}_{C}\, h^{(\dot{B})}_{\dot{D}}\, \sigma^{\mu C\dot{D}} \tag{18}$$

Under a Lorentz transformation the quantities $\sigma^{\mu(A)(\dot{B})}$ transform like a four vector, viz,

$$\sigma'^{\mu(A)(\dot{B})} = \Lambda^{\mu}{}_{\nu}\, \sigma^{\nu(A)(\dot{B})} \tag{19}$$

Moreover, we may easily show $\sigma^{\mu(A)(\dot{B})}\, \sigma_{\mu(C)(\dot{D})} = 2\, \delta^{(A)}_{(C)}\, \delta^{(\dot{B})}_{(\dot{D})}$ and $\sigma^{\mu(A)(\dot{B})}\, \sigma^{\nu}{}_{(A)(\dot{B})} = 2\, g^{\mu\nu}$ so that $\sigma^{\mu(1)(\dot{1})}$, $\sigma^{\mu(1)(\dot{2})}$, $\sigma^{\mu(2)(\dot{1})}$ and $\sigma^{\mu(2)(\dot{2})}$ are linearly independent set. We may thus expand [32] any four-vector $U^{\mu}$ in terms of them

$$U^{\mu} = u_{(A)(\dot{B})}\, \sigma^{\mu(A)(\dot{B})} = u_{A\dot{B}}\, \sigma^{\mu A\dot{B}} \tag{20}$$

where

$$u_{(A)(\dot{B})} = \frac{1}{2}\, \sigma_{\mu(A)(\dot{B})}\, U^{\mu} \tag{21}$$

The explicity expressions for $\sigma^{\mu(A)(\dot{B})}$ are

$$\begin{aligned} \sigma^{0(A)(\dot{B})} &= h^{(A)}_{1}\, h^{(\dot{B})}_{\dot{1}} + h^{(A)}_{2}\, h^{(\dot{B})}_{\dot{2}}, \quad & \sigma^{1(A)(\dot{B})} &= h^{(A)}_{1}\, h^{(\dot{B})}_{\dot{2}} + h^{(A)}_{2}\, h^{(\dot{B})}_{\dot{1}} \\ \sigma^{2(A)(\dot{B})} &= i(-h^{(A)}_{1}\, h^{(\dot{B})}_{\dot{2}} + h^{(A)}_{2}\, h^{(\dot{B})}_{\dot{1}}), \quad & \sigma^{3(A)(\dot{B})} &= h^{(A)}_{1} h^{(\dot{B})}_{\dot{1}} - h^{(A)}_{2} h^{(\dot{B})}_{\dot{2}} \end{aligned} \tag{22}$$

We also note that

$$\sigma^{\mu(A)(\dot{B})}\, \sigma_{\mu}{}^{(A)(\dot{B})} = 0\ , \qquad (A),\ (\dot{B})\ \text{fixed.} \tag{23}$$

Thus $\sigma^{\mu(A)(\dot{B})}$ constitute a basis in Minkowski space of four null tetrad of vectors, two of which are real e.g. $\sigma^{\mu(1)(\dot{1})}$ and $\sigma^{\mu(2)(\dot{2})}$ and $\sigma^{\mu(1)(\dot{2})}$ and $\sigma^{\mu(2)(\dot{1})}$ are complex conjugate of each other. [33]

6. REPRESENTATION OF FOUR GROUP IN SPINOR SPACE

The following 4x4 matrices $\Lambda_s$, $\Lambda_t$, $\Lambda_{st}$ together with identity matrix I constitute an Abelian group called the Four-group:

$$\Lambda_s = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix} \quad \Lambda_t = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix} \quad \Lambda_{st} \equiv \Lambda_s \Lambda_t = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{1}$$

They correspond to space reflection, time inversion and space-time inversion in Minkowski space. Combined with the restricted Lorentz group we obtain the Full or Extended Lorentz group. We can show that, if we stick to linear transformations in representation space, it is impossible to represent the four group by 2x2 matrices while maintaining the mixed quantities $\sigma^{\mu A\dot{B}}$ fixed according to equation (4.26). For $\Lambda_s$ we have,

$$(\sigma^{kA\dot{B}}) = - S_1(\Lambda_s)\,(\sigma^{kA\dot{B}})\,S_1^{\dagger}(\Lambda_s); \qquad k = 1, 2, 3 \tag{2}$$
$$(\sigma^{oA\dot{B}}) = S_1(\Lambda_s)(\sigma^{oAB})\,S_1^{\dagger}(\Lambda_s)$$

while for $\Lambda_t$,

$$(\sigma^{kA\dot{B}}) = + S_1(\Lambda_t)\,(\sigma^{kA\dot{B}})\,S_1^{\dagger}(\Lambda_t); \qquad k = 1, 2, 3$$
$$(\sigma^{oA\dot{B}}) = - S_1(\Lambda_t)\,(\sigma^{oA\dot{B}})\,S_1^{\dagger}(\Lambda_t) \tag{3}$$

In either case we require $S_1 \sigma^k S_1^{-1} = -\sigma^k$ (or $S_1 \sigma^k = -\sigma^k S_1$) for k=1,2,3. It is easily verified that this is not possible to attain this in terms of 2x2 matrices for which $(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ is a complete set. The situation is different in 4-dimensional spinor space and the improper transformation can be represented by linear transformations through 4x4 matrices.

The restricted Lorentz group is invariant sub-group of the full group and one verifies the following relations:

$$\Lambda_s^{-1} \Lambda_R \Lambda_s = \Lambda_R \qquad \Lambda_t^{-1} \Lambda_R \Lambda_t = \Lambda_R$$
$$\Lambda_s^{-1} \Lambda_L \Lambda_s = \Lambda_L^{-1} \qquad \Lambda_t^{-} \Lambda_L \Lambda_t = \Lambda_L^{-1} \tag{4}$$
$$\Lambda_{st}^{-1} \Lambda_R \Lambda_{st} = \Lambda_R \qquad \Lambda_{st}^{-1} \Lambda_L \Lambda_{st} = \Lambda_L$$

where $\Lambda_R$ is a space rotation and $\Lambda_L$ pure Lorentz transformation, say, in (01) plane. Hence we require the corresponding representation matrices in spinor space to satisfy:

$$S^{-1}(\Lambda_s)\, \gamma^k \gamma^\ell\, S(\Lambda_s) = \gamma^k \gamma^\ell = S^{-1}(\Lambda_t)\, \gamma^k \gamma^\ell\, S(\Lambda_t)$$
$$S^{-1}(\Lambda_s)\, \gamma^0 \gamma^k\, S(\Lambda_s) = -\gamma^0 \gamma^k = S^{-1}(\Lambda_t)\, \gamma^0 \gamma^k\, S(\Lambda_t) \tag{5}$$

At the same time we require that the $\gamma^{\mu a}{}_b$ behave as invariant "mixed quantity" under the full group, according to equation ( 3.1 ). This leads to

$$S(\Lambda_s)\, \gamma^k\, S^{-1}(\Lambda_s) = -\gamma^k$$
$$S(\Lambda_s)\, \gamma^0\, S^{-1}(\Lambda_s) = \gamma^0 \tag{6}$$

and

$$S(\Lambda_t)\,\gamma^k\,S^{-1}(\Lambda_t) = \gamma^k$$
$$S(\Lambda_t)\,\gamma^o\,S^{-1}(\Lambda_t) = -\gamma^o \quad (7)$$

It is easily shown that these imply [34]

$$\gamma_5' \equiv S(\Lambda)\,\gamma_5\,S^{-1}(\Lambda) = (\det\Lambda)\,\gamma_5 \quad (8)$$

or, written explicitly,

$$\gamma_5'{}^a{}_b \equiv S(\Lambda)^a{}_c\,S^{-1}(\Lambda)^d{}_b\,\gamma_5^c{}_d = (\det\Lambda)\,\gamma_5^a{}_b \quad (9)$$

From equation (2.18), it follows that we may choose

$$S(\Lambda_s) = a\,\gamma^o$$
$$S(\Lambda_t) = b\,\gamma_5\,\gamma^o \quad (10)$$

then

$$S(\Lambda_{st}) \equiv S(\Lambda_s)\,S(\Lambda_t) = -ab\,\gamma_5$$
$$= -S(\Lambda_t)\,S(\Lambda_s) \equiv -S(\Lambda_{ts}) \quad (11)$$

We note that, though $\Lambda_{st} = \Lambda_{ts}$ but $S(\Lambda_{st}) = -S(\Lambda_{ts})$. Hence, we have double valued representation of the four group in spinor space [35]. The constants 'a' and 'b' may be fixed by requiring that the parity and time inversion operations applied twice lead the identity transformation upto a $\pm$ sign due to double-valuedness of the representation. Thus

$$\left[S(\Lambda_s)\right]^2 = a^2 I = \pm I \qquad \text{or} \qquad a^2 = \pm 1$$
$$\left[S(\Lambda_t)\right]^2 = b^2 I = \pm I \qquad \text{or} \qquad b^2 = \pm 1 \tag{12}$$

so that [36] $a = \pm 1, \pm i$; $b = \pm 1, \pm i$; $|a|^2 = 1$, $|b|^2 = 1$, $a^4 = 1$ and $b^4 = 1$. We find then the following relations. [37]

$$A\, S(\Lambda_s) A^{-1} = a\, \gamma^{0\dagger} = aa^*(a\, \gamma^0)^{\dagger -1} = S(\Lambda_s)^{-1\dagger}$$

$$A\, S(\Lambda_t) A^{-1} = b\, \gamma_5^\dagger\, \gamma^{0\dagger} = bb^*(b\ \ \gamma_5^{-1}\, \gamma^{0-1})^{\dagger -1} = -(b\, \gamma_5\ \ \gamma^0)^{\dagger -1} = -S(\Lambda_t)^{\dagger -1} \tag{13}$$

which may be combined with equation (3.7) as

$$A\, S(\Lambda)\, A^{-1} = S(\Lambda)^{\dagger -1} \qquad \Lambda^0{}_0 \geqslant 1$$
$$= -S(\Lambda)^{\dagger -1} \qquad \Lambda^0{}_0 \leqslant -1 \tag{14}$$

for the full group. This may be interpreted as the transformation of matrix A,

$$A' \equiv S(\Lambda)^{\dagger -1}\, A\, S^{-1}(\Lambda) = \mathrm{Sgn}(\Lambda^0{}_0)\, A \tag{15}$$

or

$$A'_{\dot{a}b} \equiv (S^{-1}(\Lambda))^{\dot{c}}{}_{\dot{a}}\, (S^{-1}(\Lambda))^d{}_b\, A_{\dot{c}d} = \mathrm{Sgn}(\Lambda^0{}_0)\, A_{\dot{a}b} \tag{16}$$

where $\mathrm{Sgn}(\Lambda^0{}_0) = \pm 1$ according as $\Lambda^0{}_0 \geqslant 1$ or $\Lambda^0{}_0 \leqslant -1$.

The metric matrix B in the spinor representation of $\gamma$ matrices of section 4 is given by $B = -\gamma_5\, \gamma^0\, \gamma^2$. We find

$$B' \equiv S^{-1T}(\Lambda_s)\, B\, S^{-1}(\Lambda_s) = a^2\, B$$
$$B' \equiv S^{-1T}(\Lambda_t)\, B\, S^{-1}(\Lambda_t) = -b^2\, B \tag{17}$$

or $\quad B'_{ab} = S^{-1}(\Lambda_s)^c{}_a\, S^{-1}(\Lambda_s)^d{}_b\, B_{cd} = a^2\, B_{ab}$ etc. Taking inverse of equation (6.17) we find similar relations for $B^{ab}$. We note that $B'^{ac} B'_{cb} = B^{ac} B_{cb} = - \delta^a_b$ since the Kronecker delta is an invariant tensor under the full group. The tensors $A_{\dot{a}b}$, $B_{ab}$ and $B^{ab}$ are invariant only upto a sign under the four group.

## 7. TRANSFORMATION OF SPINOR AND BILINEAR INVARIANTS

The transformation of spinors given by equations (2.23), (2.25),(2.27) and (2.29) reads in terms of 2-spinors of section 4 as follows:

$$\text{Parity:} \quad u'^A = a\, v_{\dot{A}}, \quad v'^A = -a^{*-1} u_{\dot{A}}, \quad u'_A = -a^{-1} v^{\dot{A}}, \quad v'_A = a^* u^{\dot{A}} \tag{1}$$

$$\text{Time-inversion:} \quad u'^A = i\, b\, v_{\dot{A}}, \quad v'^A = i\, b^{-1*} u_{\dot{A}}, \quad u'_A = ib^{-1} v^{\dot{A}}, \quad v'_A = ib^* u^{\dot{A}} \tag{2}$$

and the relations obtained by taking their complex conjugate. From equations (4.12) and (6.17) it follows

$$(\varepsilon_{AB})' = a^2(\varepsilon^{\dot{A}\dot{B}}) \qquad (\varepsilon^{AB})' = a^2(\varepsilon_{\dot{A}\dot{B}}) \tag{3}$$

for parity and

$$(\varepsilon_{AB})' = - b^2(\varepsilon^{\dot{A}\dot{B}}) \qquad (\varepsilon^{AB})' = -b^2(\varepsilon_{\dot{A}\dot{B}}) \tag{4}$$

for time inversion. [39]

The bilinear invariants of section 3 take the following form in spinor representation

$$S\ (\text{Scalar}) = \xi^{\dot{a}} A_{\dot{a}b}\, \eta^b = u^{\dot{A}} \chi_{\dot{A}} + v_A\, \phi^A \tag{5}$$

$$\text{where} \qquad \xi^a = \binom{u^A}{v_{\dot{A}}} \quad \text{and} \quad \eta^a = \binom{\phi^A}{\chi_{\dot{A}}} \tag{6}$$

$$P\ (\text{Pseudoscalar}) = \xi^{\dot{a}} A_{\dot{a}b}\ \gamma_5{}^b{}_c\ \eta^c = -i(u^{\dot{A}} \chi_{\dot{A}} - v_A\, \phi^A) \tag{7}$$

V (4-Vector):

$$\begin{aligned} V^\mu &= \frac{1}{2}\ \xi^{\dot{a}}\ A_{\dot{a}c}\ \gamma^{\mu c}{}_b\ \eta^b \\ &= \frac{1}{2}\,\sigma^\mu_{B\dot{A}}(\phi^B\ u^{\dot{A}} + v^B\ \chi^{\dot{A}}) \end{aligned} \tag{8}$$

A (Pseudo vector):

$$A^\mu = \frac{1}{2}\,\xi^{\dot{a}}\ A_{\dot{a}c}(i\ \gamma_5\ \gamma^\mu)^c{}_b\ \eta^b = \frac{1}{2}\,\sigma^\mu{}_{B\dot{A}}\ (\phi^B\ u^{\dot{A}} - v^B\ \chi^{\dot{A}}) \tag{9}$$

T (Antisymmetric tensor): $F^{\mu\nu} = \xi^{\dot{a}}\ A_{\dot{a}c}(\Sigma^{\mu\nu})^c{}_b\ \eta^b$ is apart from a factor $\simeq \left[u^{\dot{A}}\sigma^\mu{}_{\dot{A}B}\ \sigma^{\nu B\dot{C}}\ \chi_{\dot{C}} + v_A\ \sigma^{\mu A\dot{B}}\ \sigma^\nu_{\dot{B}C}\ \phi^C\right] - (\mu \leftrightarrow \nu)$.

For

$$\begin{aligned} \xi^a &= \eta^a \\ S &= u^{\dot{A}}\ v_{\dot{A}} + u^A\ v_A = S^* \\ P &= -i(u^{\dot{A}}\ v_{\dot{A}} - u^A\ v_A) = P^* \\ V^\mu &= \frac{1}{2}\,\sigma^\mu_{B\dot{A}}\ (u^B\ u^{\dot{A}} + v^B\ v^{\dot{A}}) = V^{\mu *} \\ A^\mu &= \frac{1}{2}\,\sigma^\mu_{B\dot{A}}\ (u^B\ u^{\dot{A}} - v^B\ v^{\dot{A}}) = A^{\mu *} \end{aligned} \tag{10}$$

We observe that the invariants S and P vanish for 4-spinors of type $\binom{u^A}{\lambda u_{\dot{A}}} = \xi = \eta$. It is easily shown that $V^\mu V_\mu = A^\mu A_\mu = 4(u^{\dot{A}} \chi_{\dot{A}})(\phi^B v_B)$ which for $\xi = \eta$ reduces to $4(u^{\dot{A}} v_{\dot{A}})(u^B v_B)$ and is real. Hence $\xi = \eta = \binom{u^A}{\lambda u_{\dot{A}}}$ defines a real null vector, or that a 2-spinor defines a real null vector [2]. The two scalars S and P and the two vectors V and A behave differently under improper transformations. For example,

$$(\xi^\dagger A \; \gamma^\mu \xi)' \equiv \xi'^\dagger A' \gamma'^\mu \; \xi' = \xi^\dagger S^\dagger S^{-1\dagger} AS^{-1} \Lambda^\mu{}_\nu S \gamma^\nu S^{-1} S \xi$$

$$= \Lambda^\mu{}_\nu (\xi^\dagger A \gamma^\nu \; \xi) = \xi'^\dagger A' \gamma^\mu \xi' \qquad (11)$$

the last equality following from the invariance of $\gamma^{\mu a}{}_b$ under full group. From equation (6.15) it follows:

$$\begin{aligned} \xi'^\dagger \; A \gamma^\mu \xi' &= \Lambda^\mu{}_\nu (\xi^\dagger A \gamma^\nu \xi) && \Lambda \in \mathcal{L}^\uparrow_+ \\ &= - \xi^\dagger A \gamma^k \xi; && k = 1, 2, 3 \quad \Lambda = \Lambda_s \text{ or } \Lambda_t \\ &= + \xi^\dagger A \gamma^o \xi && \mu = 0 \\ &= + \xi^\dagger A \gamma^\mu \xi && \Lambda = \Lambda_{st} \end{aligned} \qquad (12)$$

For pseudo-vector [40] $\xi^\dagger \; A \gamma_5 \gamma^\mu \xi$ we have opposite sign for $\Lambda = \Lambda_s$ or $\Lambda_t$. For scalar and pseudo scalar we obtain

$$\begin{aligned} \xi'^\dagger A \xi' &= \xi^\dagger A \xi && \text{for} \quad \Lambda = \Lambda_s \\ &= - \xi^\dagger A \xi && \text{for} \quad \Lambda = \Lambda_t, \Lambda_{st} \\ \xi'^\dagger A \gamma_5 \xi' &= - \xi^\dagger A \gamma_5 \xi && \text{for} \quad \Lambda = \Lambda_s, \Lambda_{st} \\ &= + \xi^\dagger A \gamma_5 \xi && \text{for} \quad \Lambda = \Lambda_t \end{aligned} \qquad (13)$$

and for tensor case

$$\xi'^{\dagger} A \gamma^k \gamma^\ell \xi' = \pm \xi^\dagger A \gamma^k \gamma^\ell \xi$$

$$\xi' \ A \gamma^o \gamma^k \xi' = \mp \xi^\dagger A \gamma^o \gamma^k \xi$$

the upper sign holding for $\Lambda_s$ and lower for $\Lambda_t$.

The choice of the phases factors 'a' and 'b' may be narrowed by appealing to antilinear operation of charge conjugation associated with Dirac equation:

$$(i \gamma^{\mu a}{}_b \partial_\mu - m \delta^a{}_b)\xi^b = e A_\mu \gamma^{\mu a}{}_b \xi^b \tag{14}$$

On taking the complex conjugate, multiplying by $A_{\dot{a}c}$ and using $A_{\dot{a}c} \gamma^{\mu c}{}_b = A_{\dot{c}b} \gamma^{\mu \dot{c}}{}_{\dot{a}}$ we obtain

$$i A_{\dot{b}d} \gamma^{\mu d}{}_c \partial_\mu \xi^{*b} + m A_{c\dot{b}} \xi^{*b} = -e A_\mu A_{\dot{b}d} \gamma^{\mu d}{}_c \xi^{*b}$$

From $\left\{ -\gamma^{\mu T}, -\gamma^{\nu T} \right\} = 2 g^{\mu\nu}$ it follows that there exists a monsingular matrix $\mathbb{C}$ such that[41] $\mathbb{C} \gamma^\mu \mathbb{C}^{-1} = -\gamma^{\mu T}$ or $\gamma^{\mu a}{}_c \mathbb{C}^{cb} = -\mathbb{C}^{ac} \gamma^{\mu b}{}_c$ where $\mathbb{C}^{-1} \equiv (\mathbb{C}^{ab})$ and whose invariance under restricted group may easily be verified. Hence

$$(i \gamma^{\mu a}{}_b \partial_\mu - m \delta^a{}_b)\eta^b = - e A_\mu \gamma^{\mu a}{}_b \eta^b \tag{15}$$

where the charge conjugate spinor $\eta$ is given by

$$\eta^a \equiv \mathbb{C}^{ca} A_{\dot{b}c} \xi^{*b} \tag{16}$$

It corresponds to a Dirac particle with charge (-e). A candidate for $\mathbb{C}$ is

$$\mathbb{C} = \lambda \gamma_5 B^{-1} \tag{17}$$

Requiring that charge conjugation applied twice leads back to original spinor leads to $|\lambda|^2 = 1$.

We verify [42] that under a restricted transformation

$$\eta'^{a} = \mathbb{C}^{ca} A_{\dot{b}c} \xi^{*'b} = \mathbb{C}^{ca} A_{\dot{b}c} S(\Lambda)^{\dot{b}}{}_{\dot{d}} \xi^{\dot{d}} = S(\Lambda)^{a}{}_{b} \eta^{b} \qquad (18)$$

and now impose [8] that $\eta'^{a} \equiv \mathbb{C}^{ca} A_{\dot{b}c} \xi^{*'b}$ satisfies the same relation under the improper transformations as well which leads to

$$\begin{aligned} \mathbb{C}'^{-1} \equiv S(\Lambda)\, \mathbb{C}^{-1}\, S^{T}(\Lambda) &= \mathbb{C}^{-1} && \text{for } \Lambda^{0}{}_{0} \geqslant 1 \\ &= -\mathbb{C}^{-1} && \text{for } \Lambda^{0}{}_{0} \leqslant -1 \end{aligned} \qquad (19)$$

This in turn requires $a^2 = -1$, $b^2 = -1$ e.g., $a = \pm i$, $b = \pm i$.

* * *

## REFERENCES AND FOOTNOTES †


* 1. Exhaustive list of references on spinor analysis may be found in W. L. Bade and H. Jehle, Rev. Mod. Phys. 25, 714 (1953) and

W. C. Parke and H. Jehle, Lectures in Theoretical Physics Vol. VII a, Univ. of Boulder Press (1964) p. 297. See also

E. M. Corson; Tensors, Spinors and Relativitic Wave Equations (Blackie and Sons Ltd. London, 1953) and

J. Aharoni, The Special Theory of Relativity, (Oxford University Press 1965).

* 2. See for example, F. A. E. Pirani, Lectures in General Relativity, (Prentice Hall, N. J., 1965); C. J. Isham, A. Salam and J. Strathdee, IC/72/123 (1972).


3. The first or upper index labels row and the second or lower index labels coloums when $\Lambda^{\mu}{}_{\nu}$ or $g_{\mu\nu}$ are written as matrices. We will avoid using matrices corresponding to $\Lambda_{\mu}{}^{\nu}$.

4. See section 7 where representations of Four group are considered.

5. It follows that $g^{\mu\nu}$ is a contravariant tensor in indices $\mu$ and $\nu$.

6. $g^{\mu\nu}\, g_{\mu\nu} = g_{\alpha\beta}\, \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}\, g^{\mu\nu} \equiv g_{\alpha\beta}\, g^{\alpha\beta}$ or $g^{\alpha\beta} = g^{\mu\nu}\, \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}$.

7. $(\gamma^{o})^2 = I$ (identity), $(\gamma^{k})^2 = -I$, $k = 1, 2, 3$. The tacit assumption that '$\mu$' appearing in $\gamma^{\mu}$ is Minskowski space index will be shown below.

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† *The references are marked by an asterisk.*

*8. See for example, S. S. Schweber: Relativistic Quantum Field Theory (Row Peterson and Co., 1961), Chapters 1 and 4.

9. Due to the appearence of half-angles the representation is double-valued for space rotations, e.g. both the matrices $\pm S(\Lambda)$ represent the rotation. There is no ambiguity in sign due to half-angles for pure Lorentz transformations. We will adopt the normalization $\det S(\Lambda) = 1$.

10. Note that except the identity matrix all other elements of Clifford algebra are traceless. Note also $\omega_{\mu\nu} + \omega_{\nu\mu} + g_{\alpha\beta}\, \omega^{\alpha}{}_{\mu}\, \omega^{\beta}{}_{\nu} = 0$. The relations $S(\Lambda)\, S(\Lambda') = S(\Lambda\Lambda')$ and $S(\Lambda^{-1}) = S^{-1}(\Lambda)$ may easily be verified.

11. T indicates transpose and $(A^T)^a{}_b = A^b{}_a$. The greek letters label space-time indices while roman letters the spinor indices. It is clear that $\{S^{-1T}(\Lambda)\}$, $\{S^*(\Lambda)\}$ and $\{S^{-1\dagger}(\Lambda)\}$ constitute representations isomorphic to the group $\{S(\Lambda)\}$.

12. We also have matrices corresponding to a negative sign on the right hand side. Note that $A\gamma^{\mu} A^{-1} = \gamma^{\mu T*} = (B\, \gamma^{\mu}\, B^{-1})^* = B^* C\, \gamma^{\mu}\, C^{-1} B^{-1*}$. Hence $(A^{-1}\, B^*\, C)$ is a multiple of identity. If we impose $\det(A) = \det(B) = \det(C) = 1$, these matrices are defined only upto a factor $\pm 1$, $\pm i$.

13. It cannot be chosen to be symmetric as it will lead to ten antisymmetric linearly independent 4x4 matrices $(B\, \gamma_5\, \gamma^{\mu})$, $(B\, \Sigma^{\mu\nu})$.

14. Thus $\underline{Z}_a \cdot \underline{Z}_b = 0$ when $a = b$ and $\underline{Z}_a \cdot \underline{Z}_b = - \underline{Z}_b \cdot \underline{Z}_a$. Also equations (3.16) ensure that $\underline{Z}_a$ are linearly independent vectors (when $B_{ab} = - B_{ba}$ is non-singular matrix).

15. These tensors are like metric tensor and may be used to relate the dotted basis with undotted basis vectors.

16. We have $\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$, $\gamma_5^\dagger = A \gamma_5 A^{-1}$, $\Sigma^{\mu\nu\dagger} = A \Sigma^{\mu\nu} A^{-1}$, $\{\gamma_5, \gamma^\mu\}_+ = 0$.

17. In fact $\eta_{\dot{a}b}$ regarded as (4x4) matrix can be decomposed in terms of the 16 linearly independent matrices $A$, $A\gamma^\mu$, $A\Sigma^{\mu\nu}$, $iA\gamma_5\gamma^\mu$, $A\gamma_5$ which form a basis for all 4-dim. matrices of the form $(\eta_{\dot{a}b})$. Similar situation holds for $(\eta_{ab})$, $(\eta_{\dot{a}\dot{b}})$ etc.

18. Since $\gamma$'s constitute an irreducible representation they can not be all chosen to be even matrices.

19. They become identical for space rotation sub-group. The representation $\{S_1(\Lambda)\}$ is the so called $D(\frac{1}{2}, 0)$ representation while $\{S_1^{\dagger -1}(\Lambda)\}$ is $D(0, \frac{1}{2})$ of the SL(2, C) group. $\{S(\Lambda)\}$ corresponds to $D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$. See for example Corson [1] or Schweber [8].

* 20. M. A. Naimark, Linear Representations of Lorentz Group, (Pergamon Press N.Y. 1964); I. M. Gel'fand, M. I. Graev and N. Ya. Vilenkin, Integral Geometry and Representation Theory (Academic Press, N. Y., 1966); M. Carmeli: J. Math. Phys. 11, 1917 (1970).

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$= \det S(\Lambda_t) = \det S(\Lambda_{st}) = 1.$

37. These relations (as well as the expressions for $S(\Lambda_s)$ and $S(\Lambda_t)$) are derived using equation (2.18), $A \gamma^\mu A^{-1} = \gamma^{\mu\dagger}$ and the definition $\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

21. The minus sign for $\xi_3$ and $\xi_4$ is for convenience.

22. $\varepsilon_{12} = \varepsilon^{12} = 1$, $\varepsilon_{AB} = - \varepsilon_{BA}$, $\varepsilon_{AB} = \varepsilon^{AB}$ and same definition for dotted indices.

23. In matrix notation $\varepsilon^{-1} S_1 \, \varepsilon = S_1^{T-1}$ where $\varepsilon = (\varepsilon^{AB}) = (\varepsilon_{AB})$.

24. Use $\gamma^\mu \, S(\Lambda) = \Lambda^\mu{}_\nu \, S(\Lambda) \, \gamma^\nu$ and $(\Lambda^k{}_\ell \, \sigma^\ell + \Lambda^k{}_o \, \sigma^o)$.

$(\Lambda^k{}_m \, \sigma^m - \Lambda^k{}_o \, \sigma^o) = (\Lambda^k{}_1) \;\; + (\Lambda^k{}_2)^2 + (\Lambda^k{}_3)^2 - (\Lambda^k{}_o)^2 = 1$ etc.

25. In fact $\sigma_{\mu C\dot{D}} \, \sigma^{\mu A\dot{B}} = 2 \, \delta^A_C \, \delta^{\dot{B}}_{\dot{D}} + F^{A\dot{B}}_{C\dot{D}}$ such that $F^{A\dot{B}}_{C\dot{D}} \, \sigma^\nu_{A\dot{B}} = 0$. It is easily shown $F^{A\dot{B}}_{C\dot{D}} \equiv 0$.

26. $U_\mu \, U^\mu = \det \, (u^{A\dot{B}})$. Note $g^{\mu\nu} = \frac{1}{4} \, \sigma^\mu_{A\dot{B}} \, \sigma^\nu_{C\dot{D}} \, (2\varepsilon^{AC} \, \varepsilon^{\dot{B}\dot{D}})$ implies $g^{A\dot{B},C\dot{D}} = 2 \, \varepsilon^{AC} \, \varepsilon^{\dot{B}\dot{D}}$.

27. The first relation leads to the second by noting $g^{\mu\nu} \, g_{(\alpha)(\beta)} \, h^{(\alpha)}_\mu \, h^{(\beta)}_\nu = g^{\mu\nu} \, g_{\mu\nu} = g_{(\alpha)(\beta)} \, g^{(\alpha)(\beta)}$.

28. $\underline{N}_{(\alpha)} = R^{(\beta)}{}_{(\alpha)} \, \underline{n}_{(\beta)}$ then $h^\mu_{(\alpha)} \longrightarrow H^\mu_{(\alpha)} = R^{(\beta)}_{(\alpha)} \, h^\mu{}_{(\beta)}$ and

$A^{(\alpha)} \longrightarrow (R^{-1})^{(\alpha)}{}_{(\beta)} \, A^{(\beta)} \equiv \mathit{A\!\!/}^{(\alpha)}$. Note $\underline{A} = A^{(\alpha)} \, \underline{n}_{(\alpha)} = \mathit{A\!\!/}^{(\alpha)} \, \underline{N}_{(\alpha)}$.

29. The label (A) = (1) or (2) is convenient. We could, of course, use any other labelling, say (A) = (0) or (1) etc.

These are exactly the results obtained in the non-degenerate case [1].

ii) Identical sub-bands

We now considers two identical sub-bands, one gets the following results:

$$\chi^{(\alpha)}_{(o)}(k,q) = \chi^{(\beta)}_{(o)}(k,q) = \chi^{(d)}_{(o)}(k,q)$$

$$\chi^{\alpha\beta}_{(1)}(k,q) = \chi^{\beta\alpha}_{(1)}(k,q) = \chi_{(1)}(k,q) \tag{56}$$

$$U_\alpha = U_\beta = U \qquad \text{and} \qquad V^{(\alpha)}_{sd}(k) = V^{(\beta)}_{sd}(k)$$

The results (55) imply that: $\tilde{\tilde{J}}_{(\alpha)}{}^{(d)}(k+q,k) = J_{(\beta)}{}^{(d)}(k+q,k)$ and consequently $v^{(d)}_{(\alpha)}(k,q) = 1$.

The susceptibility $\chi^{(\alpha)}_{(J)} = \chi^{(\beta)}_{(J)}$ assume a particularly interesting form, namely:

$$\chi_{(J)}(k,q) = \frac{\chi^{(d)}_{(o)}(k,q)}{1+(U^{(d)}_{eff}(q) + \bar{J}_{\alpha\beta}(q))\chi^{(d)}_{(o)}(q)} \tag{57-a}$$

where the effective interactions are defined by:

$$U^{(d)}_{eff}(q) = U\left\{1 + \frac{J_{\alpha\beta}}{U}\,\frac{\chi_{mix}(q)}{\chi^{(d)}_{(o)}(q)}\right\} \tag{57-b}$$

$$\bar{J}_{\alpha\beta}(q) = J_{\alpha\beta}\left\{1 + \frac{U}{J_{\alpha\beta}}\,\frac{\chi_{mix}(q)}{\chi^{(d)}_{(o)}(q)}\right\}$$

As for the s-electrons are concerned, since $\bar{J}^{(d)}_{(\alpha)} = \bar{J}^{(d)}_{(\beta)}$ the effective ex-

38. Thus we need two kinds of 2-spinors to represent parity and time inver corresponding to inequivalent representations $D(\frac{1}{2}, 0)$ and $D(0, \frac{1}{2})$. Under space-time inversion $u'^{A} = i(ab)u^{A}$, $v'_{\dot{A}} = -i(ab)v_{\dot{A}}$.

39. $(\varepsilon^{\dot{A}\dot{B}})' = (\varepsilon^{AB})'^{*} = a^{2}(\varepsilon_{AB})$ etc. and $u'_{A} = -(\varepsilon_{AB})' u'^{B}$ etc.

40. $\gamma'_5 = (\det \Lambda)\gamma_5$.

41. $\mathbb{C}$ must be antisymmetric just like B matrix.

42. $\mathbb{C}^{-1} = S(\Lambda)\, \mathbb{C}^{-1}\, S(\Lambda)^{T}$; $S^{\dagger}(\Lambda)\, A\, S(\Lambda) = A$.