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by

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WEIGHTED APPROXIMATION FOR ALGEBRAS AND MODULES OF CONTINUOUS FUNCTIONS: REAL AND SELF-ADJOINT COMPLEX CASES

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1. INTRODUCTION

The purpose of the present article is to prove some general results concerning the weighted approximation for algebras and modules of continuous scalar-valued functions on a completely regular space E. We shall deal with uniform approximation over the whole E. The fact that E need not be compact and, accordingly, a continuous scalar-valued function on E need not be bounded, calls for the use of weights. To build a simple theory, which is reasonably general so that it will subsume important classical cases, we are led to introduce two kinds of weights.

First of all, we introduce a set V of upper-semi-continuous positive real-valued functions on E, whose elements are called weights. We may assume that V is directed, in the sense that, given $v_1, v_2 \in V$, there are $\lambda > 0$ and $v \in V$ such that $v_1 \leqslant \lambda v$, $v_2 \leqslant \lambda v$. In terms of V, we define a weighted locally convex space $\mathcal{C}V_\infty(E)$. It consists of all continuous scalar-valued functions f on E such that vf vanishes at infinity for any $v \in V$. The topology on $\mathcal{C}V_\infty(E)$ is determined by the family of seminorms $f \rightarrow \|f\|_v = \sup\{v(x).|f(x)| \mid x \in E\}$, where v varies in V. To exemplify two instances, let us say immediately that:

(1) when V is the set of characteristic functions of all compact subsets of E, then $\mathcal{C}V_\infty(E)$ is the algebra $\mathcal{C}(E)$ of all continuous scalar-valued functions on E endowed with the compact open topology;

(2) when E is locally compact and V is reduced to the constant function 1, then $\mathcal{C}V_\infty(E)$ is the algebra $\mathcal{C}_\infty(E)$ of all continuous scalar-valued functions on E vanishing at infinity, endowed with the topology of the supremum on E.

Turning back to the general case of $\mathcal{C}V_\infty(E)$, we consider a subalgebra $\mathcal{A}$ containing 1 of $\mathcal{C}(E)$ and a vector subspace $\mathcal{W}$ of $\mathcal{C}V_\infty(E)$. We assume that $\mathcal{W}$ is an $\mathcal{A}$-module, that is $\mathcal{A}\mathcal{W} \subset \mathcal{W}$. The weighted approximation problem consists of asking for a description of the closure of $\mathcal{W}$ in $\mathcal{C}V_\infty(E)$ under such circumstances. This problem has not yet been solved even in some classical special cases. Notice that since $\mathcal{A}$ does not necessarily lie in $\mathcal{C}V_\infty(E)$, the elements of $\mathcal{W}$ appear as weights which, when multiplied by the elements of $\mathcal{A}$, lead to elements of $\mathcal{C}V_\infty(E)$.

A more precise form of this problem to be discussed in the present article is as follows. The algebra $\mathcal{A}$ defines an equivalence relation $E/\mathcal{A}$ on E if we set $x_1 \sim x_2$ for $x_1, x_2 \in E$, provided $f(x_1) = f(x_2)$ for any $f \in \mathcal{A}$. We then say that $\mathcal{W}$ is localisable under $\mathcal{A}$ in $CV_\infty(E)$ if the following condition holds true: an element f belonging to $CV_\infty(E)$ is in the closure of $\mathcal{W}$ in $CV_\infty(E)$ if (and always only if), for any $v \in V$, any $\varepsilon > 0$ and any equivalence class $X \subset E$ modulo $E/\mathcal{A}$, there exists some $w \in \mathcal{W}$ such that $v(x).|w(x) - f(x)| < \varepsilon$ for any $x \in X$. In defining localisability, we look for a natural description of the closure of $\mathcal{W}$ in $CV_\infty(E)$ using $E/\mathcal{A}$. The strict weighted approximation problem consists of finding necessary and sufficient conditions for $\mathcal{W}$ to be localisable under $\mathcal{A}$ in $CV_\infty(E)$. This problem has not yet been solved even in some classical special cases. On the other hand, the classical necessary and sufficient conditions due to Pollard, Melgelyan, etc. (see [8], [4]) for a continuous scalar-valued function on the real line to be a fundamental weight in the sense of Serge Bernstein may be regarded as being necessary and sufficient conditions for localisability in the situations in question.

The present article is devoted to the proof of some fairly general sufficient conditions for localisability which, in suitable senses not described here, are fairly close to being necessary too. Notice that $\mathcal{W}$ will be dense in $CV_\infty(E)$ whenever:

(1) $\mathcal{W}$ satisfies a sufficient condition for localisability under $\mathcal{A}$ in $CV_\infty(E)$;

(2) $\mathcal{Q}$ is separating on E.

(3) $\mathcal{W}$ is everywhere different from zero on E.

We point out also that, in the complex case, $\mathcal{Q}$ is always assumed here to be self-adjoint.

Our main general result is Theorem 2. It reduces the search of sufficient conditions for localisability on a completely regular space to the search of sufficient conditions for a weight on R^n to be fundamental in the classical sense of **Bernstein.** The next general result which follows from Theorem 2, is Theorem 4. It reduces the search of sufficient conditions for localisability on a completely regular space to the search of sufficient conditions for a weight on R to be fundamental. Theorems 3 and 5 are slight variations of Theorems 2 and 4, respectively. We note that the proof of Theorem 2 as given here is based on Theorem 1 (a different proof of Theorem 2 is given in [7]). According to Theorem 1, there is always localisability in the so-called bounded case, which essentially is the case where all functions belonging to $\mathcal{Q}$ are bounded. The emphasis that we put here on the bounded case, as a bridge to reach general results, has its motivation in the Fourier transform. As a matter of fact, if we look at some classical proofs of sufficiency of conditions for a weight on R^n to be fundamental, we realize that the success of the use of the Fourier transform is in part due to the fact that the function $t \longrightarrow \exp(it)$ is bounded on R. Finally, by combining Theorem 5 with classical sufficient conditions for a weight on R to be fundamental, we get some fairly general sufficient conditions for localisability of a more practical nature; see Theorems 6, 7 and 8.

The results of this article were summarized in [5], in the real case, under the assumption that E is locally compact and V is reduced to the constant function 1, hence $\mathcal{C}V_\infty(E) = \mathcal{C}_\infty(E)$. Under such assumptions, Malliavin [3] had proved the corollary to Theorem 7 which asserts the following: if $\mathcal{A}$ is separating on E, $\mathcal{W}$ has one generator w which is everywhere different from zero on E, and the divergence condition in Theorem 7 holds for this fixed w and all $a \in A$, then $\mathcal{W}$ is dense in $\mathcal{C}_\infty(E)$. Malliavin's methods are entirely different from ours, as he does not prove our Theorems 2, 3, and 5. We would like also to point out that Theorem 7 of this article follows from the main theorem on quasi-analytic mappings indicated in [6]. This approach will be developed elsewhere.

2. NOTATION AND TERMINOLOGY

We shall denote by R and C the systems of all real and of all complex numbers, respectively. We shall refer to R or C indistinctly by K. The elements of K are called scalars.

Throughout this article, E will denote a completely regular space. $\mathcal{C}(E)$ is the topological algebra of all continuous K-valued functions on E, endowed with the compact-open topology. $\mathcal{C}_b(E)$ is the Banach algebra, actually a subalgebra of $\mathcal{C}(E)$, of all functions belonging to $\mathcal{C}(E)$ that are bounded on E, endowed with the supremum norm.

A K-valued function f on E is said to vanish at infinity if, for any $\varepsilon > 0$, the set $\{|f| \geqslant \varepsilon\} = \{x | x \in E, |f(x)| \geqslant \varepsilon\}$ is relatively compact. This is surely the case when the support of f is compact. When f is continuous, the above definition is of interest only if E is locally compact. We shall use the above definition in the case of f such that $|f|$ is upper-semi-continuous, without assuming E locally compact: then the closed set $\{|f| \geqq \varepsilon\}$ must be compact, for any $\varepsilon > 0$; it follows that f is bounded.

Assume now that E is locally compact. $\mathcal{C}_\infty(E)$ is the closed ideal of $\mathcal{C}_b(E)$, hence a Banach algebra, of all functions belonging to $\mathcal{C}(E)$ that vanish at infinity, endowed with the norm induced by $\mathcal{C}_b(E)$. Moreover $\mathcal{K}(E)$ is the ideal in $\mathcal{C}(E)$ of all functions belonging to $\mathcal{C}(E)$ with compact supports. Notice that $\mathcal{K}(E)$ is a topological algebra with respect to its natural inductive limit topology.

In case it appears necessary to avoid any misunderstanding, we shall include K in our notation by writing $\mathcal{C}(E; K)$ in place of $\mathcal{C}(E)$; and similarly for the other spaces introduced above and defined below. Except as stated otherwise, we follow the terminology of Bourbaki [1].

3. WEIGHTED LOCALLY CONVEX SPACES OF CONTINUOUS FUNCTIONS

Definition 1. Introduce a set V of upper-semi-continuous positive real valued functions on E, whose elements shall be referred to as being weights. The functions $f \in \mathcal{C}(E)$ such that vf is bounded on E for any $v \in V$ form a vector subspace $\mathcal{C}V_b(E) \subset \mathcal{C}(E)$. If $v \in V$, then $f \longrightarrow \|f\|_v = \sup\{v(x).|f(x)| \,|\, x \in E\}$ is a semi-norm on $\mathcal{C}V_b(E)$. We shall endow $\mathcal{C}V_b(E)$ with the so called weighted topology, which is determined by the family of all such semi-norms corresponding to $v \in V$. Thus $\mathcal{C}V_b(E)$ becomes a locally convex space. Notice that $\mathcal{C}V_b(E)$ is a sub-module of $\mathcal{C}(E)$ over the algebra $\mathcal{C}_b(E)$.

Definition 2. In the notation of Definition 1, consider the vector subspace $\mathcal{C}V_\infty(E)$ of all $f \in \mathcal{C}(E)$ such that vf vanishes at infinity for any $v \in V$. It is clear that $\mathcal{C}V_\infty(E)$ is a closed vector subspace of $\mathcal{C}V_b(E)$. We shall endow $\mathcal{C}V_\infty(E)$ with the weighted topology induced by $\mathcal{C}V_b(E)$, so that $\mathcal{C}V_\infty(E)$ is a locally convex space. Notice that $\mathcal{C}V_\infty(E)$ is a sub-module of $\mathcal{C}V_b(E)$ over the algebra $\mathcal{C}_b(E)$.

We shall refer to $\mathcal{C}V_b(E)$ and $\mathcal{C}V_\infty(E)$ as weighted locally convex spaces of continuous scalar-valued functions. Actually we shall be more interested in $\mathcal{C}V_\infty(E)$, as $\mathcal{C}V_b(E)$ has merely an auxiliary role.

Remark 1. V will be said to be directed if, given v_1, $v_2 \in V$, there exist $\lambda > 0$ and $v \in V$ such that $v_1 \leq \lambda v$, $v_2 \leq \lambda v$. Without affecting the locally convex spaces $\mathcal{C}V_b(E)$ and $\mathcal{C}V_\infty(E)$, we may replace V by a larger directed set. Accordingly there is no essential loss of generality in assuming V to be directed, as we shall do.

Remark 2. When V is reduced to a single function v, we shall write $\mathcal{C}v_b(E)$ and $\mathcal{C}v_\infty(E)$ in place of $\mathcal{C}V_b(E)$ and $\mathcal{C}V_\infty(E)$

respectively. Then we shall consider these two vector spaces as being semi-normed by $f \longrightarrow \|f\|_v$. Actually the consideration of $\mathcal{C}v_\infty(E)$ is interesting only when E is locally compact; then $\mathcal{C}v_\infty(E)$ is the closure of $\mathcal{K}(E)$ in $\mathcal{C}v_b(E)$.

Remark 3. When V is the set of characteristic functions of all compact subsets of E, then $\mathcal{C}V_b(E) = \mathcal{C}V_\infty(E) = \mathcal{C}(E)$ as locally convex spaces. If V is reduced to the constant function 1, then $\mathcal{C}V_b(E) = \mathcal{C}_b(E)$; if moreover E is locally compact, then $\mathcal{C}V_\infty(E) = \mathcal{C}_\infty(E)$.

4. THE WEIGHTED APPROXIMATION PROBLEM

Definition 3. Let V and $\mathcal{C}V_\infty(E)$ be as in Definition 2. Consider a subalgebra $\mathcal{A} \subset \mathcal{C}(E)$ and a vector subspace $\mathcal{W} \subset \mathcal{C}V_\infty(E)$. Assume that $\mathcal{W}$ is an $\mathcal{A}$-module, that is $\mathcal{A}\mathcal{W} \subset \mathcal{W}$. Without essential loss of generality, we may assume that $\mathcal{A}$ contains the unit function 1. The weighted approximation problem consists of asking for a description of the closure of $\mathcal{W}$ in $\mathcal{C}V_\infty(E)$ under such circumstances.

In the special case in which $\mathcal{A}$ consists only of constants, $\mathcal{W}$ is the most general vector subspace of $\mathcal{C}V_\infty(E)$. In such a case, there is not much that we can say to describe the closure of $\mathcal{W}$ in $\mathcal{C}V_\infty(E)$: we may at most describe the dual space of $\mathcal{C}V_\infty(E)$ and apply the Hahn-Banach theorem. The attack on the weighted approximation problem that we shall discuss in the present article consists precisely in reducing the general case to the special one just mentioned. This will be achieved by looking at the subsets of E on which the functions belonging to $\mathcal{A}$ are constant. Such a viewpoint materializes in the following definition.

Definition 4. In the notation of Definition 3, $\mathcal{A}$ defines an equivalence relation $E/\mathcal{A}$ on E if we write $x_1 \sim x_2$, whenever $x_1, x_2 \in E$ and $f(x_1) = f(x_2)$ for any $f \in \mathcal{A}$. Given an equivalence class $X \subset E$ modulo $E/\mathcal{A}$, we may consider the set $V|X$ of the restrictions to X of all functions belonging to V and the locally convex space $\mathcal{C}(V|X)_\infty(X)$, which contains as a vector subspace the set $\mathcal{W}|X$ of the restrictions to X of all functions belonging to $\mathcal{W}$. We shall say that $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$ when the following condition holds true: a function f belonging to $\mathcal{C}V_\infty(E)$ is in the closure of $\mathcal{W}$ in $\mathcal{C}V_\infty(E)$ if (and always only if) its restriction $f|X$ to X belongs to the closure of $\mathcal{W}|X$ in $\mathcal{C}(V|X)_\infty(X)$, for any equivalence class X modulo $E|\mathcal{A}$. This condition means the following: a function f belonging to $\mathcal{C}V_\infty(E)$ is in the closure of $\mathcal{W}$ in $\mathcal{C}V_\infty(E)$ if (and only if), for any $v \in V$, any $\varepsilon > 0$ and any equivalence class X modulo $E/\mathcal{A}$, there exists some $w \in \mathcal{W}$ such that $v(x).|w(x) - f(x)| < \varepsilon$ for any $x \in X$. The strict weighted approximation problem consists of asking for necessary and sufficient conditions in order that $\mathcal{W}$ be localisable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$.

Remark 4. $\mathcal{W}$ will be dense in $\mathcal{C}V_\infty(E)$ provided the following conditions are satisfied:

(1) $\mathcal{W}$ satisfies a sufficient condition for localisability

under $\mathcal{A}$ in $\mathcal{C}V_{\infty}(E)$;

(2) $\mathcal{A}$ is separating on E, that is, if $x_1, x_2 \in E$, $x_1 \neq x_2$, there exists some $a \in \mathcal{A}$ such that $a(x_1) \neq a(x_2)$;

(3) $\mathcal{W}$ is everywhere different from zero on E, that is, if $x \in E$, there exists some $w \in \mathcal{W}$ such that $w(x) \neq 0$.

Therefore, corresponding to every sufficient condition for localisability established below, there is a corollary of density.

5. THE BOUNDED CASE OF THE WEIGHTED APPROXIMATION PROBLEM

Definition 5. In the notation of Definition 3, the bounded case of the weighted approximation problem is the one in which every $a \in \mathcal{A}$ is bounded on the support of every $v \in V$. Each of the following assumptions leads to a noteworthy instance of the bounded case:

(1) $\mathcal{A} \subset \mathcal{C}_b(E)$;

(2) each $v \in V$ has a compact support.

LEMMA 1. Let $\mathcal{A}$ be a closed subalgebra containing 1 of $\mathcal{C}_b(E)$, where $\mathcal{A}$ is assumed to be self-adjoint in the complex case. For every equivalence class X modulo $E/\mathcal{A}$, let there be associated a compact subset K_X of E disjoint from X. Then there exist a finite subset L of the set of such equivalence classes and functions $\varphi_X \in \mathcal{A}$ such that $\varphi_X \geq 0$, $\varphi_X | K_X = 0$ for $X \in L$, and $\sum_{X \in L} \varphi_X = 1$.

PROOF. Introduce the Stone-Čech compactification $\beta E \supset E$. We have the natural Banach algebra isomorphism $\beta : \mathcal{C}_b(E) \to \mathcal{C}(\beta E)$ which to every $f \in \mathcal{C}_b(E)$ associates its unique continuous K-valued extension $\beta f \in \mathcal{C}(\beta E)$. Let $\beta\mathcal{Q}$ be the closed image of $\mathcal{Q}$ under β. It determines the equivalence relation $\beta E/\beta\mathcal{Q}$ on βE. Call F the compact quotient space of βE by $\beta E/\beta\mathcal{Q}$ and let $\pi: \beta E \to F$ be the corresponding natural continuous mapping. If $Y \in F$ and $Y \cap E \neq \emptyset$, that is $Y \in \pi(E)$, then $Y \cap E$ is an equivalence class in E modulo $E/\mathcal{Q}$. In this way, we set a one-to-one natural correspondence between the points of $\pi(E)$ and the equivalence classes of E modulo $E/\mathcal{Q}$. If $Y \in \pi(E)$, it follows from our **assumptions** that $Y \cap E$ and $K_{Y \cap E}$ are disjoint; hence Y does not belong to the compact subset $\pi(K_{Y \cap E})$ of $\pi(E)$. This implies that $\bigcap_{Y \in \pi(E)} \pi(K_{Y \cap E}) = \emptyset$. By the finite intersection property for compact subsets, there is a finite subset M of $\pi(E)$ such that $\bigcap_{Y \in M} \pi(K_{Y \cap E}) = \emptyset$. By the method of the continuous partition of the unity, we may find functions $\Psi_Y \in \mathcal{C}(F)$ such that $\Psi_Y \geq 0$, $\Psi_Y | \pi(K_{Y \cap E}) =$ for $Y \in M$, and $\sum_{Y \in M} \Psi_Y = 1$. Put $\Phi_Y = \Psi_Y \pi \in \mathcal{C}(\beta E)$, so that we have $\Phi_Y \in \beta\mathcal{Q}$ for all $Y \in M$, by the Weierstrass-Stone theorem. Call L the finite subset of the set of equivalence classes of E modulo $E/\mathcal{Q}$ that corresponds to the finite subset M of $\pi(E)$. It is then clear that $\varphi_X = \Phi_Y | E$ for $X = Y \cap E \in L$, where $Y \in M$, satisfies all the requirements q.e.d.

THEOREM 1. In the notation and terminology of Definitions 4 and 5, there is always localisability in the bounded case of the weighted approximation problem, provided we assume K = R, or

$K = C$ and $\mathcal{Q}$ self-adjoint.

PROOF. It is sufficient to prove the theorem by assuming that V is reduced to one function v. It is also sufficient to prove the theorem in case $\mathcal{Q} \subset \mathcal{C}_b(E)$, for we may reduce ourselves to such a case by replacing E by the support of v. Let then $f \in \mathcal{C}v_\infty(E)$ be such that, given any $\varepsilon > 0$ (to be kept fixed until the end of the proof) and any equivalence class $X \subset E$ modulo $E/\mathcal{Q}$, there exists some $w_X \in \mathcal{W}$ such that $v(x).|w_X(x) - f(x)| < \varepsilon$ for any $x \in X$. We want to conclude that f belongs to the closure of $\mathcal{W}$ in $\mathcal{C}v_\infty(E)$. The closed set $K_X = \{x | x \in E,\ v(x).|w_X(x) - f(x)| \geqslant \varepsilon\}$ is compact, since $w_X - f \in \mathcal{C}v_\infty(E)$. Moreover X and K_X are adjoint. By Lemma 1, there is a finite set L of equivalence classes in E modulo $E/\mathcal{Q}$ and, for each $X \in L$, there is φ_X belonging to the closure of $\mathcal{Q}$ in $\mathcal{C}_b(E)$ such that $\varphi_X \geq 0$, $\varphi_X | K_X = 0$ for $X \in L$ and $\sum_{X \in L} \varphi_X = 1$. We then notice that

$$\varphi_X(x)v(x).|w_X(x) - f(x)| \leq \varepsilon \cdot \varphi_X(x) \qquad \text{for any } x \in X \text{ and } X \in L. \tag{1}$$

In fact, either $x \in K_X$ and then $\varphi_X(x) = 0$; or else $x \notin K_X$, which means $v(x).|w_X(x) - f(x)| < \varepsilon$. In both cases, (1) holds true. From it we get

$$v(x).|\textstyle\sum_{X \in L} \varphi_X(x) w_X(x) - f(x)| \leqq \varepsilon \qquad \text{for any } x \in E. \tag{2}$$

Given any $\delta > 0$, there exists some $a_X \in \mathcal{Q}$ for each $X \in L$, such that $|a_X(x) - \varphi_X(x)| \leq \delta$ for any $x \in E$. Noticing that each vw_X is bounded on E, we get

$$v(x).|\textstyle\sum_{X \in L} a_X(x) w_X(x) - f(x)| \leqq 2\varepsilon \text{ for any } x \in E\ ,$$

provided δ is taken small enough. There remains to notice that $a_X w_X \in \mathcal{A}\mathcal{W} \subset \mathcal{W}$ for any $X \in L$, to conclude that $v(x).|w(x) - f(x)| \leq 2\varepsilon$ for any $x \in E$, where $w = \sum_{X\in L} a_X w_X \in \mathcal{W}$. Hence f belongs to the closure of $\mathcal{W}$ in $\mathcal{C}V_{\infty}(E)$, q.e.d.

6. REDUCTION OF THE TOPOLOGICAL CASE TO THE FINITE DIMENSIONAL CASE

DEFINITION 6. Let E be a real vector space of finite dimension n. Let us denote by $\mathcal{P}(E)$ the algebra of all K-valued polynomials on E (that is the subalgebra of the algebra $\mathcal{C}(E)$ generated by the K-valued constant functions on E and the K-valued linear froms on E). A weight ω on E, that is an upper-semi-continuous positive real-valued function on E, determines (see Remark 2) the semi-normed spaces $\mathcal{C}\omega_b(E)$ and $\mathcal{C}\omega_{\infty}(E)$; moreover $\mathcal{C}\omega_{\infty}(E)$ is the closure of $\mathcal{K}$ (E) in $\mathcal{C}\omega_b(E)$. The weight ω is said to be rapidly decreasing at infinity when $\mathcal{P}(E) \subset \mathcal{C}\omega_b(E)$, or equivalently $\mathcal{P}(E) \subset \mathcal{C}\omega_{\infty}(E)$. If, in addition to this, $\mathcal{P}(E)$ is dense in the semi-normed space $\mathcal{C}\omega_{\infty}(E)$, then ω is called a fundamental weight in the sense of Serge Bernstein. We shall denote by $\Omega(E)$ the set of all fundamental weights on E. When $E = R^n$, we write simply $\mathcal{P}_n = \mathcal{P}(R^n)$ and $\Omega_n = \Omega(R^n)$. For future reference, notice that $\mathcal{C}_b(E) \subset \mathcal{C}\omega_{\infty}(E)$ provided the weight ω vanishes at infinity, this being the case if ω is rapidly decreasing, hence if ω is fundamental.

DEFINITION 7. Let V, $\mathcal{C}V_\infty(E)$, $\mathcal{A}$ and $\mathcal{W}$ be as in Definition 3. We shall denote by A a subset of $\mathcal{A}$ which topologically generates $\mathcal{A}$ as an algebra over K with unit, that is such that the subalgebra over K of $\mathcal{A}$ generated by A and 1 is dense in $\mathcal{A}$ for the topology of $\mathcal{C}(E)$. We shall also introduce a subset W of $\mathcal{W}$ which topologically generates $\mathcal{W}$ as an $\mathcal{A}$-module, that is the $\mathcal{A}$-submodule of $\mathcal{W}$ generated by W is dense in $\mathcal{W}$ for the topology of $\mathcal{C}V_\infty(E)$.

THEOREM 2. In the notation and terminology of Definitions 4, 6 and 7, let us assume that $K = R$, or that $K = C$ and A consists only of real functions. Let us assume also that, for every $v \in V$, every $a_1, \dots, a_n \in A$ and every $w \in W$, there are $N \geqq n$, $a_{n+1}, \dots, a_N \in A$ and $\omega \in \Omega_N$ such that

$$v(x).|w(x)| \leqq \omega\left[a_1(x), \dots, a_n(x), \dots, a_N(x)\right] \quad \text{for any } x \in E. \quad (1)$$

Then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$.

PROOF. We start with the following remark, to be used at the end of the proof. If $v \in V$, $a_1, \dots, a_n \in A$, $w \in W$, $\alpha \in \mathcal{C}_b(R^n)$ and $\delta > 0$ are given, there exists some $w' \in \mathcal{W}$ such that

$$v(x).|w'(x) - \alpha\left[a_1(x), \dots, a_n(x)\right]w(x)| < \delta \quad \text{for any } x \in E \ . \quad (2)$$

In fact, by virtue of the assumption made in the statement of the theorem there are $N \geqq n$, $a_{n+1}, \dots, a_N \in A$ and $\omega \in \Omega_N$ such that (1) holds true. Define $\alpha' \in \mathcal{C}_b(R^N)$ by $\alpha'(t_1, \dots, t_n, \dots, t_N) = \alpha(t_1, \dots, t_n)$ for $t_1, \dots, t_n, \dots, t_N \in R$. Since $\mathcal{C}_b(R^N) \subset \mathcal{C}\omega_\infty(R^N)$, there exists some $p \in \mathcal{P}_N$ such that

$$\omega(t_1,\ldots, t_n,\ldots,t_N).|p(t_1,\ldots,t_n,\ldots,t_N)-\alpha(t_1,\ldots,t_n)|<\delta$$
$$\text{for any } t_1,\ldots,t_n,\ldots,t_N\in R \,. \qquad (3)$$

From (1) and (3) we get, letting $a = p(a_1,\ldots,a_n,\ldots,a_N)\in\mathcal{A}$ and $w' = aw\in \mathcal{A}\mathcal{W}\subset\mathcal{W}$,

$$v(x).|w'(x) - \alpha\left[a_1(x),\ldots,a_n(x)\right]w(x)|$$
$$\leqq\omega\left[a_1(x),\ldots,a_n(x),\ldots,a_N(x)\right].\left|a(x)-\alpha\left[a_1(x),\ldots,a_n(x)\right]\right|<\delta$$
$$\text{for any } x\in E \,,$$

which proves (2).

Let us now prove the theorem. Let $f\in \mathcal{C}V_\infty(E)$ be such that corresponding to any $v\in V$, any $\varepsilon>0$ and any equivalence class $X\subset E$ modulo $E/\mathcal{A}$, there exists some $w\in\mathcal{W}$ such that $v(x).|w(x)-f(x)|<\varepsilon$ for all $x\in X$. We may assume that w belongs to the vector subspace of $\mathcal{W}$ generated by W, that is w is a finite linear combination of elements of W with scalar coefficients. This is due to the fact that the $\mathcal{A}$-submodule generated by W is dense in $\mathcal{W}$ for the topology of $\mathcal{C}V_\infty(E)$ and to the fact that the functions belonging to $\mathcal{A}$ are constant on X. Let us introduce the subalgebra $\mathcal{A}'$ of $\mathcal{C}(E)$ of all functions of the form $\alpha(a_1,\ldots,a_n)$, where $n\geqq 1$, $a_1,\ldots,a_n\in A$ and $\alpha\in \mathcal{C}_b(R^n)$ are arbitrary. Notice that $\mathcal{A}'\subset \mathcal{C}_b(E)$. Next let us call $\mathcal{W}'$ the $\mathcal{A}'$-submodule of $\mathcal{C}(E)$ generated by W. It is formed by all functions of the form $a_1w_1+\ldots+a_nw_n$, where $n\geqq 1$, $a_1,\ldots,a_n\in\mathcal{A}'$ and $w_1,\ldots,w_n\in W$ are arbitrary. Since $W\subset \mathcal{C}V_\infty(E)$ and $\mathcal{C}V_\infty(E)$ is a $C_b(E)$-module, it follows that $\mathcal{W}'\subset \mathcal{C}V_\infty(E)$. Notice that the equivalence relations $E/\mathcal{A}$ and

and $E/\mathcal{A}'$ coincide because the subalgebra of $\mathcal{A}$ generated by A and 1 is dense in $\mathcal{A}$. Notice also that the vector subspace of $\mathcal{W}$ generated by W is contained in $\mathcal{W}'$ too. By Theorem 1, $\mathcal{W}'$ is localisable under $\mathcal{A}'$ in $\mathcal{C}V_\infty(E)$. It then follows from our assumptions and all that has been asserted, that f belongs to the closure of $\mathcal{W}'$ in $\mathcal{C}V_\infty(E)$. Using then the remark made at the beginning of this proof, we readily deduce from this that f belongs to the closure of $\mathcal{W}$ in $\mathcal{C}V_\infty(E)$, q.e.d.

COROLLARY 1. In the notation and terminology of Theorem 2, let us assume that A consists of $a_1,\ldots,a_n$ and W of $w_1,\ldots,w_m$. Let us assume also that K = R, or that K = C and A consists only of real functions. If for every $v \in V$ and every $i = 1,\ldots,m$, there exists some $\omega \in \Omega_n$ such that

$$v(x).|w_i(x)| \leq \omega\left[a_1(x),\ldots,a_n(x)\right] \quad \text{for every } x \in E ,$$

then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$.

REMARK 5. A different approach to the proof of Theorem 2 not using Theorem 1 (hence not using the Weierstrass-Stone theorem) is to be found in [7].

In the complex case of Theorem 2, we have assumed that A consists only of real functions. In order to drop this assumption, we introduce the following definition.

DEFINITION 8. We shall consider R^n as a vector lattice in the usual way: if $x = (x_1,\ldots,x_n) \in R^n$ and $y = (y_1,\ldots,y_n) \in R^n$, write $x \leqq y$ provided $x_i \leqq y_i$ $(i=1,\ldots,n)$; and put $|x| = (|x_1|,\ldots,|x_n|) \in R^n$. We shall denote by Ω_n^d the set of all $\omega \in \Omega_n$ (see Definition 6) which are modulus-decreasing, that is such that $x, y \in R^n$ and $|x| \leqq |y|$ imply $\omega(x) \geqq \omega(y)$.

THEOREM 3. In the notation and terminology of Definitions 4, 7 and 8, let us assume that $K = R$, or that $K = C$ and $\mathcal{A}$ is self-adjoint. Let us assume also that, for every $v \in V$, every $a_1,\ldots,a_n \in A$ and every $w \in W$, there are $N \geqq n$, $a_{n+1},\ldots,a_N \in A$ and $\omega \in \Omega_N^d$ such that

$$v(x).|w(x)| \leqq \omega\left[|a_1(x)|,\ldots,|a_n(x)|,\ldots,|a_n(x)|\right] \text{ for any } x \in E.$$

Then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$.

PROOF. The case $K = R$ is uninteresting, as it is trivially subsumed by Theorem 2 for $K = R$. Let $K = C$. Every $a \in A$ may be written uniquely as $a = a' + ia''$, where a' and a'' are real. Call A' and A'' the sets of all such a' and a'', respectively. Since $\mathcal{A}$ is self-adjoint, $A' \cup A''$ and 1 generate a subalgebra over C which is dense in $\mathcal{A}$. If we notice that $\omega \in \Omega_N^d$ and that $|a'| \leqq |a|, |a''| \leqq |a|$, we see that we may apply Theorem 2, with $K = C$ and A replaced by $A' \cup A''$, q.e.d.

COROLLARY 2. To Theorem 3 there corresponds a corollary which is analogous to Corollary 1 to Theorem 2.

REMARK 6. Although Theorems 2 and 3 will be sufficient in the next section by taking $N = n$, we notice that they are not sufficient for the proofs of Corollaries 1 and 2, if we take $N = n$.

7. REDUCTION OF THE TOPOLOGICAL CASE TO THE ONE-DIMENSIONAL CASE

LEMMA 2. Let $E_1,\dots,E_n$ be finite dimensional real vector spaces and $E = E_1\times\dots\times E_n$. Consider $\omega_i \in \Omega(E_i)(i=1,\dots,n)$ and define the tensor product ω on E by $\omega(t) = \omega_1(t_1)\dots\omega_n(t_n)$, where $t = (t_1,\dots,t_n) \in E$. Then $\omega \in \Omega(E)$.

PROOF. It is clear that ω is an upper-semi-continuous positive real-valued function on E. It is also clear that ω is rapidly decreasing at infinity. Consider the following **commutative** diagram of spaces and mappings

$$\begin{array}{ccc}
\mathcal{P}(E_1)\times\dots\times\mathcal{P}(E_n) & \xrightarrow{\tau_2} & \mathcal{P}(E) \\
{\scriptstyle j_2}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\
(\mathcal{C}\omega_1)_\infty(E_1)\times\dots\times(\mathcal{C}\omega_n)_\infty(E_n) & \xrightarrow{\tau} & \mathcal{C}\omega_\infty(E) \\
{\scriptstyle j_1}\big\uparrow & & \big\uparrow{\scriptstyle i_1} \\
\mathcal{K}(E_1)\times\dots\times\mathcal{K}(E_n) & \xrightarrow[\tau_1]{} & \mathcal{K}(E)
\end{array}$$

The vertical mappings i_1, i_2, j_1 and j_2 are inclusion mappings. The horizontal mapping τ is defined by tensor multiplication as follows. If $f_i \in (\mathcal{C}\omega_i)_\infty(E_i)$ $(i=1,\dots,n)$, then $f = \tau(f_1,\dots,f_n) \in \mathcal{C}\omega_\infty(E)$ is defined by $f(t) = f_1(t_1)\dots f_n(t_n)$, where $t = (t_1,\dots,t_n) \in E$. The horizontal mappings τ_1 and τ_2 are defined as restrictions of τ. We note that the vector spaces $\mathcal{K}(E_i)(i=1,\dots,n)$ and $\mathcal{K}(E)$ will be endowed with their natural inductive limit topologies. However, it is trivial to modify our argument so as to avoid the use of these topologies; this is done by focusing attention on their vector subspaces of functions having supports contained in fixed compact subsets. The mapping τ is continuous. In fact, it is multilinear and $\| \tau(f_1,\ldots,f_n)\|_\omega = \|f_1\|_{\omega_1}\ldots\|f_n\|_{\omega_n}$ for $f_i \in (\mathcal{C}\omega_i)_\infty(E_i)$ $(i = 1,\ldots,n)$. It is clear that τ_1, i_1 and j_1 are continuous too. The image of i_1 is dense. The vector subspace generated by the image of τ_1 is dense. The image of j_1 is dense. From these remarks and from $\tau j_1 = i_1 \tau_1$, it follows that the vector subspace generated by the image of τ is dense. On the other hand, the image of j_2 is dense. The vector subspace generated by the image of τ_2 is the whole space. From these remarks and from $i_2 \tau_2 = \tau j_2$, it follows that the image of i_2 is dense, that is ω is a fundamental weight, q.e.d.

DEFINITION 9. Let E be a real vector space of finite dimension n. We shall denote by $\Gamma(E)$ the set of all weights γ, that is upper-semi-continuous positive real-valued functions, on E such that γ^h is a fundamental weight on E for any $h > 0$. Notice that, if $\gamma^h \in \Omega(E)$ for some $h > 0$, then $\gamma^k \in \Omega(E)$ for all $k > h$. We have obviously $\Gamma(E) \subset \Omega(E)$, and it is known that this set inclusion is proper. When $E = R^n$, we shall write $\Gamma_n = \Gamma(R^n)$ for short.

THEOREM 4. In the notation and terminology of Definitions 4, 7 and 9, let us assume that K = R, or that K = C and A consists only of real functions. Let us assume also that, for every $v \in V$, every $a \in A$ and every $w \in W$, there exists some $\gamma \in \Gamma_1$

such that

$$v(x).|w(x)| \leq \gamma[a(x)] \qquad \text{for any } x \in E.$$

Then $\mathcal{W}$ is localisable under $\mathcal{Q}$ in $\mathcal{C}V_\infty(E)$.

PROOF. We shall prove that Theorem 2 is applicable with $N = n$. Let $v \in V$, $a_1, \ldots, a_n \in A$ and $w \in W$ be given. By the assumption, there are $\gamma_i \in \Gamma_1$ such that $v(x).|w(x)| \leq \gamma_i[a_i(x)]$ for any $i = 1, \ldots, n$ and any $x \in E$. There results that $v(x).|w(x)| \leq \gamma[a_1(x), \ldots, a_n(x)]$ for any $x \in E$, where γ is defined on R^n by $\gamma(t) = [\gamma_1(t_1) \ldots \gamma_n(t_n)]^{1/n}$ for $t = (t_1, \ldots, t_n) \in R^n$. Since $\gamma_i \in \Gamma_1$, we have that $(\gamma_i)^{1/n} \in \Omega_1 (i = 1, \ldots, n)$, hence $\gamma \in \Omega_n$, by Lemma 2. q.e.d.

DEFINITION 10. In the notation of Definitions 8 and 9, we shall denote by Γ_n^d the set of all $\gamma \in \Gamma_n$ that are modulus-decreasing, that is such that $x, y \in R^n$ and $|x| \leq |y|$ imply $\gamma(x) \geq \gamma(y)$.

THEOREM 5. In the notation of Definitions 4, 7 and 10, let us assume that $K = R$, or that $K = C$ and that $\mathcal{Q}$ is self-adjoint. Let us assume also that, for every $v \in V$, every $a \in A$ and every $w \in W$, there exists some $\gamma \in \Gamma_1^d$ such that

$$v(x).|w(x)| \leq \gamma[|a(x)|] \qquad \text{for any } x \in E.$$

Then $\mathcal{W}$ is localisable under $\mathcal{Q}$ in $\mathcal{C}V_\infty(E)$.

PROOF. The proof consists in reducing Theorem 5 to Theorem 4, in the same way that Thoerem 3 was reduced to Theorem 2.

8. PRACTICAL SUFFICIENT CONDITIONS FOR LOCALISABILITY

The notation and terminology will be that described in Definitions 4 and 7. $\mathcal{A}$ is assumed to be self-adjoint in the complex case. The classical results concerning Bernstein's fundamental weights to be used below are to be found in [2] and [4].

THEOREM 6. If for any $v \in V$, any $a \in A$ and any $w \in W$, there are $C > 0$ and $c > 0$ such that

$$v(x).|w(x)| \leq C.\exp\left[-c.|a(x)|\right] \qquad \text{for any } x \in E,$$

then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $CV_\infty(E)$.

PROOF. This theorem results immediately from Theorem 5 and the following classical result. Let $\gamma(t) = \exp(-|t|)$ for $t \in R$ Then $\gamma \in \Gamma_1^d$, q.e.d.

THEOREM 6 gives us one of the simplest sufficient conditions for localisability. It is tied up with the concept of analytic mappings, whereas Theorem 7 below is associated with the concept of quasi-analytic mappings.

REMARK 8. If every $a \in A$ is bounded on the support of vw, for every $v \in V$ and every $w \in W$, it follows immediately from Thoerem 6 that $\mathcal{W}$ is localisable under $\mathcal{A}$ in $CV_\infty(E)$. This extends Theorem 1 and may be proved directly as Theorem 1. Therefore Theorem 1 follows from Theorem 6. When V is the set of characteristic functions of all compact subsets of E, it follows from Theorem 1, hence from Theorem 6, that we always have localisability of $\mathcal{W}$ under $\mathcal{A}$ in $\mathcal{C}(E)$. For $\mathcal{W} = \mathcal{A}$ (or more generally for $\mathcal{W}$ a subalgebra of $\mathcal{C}(E)$ and $\mathcal{A}$ the subalgebra of $\mathcal{C}(E)$ generated by $\mathcal{W}$ and 1), we then get the Weierstrass-Stone theorem as a special case of Theorem 1, hence of Theorem 6. Of course, it should be noted that Theorem 6 was proved by using Theorem 2, which was based here on Theorem 1. Once we prove Theorem 2 as noticed in Remark 5, we may say that Theorem 6 genuinely contains the Weierstrass-Stone theorem: this corresponds simply to the fact that the original Weierstrass approximation theorem is a consequence of the remark that the weight $t \longrightarrow \exp(-|t|)$ is fundamental on R.

THEOREM 7. If for any $v \in V$, any $a \in A$, and any $w \in W$, we have

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt[m]{M_m}} = \infty$$

where

$$M_m = \sup\{v(x). |[a(x)]^m w(x)| \,|\, x \in E\} \qquad (m = 0,1,\ldots),$$

then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}V_{\infty}(E)$.

PROOF. This theorem results immediately from Theorem 5 and the following classical result. Let γ be an upper-semi-continuous positive real-valued function on R. Assume that γ is rapidly decreasing at infinity. Put

$$N_m = \sup\{|t^m| \cdot \gamma(t) \,|\, t \in R\} \qquad (m = 0,\ 1,\ \ldots)$$

and assume that

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt[m]{N_m}} = \infty .$$

Then $\gamma \in \Gamma_1$. By using this result, let us prove the theorem.

Take $M_m (m = 0, 1, \ldots)$ to be the sequence defined in the statement of the theorem. Define

$$\gamma(t) = \inf \left\{ \frac{M_m}{|t^m|} \middle| m = 0, 1, \ldots \right\} \qquad \text{for any } t \in R$$

where we take $\gamma(0) = M_0$. If $N_m (m = 0, 1, \ldots)$ is the sequence defined above, we have $N_m \leqq M_m$ $(m = 0, 1, \ldots)$. It follows then from what we said above that $\gamma \in \Gamma_1^d$. Moreover we have obviously $v(x).|w(x)| \leqq \gamma\left[|a(x)|\right]$ for any $x \in E$, q.e.d.

THEOREM 8. If for any $v \in V$, any $a \in A$, and $w \in W$, there exists some $\gamma \in \mathcal{C}(R)$ satisfying the following conditions

$\gamma(t) > 0$ and $\gamma(t) = \gamma(-t)$ for $t \in R$,

$\gamma(t)$ is decreasing for $t \geqq 0$,

$\log \dfrac{1}{\gamma(t)}$ is convex in $\log t$ for $t > 0$,

$$\int^{+\infty} \frac{1}{t^2} \cdot \log \frac{1}{\gamma(t)} \cdot dt = \infty,$$

for which

$$v(x).|w(x)| \leqq \gamma\left[|a(x)|\right] \qquad \text{for any } x \in E,$$

then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}V_\infty(E)$.

PROOF. This theorem follows from Theorem 5 and the classical result that then $\gamma \in \Gamma_1^d$, q.e.d.

REMARK 8. Theorem 6 is easily seen to be a particular case of either Theorem 7 or Theorem 8.

REMARK 9. Theorem 3 remains true if ω is supposed to belong to Ω_N and to satisfy $\omega(x) \geqq \omega(y)$ provided $|x| \leqq |y|$ and x is out side a suitably large compact subset of R^N. In fact, there exists then an upper-semi-continuous positive real-valued function ω' on R^N such that $\omega'(x) = \omega(x)$ for x outside a suitably large compact subset, such that $\omega'(x) \geqq \omega(x)$ for all x, and such that $\omega'(x) \geqq \omega'(y)$ provided $|x| \leqq |y|$ for all x and y. Then $\omega' \in \Omega_N^d$, which justifies our essertion. A similar remark applies to Theorem 5. Then concerning Theorem 8 we may say the following. We may dispense altogether the assumption that $\gamma(t)$ is decreasing for $t \geqq 0$, and it is sufficient to assume that $\log 1/\gamma(t)$ is convex in $\log t$ for $t > a$, where $a \geqq 0$ is large enough. As a matter of fact this type of convexity for $\gamma(t)$ implies that $\gamma(t)$ is decreasing for $t > a$, provided a is large enough.

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