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FLAT-SPACE METRIC IN THE QUATERNION FORMULATION OF GENERAL RELATIVITY

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ABSTRACT

Using the quaternion representation of the Riemannian space-time of general relativity it is formulated the description of the gravitational field in flat space. The method presently used is an extension for the quaternion fields of a formalism introduced by Rosen. It is stablished the theorem of quasi-inertia. The equations of the gravitational field are obtained in terms of spinors which are function of quaternions defined on the flat space. The equation of motion of a test particle is also obtained.

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INTRODUCTION

L. Civita many years ago showed the possibility of associating two different metrics to a same space in a given system of coordinates [1]. The significance of this result may be interpreted in several ways. We indicated here two possible interpretations, the first which we may call a global interpretation is that of imbedding the four dimensional Riemannian manifold into a ten-dimensional flat space. A mapping is then stablished between the Riemannian space and a given four-dimensional sub-space of the ten dimensional flat space in such way that we assign the same value to the coordinates of corresponding points. The second possible interpretation is to consider the flat space metric as describing the geometrical properties of the space as if we had no gravitational fields present. In this way it is possible, in a same formalism, to compare directly the differences which arise from the simultaneous use of these two metrics. This type of interpretation may be subsequently re-interpreted by considering the $g_{\mu\nu}$ which initially was associate to the Riemannian space as the potentials of the gravitational field, the space being now flat in all its extensions, with the other metric tensor playing the role of the metric. This point of view was used by Rosen [2].

In this paper we consider the Rosen's formulation but now written in terms of quaternion fields which take over the place of the symmetric second order metric tensors of the conventional tensor formulation of general relativity. Due to this modification of the basic formalism several results of Rosen's method are presently generalized since the quaternion fields, or its corresponding fields, the spinors posess more degrees of freedom than the symmetric second rank tensors.

The results which are obtained by this method are equivalent to those belonging to the other flat space theories which are well known.

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1. QUATERNIONS IN FLAT SPACE

In the Minkowski space it is possible to write the arc element in the form

$$dl^2 = \gamma_{\mu\nu}(x)dx^\mu\, dx^\nu \tag{1}$$

where the coordinate x^μ are related to the cartesian set of coordinates by means of known functions (the $\bar{x}^\mu$ denote the cartesian coordinates),

$$x^\mu = x^\mu(\bar{x})$$

and thus,

$$\gamma_{\mu\nu}(x) = \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\,\frac{\partial \bar{x}^\beta}{\partial x^\nu}\, g_{\alpha\beta}.$$

We may introduce at each space-time point x^μ where $\gamma_{\mu\nu}$ is defined according to the above relation, a system of four quaternion fields $\Sigma_\mu(x)$ such that

$$\Sigma_\mu(x)|\,\Sigma_\nu(x) = \frac{1}{2}\left(\Sigma_\mu(x)\,\bar{\Sigma}_\nu(x) + \Sigma_\nu(x)\,\bar{\Sigma}_\mu(x)\right) = \Sigma_\nu(x)|\,\Sigma_\mu(x) \tag{2}$$

This relation defines the scalar product of quaternions. The quaternion $\bar{\Sigma}_\mu$ is the adjoint of the quaternion Σ_μ.[3] Taking as the quaternion basis the set of the three Pauli matrices together with the two-by-two identity matrix, we will have

$$\overset{\circ}{\sigma}_{(0)}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \overset{\circ}{\sigma}_{(1)}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \overset{\circ}{\sigma}_{(2)}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \overset{\circ}{\sigma}_{(3)}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\Sigma_\mu(x) = h^{(\alpha)}_{\mu}(x)\ \overset{\circ}{\sigma}_{(\alpha)}\,. \qquad (3)$$

From the equations (2) and (3) we get for the relation (1),

$$dl^2 = \Sigma_\mu(x)\,|\,\Sigma_\nu(x)dx^\mu dx^\nu\,. \qquad (4)$$

The Riemann-Christoffel tensor constructed with the metric $\gamma_{\mu\nu}(x)$ vanish over all space, and this is the condition which implies that the space is flat.

From the equations (1), (3) and (4) it follows that the coefficients h which here form the tetrad components, are

$$h^{(\alpha)}_{\mu} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \qquad (5)$$

its reciprocal matrix is given by the matrix elements

$$h^{\mu}_{(\alpha)} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\,. \qquad (6)$$

2. QUATERNIONS IN CURVED SPACES

In a Riemannian space where it is used the same coordinate system as that used previously, we define the element of arc by means of the relation

$$gs^2 = g_{\mu\nu}(x)\, dx^{\mu}\, dx^{\nu} \tag{7}$$

Similarly as before we may introduce at each space-time point a quaternion field $\sigma_{\mu}(x)$ such that

$$g_{\mu\nu}(x) = \sigma_{\mu}(x) | \sigma_{\nu}(x) \; . \tag{8}$$

The σ_{μ} is now written in terms of the quaternion basis of the previous section

$$\sigma_{\mu}(x) = k_{\mu}^{(\alpha)}(x)\; \overset{\circ}{\sigma}_{(\alpha)} \tag{9}$$

The tetrad components $h_{\mu}^{(\alpha)}(x)$ posess an inverse $k_{(\alpha)}^{\mu}$,

$$k_{\mu}^{(\alpha)}\; k_{(\alpha)}^{\nu} = \delta_{\mu}^{\nu} \tag{10-1}$$

$$k_{\mu}^{(\alpha)}\; k_{(\lambda)}^{\mu} = \delta_{(\lambda)}^{(\alpha)} \tag{10-2}$$

The use of an abstract space within which two metric fields are defined, in the same system of coordinates, is a mathematical concept first introduced by L. Civita [1]. This concept was also used by N. Rosen in the theory of general relativity [2].

3. CORRESPONDENCE BETWEEN THE CURVED AND THE FLAT SPACES

The existence of two metrics associated to the same system of coordinates implies in the possibility of stablishing a correlation among the components of these metrics. Such correlation may be obtained for the components of the affine connection associated to these two metrics. It may also be written directly for the two arc elements ds and dl. In what follows we first obtain the relationships between the two quaternion fields σ and Σ. The inverse of the relation (3) is

$$\overset{\circ}{\sigma}_{(\alpha)} = h^{\mu}_{(\alpha)} \Sigma_{\mu} \, .$$

Substitution of this into the right hand side of (9) gives

$$\sigma_{\mu} = k^{(\alpha)}_{\mu} \, h^{\lambda}_{(\alpha)} \, \Sigma_{\lambda} .$$

Introducing the composed tetrad components $H^{\lambda}{}_{\mu}$ by

$$H^{\lambda}(x)_{\mu} = h^{\lambda}(x)_{(\alpha)} \, k^{(\alpha)}(x)_{\mu} \qquad (11)$$

gives,

$$\sigma_{\mu}(x) = H^{\lambda}(x)_{\mu} \, \Sigma_{\lambda}(x) \, . \qquad (12)$$

The inverse of the matrix $H = (H^{\lambda}{}_{\mu})$ is

$$S = (S^{\lambda}{}_{\mu}) = (k^{\lambda}_{(\rho)} \, h^{(\rho)}_{\mu}) \qquad (13)$$

indeed,

$$S^{\lambda}{}_{\mu} \, H^{\mu} = S^{\lambda}{}_{\alpha} \qquad (14\text{-}1)$$

$$H^{\lambda}{}_{\mu} \, S^{\mu}{}_{\alpha} = S^{\lambda}{}_{\alpha} \qquad (14\text{-}2)$$

Thus,

$$\Sigma_{\lambda}(x) = S^{\alpha}(x)_{\lambda} \, \sigma_{\alpha}(x) \qquad (15)$$

the following formulas are of importance,

$$h^{\mu(\alpha)} = \gamma^{\mu\nu} h^{(\alpha)}_{\nu}$$

$$k^{\mu(\alpha)} = g^{\mu\nu} \, k^{(\alpha)}_{\nu}$$

$$H^{\lambda\mu} = g^{\mu\nu} \, H^{\lambda}{}_{\nu}$$

$$H_{\lambda\mu} = \gamma_{\lambda\alpha} \, H^{\alpha}{}_{\mu}$$

$$S^{\lambda\mu} = \gamma^{\mu\alpha} S^{\lambda}{}_{\alpha}$$

$$S_{\lambda\mu} = g_{\lambda\alpha} S^{\alpha}{}_{\mu}$$

$$S^{\nu}{}_{\lambda} H_{\beta\nu} = \gamma_{\beta\lambda}$$

$$S_{\nu\lambda} H^{\lambda}{}_{\mu} = g_{\mu\nu} .$$

The element of arc in the curved space may be written as

$$\overrightarrow{ds} = dx^{\nu} \sigma_{\nu}(x) = dx^{\nu} H^{\lambda}{}_{\nu}(x) \Sigma_{\lambda}(x) \qquad (16)$$

and in the flat space as

$$\overrightarrow{dl} = dx^{\nu} \Sigma_{\nu}(x) . \qquad (17)$$

In the appendix we indicate how it is possible to stablish a relation between $\overrightarrow{ds}$ and $\overrightarrow{dl}^2$. In the next section it is treated the problem of the relationships between the affine connections in both spaces.

4. THE AFFINITIES IN CURVED AND FLAT SPACES

We have seen that to the points with coordinates x and x + + dx there are associated two arc elements $\overrightarrow{ds}$ and $\overrightarrow{dl}$ given by the equations (16) and (17). They represent two distinct metrical spaces. To these spaces there are associated two types of observers, those corresponding to the quaternions $\Sigma_{\mu}(x)$ which describe a flat metrical space and those associated to the quaternion $\sigma_{\mu}(x)$ which describe a Riemannian metrical space. The transition from one representation to the other being accomplished by the tetrad field $H^{\mu}{}_{\lambda}(x)$, or its inverse $S^{\mu}{}_{\lambda}(x)$, according to the equations (12) and (15).

In the flat space we now introduce an affine connection by means of the relation

$$d\,\Sigma_\lambda(x) = dx^\gamma\,\Gamma_{\gamma\ \lambda}^{\ \beta}(x)\,\Sigma_\beta(x)\ . \qquad (18)$$

From (15) we obtain

$$d\,\Sigma_\lambda(x) = dx^\gamma \frac{\partial S^\mu_{\ \lambda}}{\partial x^\gamma}\,\sigma_\mu(x) + S^\mu_{\ \lambda}(x)\,d\sigma_\mu(x) \qquad (19)$$

the differential of σ_μ is given by [3]

$$d\,\sigma_\mu(x) = dx^\gamma \begin{Bmatrix}\alpha\\ \gamma\mu\end{Bmatrix} \sigma_\alpha(x) + (\Gamma_\gamma(x)\sigma_\mu(x) + \sigma_\mu(x)\Gamma_\gamma^\dagger(x))dx^\gamma \qquad (20)$$

imposing

$$\Gamma_\gamma + \Gamma_\gamma^\dagger = 0 \qquad (21)$$

and introducing the Hermitian quaternion

$$\Delta_{\gamma\mu} = \Gamma_\gamma\,\sigma_\mu - \sigma_\mu\,\Gamma_\gamma$$

this quaternion is written in terms of the quaternion basis σ_μ as

$$\Delta_{\gamma\mu} = \Delta_{\gamma\ \mu}^{\ \alpha}\,\sigma_\alpha\ .$$

Thus, we may write (20) as

$$d\sigma_\mu = dx^\gamma \left[\begin{Bmatrix}\alpha\\ \gamma\mu\end{Bmatrix} + \Delta_{\gamma\ \mu}^{\ \alpha}\right] \sigma_\alpha \qquad (22)$$

it should be noted that $\begin{Bmatrix}\alpha\\ \gamma\mu\end{Bmatrix}$ is symmetric in γ, μ but $\Delta_{\gamma\ \mu}^{\ \alpha}$ is skew symmetric over this pair of indices.

Substituting (22) into (19) and at the same time using the relation (12),

$$d\Sigma_\lambda = dx^\gamma \frac{\partial S^\mu_{\ \lambda}}{\partial x^\gamma} H^\beta_{\ \mu}\,\Sigma_\beta + S^\mu_{\ \lambda}\left[\begin{Bmatrix}\alpha\\ \gamma\mu\end{Bmatrix} + \Delta_{\gamma\ \mu}^{\ \alpha}\right] H^\beta_{\ \alpha}\,\Sigma_\beta\ \ dx^\gamma$$

a comparison of this equation with (18) gives

$$\Gamma_{\gamma\ \lambda}^{\ \beta} = \frac{\partial S^{\mu}_{\ \lambda}}{\partial x^{\gamma}} H^{\beta}_{\ \mu} + S^{\mu}_{\ \lambda}\left[\left\{ {\alpha \atop \gamma\mu} \right\} + \Delta_{\gamma\ \mu}^{\ \alpha}\right] H^{\beta}_{\ \alpha} . \tag{23}$$

From the equation (22) we see that the affine connection in curved space is given by the quantities

$$Y_{\gamma\ \mu}^{\ \alpha} = \left\{ {\alpha \atop \gamma\mu} \right\} + \Delta_{\gamma\ \mu}^{\ \alpha} \tag{24}$$

so that the relationship between the affinities Γ and Y is given by (23) as

$$\Gamma_{\gamma\ \lambda}^{\ \beta} = \frac{\partial S^{\mu}_{\ \lambda}}{\partial x^{\gamma}} H^{\beta}_{\ \mu} + S^{\mu}_{\ \lambda} Y_{\gamma\ \mu}^{\ \alpha} H^{\beta}_{\ \alpha} . \tag{25}$$

It is important to note that the tetrad $H^{\mu}_{\ \lambda}$ and $S^{\mu}_{\ \lambda}$ transform one representation into the other, as example, given the vector ϕ^{μ} defined in the curved space, the quantity

$$\Phi^{\lambda} = \phi^{\mu} H^{\lambda}_{\ \mu}$$

represents a vector in flat space [4]. Similarly $S^{\mu}_{\ \lambda}$ also transforms indices from one metrical space to the other. These properties may be verified easily from the structure of the $H^{\mu}_{\ \lambda}$ and $S^{\mu}_{\ \lambda}$ given in (11) and (13). They are characteristic properties of the structure of the tetrad field.

From the form of the equation (25), the presence of the tetrad $H^{\mu}_{\ \lambda}$ and $S^{\mu}_{\ \lambda}$ is interpreted similarly as a transformation of indices belonging to the curved space into indices referring to the flat space, with exception of γ which is common to both spaces due to its appearance in the differential of coordinates.

Thus, we may interpret the term

$$S^{\mu}{}_{\lambda}\, Y_{\gamma}{}^{\alpha}{}_{\mu} H^{\beta}{}_{\alpha} = J_{\gamma}{}^{\beta}{}_{\lambda}$$

as the projection of the curved space affinity into the flat space. We rewrite (25) as

$$\Gamma_{\gamma}{}^{\beta}{}_{\lambda} - J_{\gamma}{}^{\beta}{}_{\lambda} = \frac{\partial S^{\mu}{}_{\lambda}}{\partial x^{\gamma}}\, H^{\beta}{}_{\mu} \tag{26}$$

the left hand side of this equation represents the difference of the two affine connections as seen from the flat space observer. According to a theorem of L. Civita [1] this difference behaves as a tensor in flat space. Therefore, the right hand side of this equation is also a tensor in flat space, where the curvilinear coordinate x^{γ} are used.

L. Civita's theorem is presently generalized due to the fact that $J_{\gamma}{}^{\beta}{}_{\lambda}$ also contains the spinor affine connection Γ_{μ}.

Another important relation is the equation which gives $g_{\mu\nu}$ in terms of $\gamma_{\mu\nu}$ (and its inverse). Such relation is obtained by taking the scalar product of two quaternions given by (12),

$$g_{\mu\nu} = H^{\lambda}{}_{\mu}\, H^{\beta}{}_{\nu}\, \gamma_{\lambda\beta} \tag{27}$$

the inverse relation being

$$\gamma_{\mu\nu} = S^{\lambda}{}_{\mu}\, S^{\beta}{}_{\nu}\, g_{\lambda\beta}\,. \tag{28}$$

Using (11) along with the definition given for $h^{\lambda}_{(\alpha)}$ in the section (1) we write (27) in the form

$$g_{\mu\nu} = k^{(\alpha)}_{\mu}\, k^{(\rho)}_{\nu}\, \overset{\circ}{g}_{\alpha\rho}\,. \tag{29}$$

This is the usual formula of the tetrad calculus, which shows that the present results are mathematically equivalent to the usual equations of the tetrad method.

5. THE EQUATION OF MOTION AND THE THEOREM OF QUASI-INERTIA

Using the definition of the previous section we write (26) as

$$\Gamma^{\beta}_{\gamma\lambda} - \left\{ {\beta \atop \gamma\lambda} \right\}_F - \Delta_{\gamma}{}^{\beta}{}_{\lambda,F} = \frac{\partial S^{\mu}{}_{\lambda}}{\partial x^{\gamma}} H^{\beta}{}_{\mu} \tag{30}$$

where the subscript "F" was used for remembering that the quanti ty refer to the flat space.

The equation of motion of a test particle follows from the variational principle

$$\delta \int_A^B ds = 0 \tag{31-1}$$

$$ds = (g_{\mu\nu}\, dx^{\mu}\, dx^{\nu})^{\frac{1}{2}} \tag{31-2}$$

which gives as result the Euler equations

$$\frac{d^2x^{\mu}}{ds^2} + \left\{ {\mu \atop \gamma\lambda} \right\} \frac{dx^{\gamma}}{ds} \frac{dx^{\lambda}}{ds} = 0 \; . \tag{32}$$

We introduce now as independent variable the line element dl belonging to the flat space, a simple transformation gives

$$\frac{d^2x^{\mu}}{d\ell^2} + \left\{ {\mu \atop \gamma\lambda} \right\} \frac{dx^{\gamma}}{d\ell} \frac{dx^{\lambda}}{d\ell} = \left(\frac{ds}{d\ell} \right)^{-1} \frac{d^2s}{d\ell^2} \frac{dx^{\mu}}{d\ell} \; . \tag{33}$$

Using the equation (30) we rewrite (33) as

$$\frac{d^2x^{\mu}}{d\ell^2} + \Gamma^{\mu}_{\gamma\lambda} \frac{dx^{\gamma}}{d\ell} \frac{dx^{\lambda}}{d\ell} = \frac{\partial S^{\alpha}{}_{\lambda}}{\partial x^{\gamma}} H^{\mu}{}_{\alpha} \frac{dx^{\gamma}}{d\ell} \frac{dx^{\lambda}}{d\ell} + \left(\frac{ds}{d\ell} \right)^{-1} \frac{d^2s}{d\ell^2} \frac{dx^{\mu}}{d\ell} \tag{34}$$

where the antisymmetry of the spinor affine connection $\Delta_{\tau}{}^{\mu}{}_{\lambda}$ has been used.

At the neighboroud of a given point O we can set the derivatives $\frac{\partial S^{\alpha}{}_{\lambda}}{\partial x^{\tau}}$ equal to zero, so that locally the particle obeys the equation of motion of a free particle, (the differentials ds and dl can be taken as equal in this neighboroud of O)

$$\frac{d^2x^{\mu}}{d\ell^2} + \Gamma^{\mu}_{\tau\lambda} \frac{dx^{\tau}}{d\ell} \frac{dx^{\lambda}}{d\ell} = 0 . \qquad (35)$$

We now introduce the convention of considering $g_{\mu\nu}$ as the potential of the gravitational field, the metric of the space being given by the $\gamma_{\mu\nu}$. The formula (34) is an extension of a result obtained by Rosen since presently we use the quaternion formalism and the concept of vierbein.

By choosing cartesian coordinates we can set all $h^{\mu}_{(\nu)}$ equal to δ^{μ}_{ν}. In this coordinate system we choose some point O as origin and suppose that the observer is located at this point. In the neighbohood of O we can set all $k^{\mu}_{(\nu)}$ equal to δ^{μ}_{ν}.

$$\sigma_{\mu}(0) = \Sigma_{\mu}(0) . \qquad (36)$$

In this region we will get ds = dl. Thus, the equation (34) simplifies to (we also impose that $\left(\frac{\partial S^{\mu}{}_{\lambda}}{\partial x^{\tau}}\right)_0 = 0$,

$$\frac{d^2x^{\mu}}{d\ell^2} = 0 . \qquad (37)$$

The equation of motion of a particle under the action of the gravitational field as seen by the observer at O, in the immediate vicinity of this point has the form of the equation (37). From the principle of equivalence the observer at O is equivalent to a freely falling observer. At O the components of the Riemann-Christoffel tensor are the unique non-vanishing quantities associated to the gravitational field, but since they are absent from the equation of motion, we get the result that the particle falls free in the nighbohood of O.

Introducing the quaternion

$$dX = dx^{\mu} \Sigma_{\mu}(0)$$

we get for the four-dimensional velocity

$$\frac{dX}{d\ell} = \frac{dx^{\mu}}{d\ell} \Sigma_{\mu}(0) ,$$

and for the four-dimensional acceleration

$$\frac{d^2X}{d\ell^2} = \frac{d^2x^{\mu}}{d\ell^2} \Sigma_{\mu}(0) . \qquad (38)$$

From the equation (37) the four-dimensional acceleration vanishes in the vicinity of the point O. This result was called in the literature as the postulate of "quasi-inertia" [5]. However, as was shown this result is a direct consequence of the fact that locally the gravitational force vanishes.

6. THE FIELD EQUATIONS IN THE QUATERNION FORMALISM

The Lagrangian density for the Einstein's field equations may be expressed in terms of the quaternion σ_{μ} [6]. Using this result Sachs has obtained the field equations from the Palatini variational principle [7]. The expression for $\mathcal{L}$ is

$$\mathcal{L}\,(\sigma_\mu;\ \bar{\sigma}_\mu;\ \Gamma_\mu;\ \bar{\Gamma}_\mu,\ \Gamma_{\mu|\nu},\ \bar{\Gamma}_{\mu|\nu}) = R\sqrt{-g}\ .$$

The field equations in empty space have the form [7]

$$\frac{1}{4}\,(\sigma^\alpha \mathcal{R}^\dagger_{\mu\alpha} + \mathcal{R}_{\mu\alpha}\sigma^\alpha) + \frac{1}{8}\,R\,\sigma_\mu = 0\ , \tag{39}$$

where $\mathcal{R}_{\mu\alpha}$ is the quaternion curvature defined by

$$\mathcal{R}_{\mu\alpha} = -\frac{1}{4}\,\sigma^\lambda R_\lambda{}^\beta{}_{\mu\alpha}\,\bar{\sigma}_\beta = \frac{\partial}{\partial x}{}_\mu \Gamma_\alpha + \Gamma_\mu \Gamma_\alpha\ , \tag{40}$$

which satisfies [4]

$$\mathcal{R}^\dagger_{\mu\alpha} = -\mathcal{R}_{\mu\alpha} \tag{41-1}$$

$$\mathcal{R}_{\mu\alpha} = -\mathcal{R}_{\alpha\mu} \tag{41-2}$$

$$\bar{\mathcal{R}}_{\mu\alpha} = -\mathcal{R}_{\mu\alpha} \tag{41-3}$$

The scalar curvature may be expressed in function of quaternions as

$$\mathring{\sigma}_{(0)}\,R = -\,\sigma^\alpha \,|(\mathcal{R}_{\mu\alpha}\,\sigma^\mu + \sigma^\mu \mathcal{R}^\dagger_{\mu\alpha})\ . \tag{42}$$

Since in empty spaces R vanishes, we obtain from (39),

$$\sigma^\alpha \mathcal{R}^\dagger_{\mu\alpha} + \mathcal{R}_{\mu\alpha}\sigma^\alpha = 0\ . \tag{43}$$

The components of the Ricci tensor $R_{\mu\nu}$ are given in terms of quaternion as

$$R^\mu{}_\nu\,\mathring{\sigma}_{(0)} = \sigma^\mu |(\mathcal{R}_{\alpha\nu}\,\sigma^\alpha + \sigma^\alpha \mathcal{R}^\dagger_{\alpha\nu})$$

since these components also vanish, we have the pair of equations

$$\begin{cases} \sigma^\alpha \mathcal{R}_{\mu\alpha} - \mathcal{R}_{\mu\alpha}\,\sigma^\alpha = 0 \\ \sigma^\mu |(\mathcal{R}_{\alpha\nu}\sigma^\alpha - \sigma^\alpha \mathcal{R}_{\alpha\nu}) = 0\ . \end{cases}$$

These equations imply in the condition

$$\mathcal{R}_{\mu\nu}\,(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}) = 0 \ . \tag{44}$$

Using (41-1) along with (12) we write (43) as

$$\mathcal{R}^{\mu\nu}\, H^{\lambda}_{\ \nu}\, \Sigma_{\lambda} - \Sigma_{\lambda}\, H^{\lambda}_{\ \nu}\, \mathcal{R}^{\mu\nu} = 0$$

multiplying by $H^{\alpha}_{\ \mu}$ we obtain

$$\Pi^{\alpha\lambda}\Sigma_{\lambda} - \Sigma_{\lambda}\,\Pi^{\alpha\lambda} = 0 \tag{45}$$

where

$$\Pi^{\alpha\lambda} = H^{\alpha}_{\ \mu}\, H^{\lambda}_{\ \nu}\, \mathcal{R}^{\mu\nu} \tag{46}$$

The equation (45) represents the projection of the equation (43) over the flat space, and the $\Pi^{\alpha\lambda}$ is the projection of the quaternion curvature $\mathcal{R}^{\alpha\lambda}$ over the flat space. The quantities $\Pi^{\alpha\lambda}$ are not interpreted as a curvature but as physical quantities associated to the gravitational field, similarly to the $g_{\mu\nu}$ (or the σ_{μ}) which now represents the gravitational potentials. Thus, in the flat space theory the gravitational field is described by the $\Pi^{\alpha\lambda}$ which are solutions of the field equations (45), the role of the gravitational potentials being taken over by the quaternions σ_{μ}.

Similarly the conditions (44) may be written in the flat space theory as

$$\Pi_{\mu\nu}(\Sigma^{\mu}\bar{\Sigma}^{\nu} - \Sigma^{\nu}\bar{\Sigma}^{\mu}) = 0 \ . \tag{47}$$

In the literature [5] from the use of the theorem of quasi-inertia and by postulating the equation (47) it is possible to derive the three tests of general relativity. We call attention to the fact that presently none of the two above postulates need to be done, both results are direct consequence of the theory.

7. CONCLUSION

With the help of the "two metric formalism" of L. Civita and using the basic ideas of the Rosen's model with the proper modifications which are needed for the quaternion formulation it is possible to derive a natural correspondence between the theory of gravitation in curved and in flat spaces. The equation of motion and the field equations are obtained in the flat space theory in a very simple and elegant form. All results are direct consequence of Rosen's interpretation and of the Sachs variational principle. The corrections associated to the spinor degree of freedom appear naturally. Since the present method deals with tetrad fields we may introduce interactions with fermion fields and thus obtain a more complete theory.

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APPENDIX

THE RELATION BETWEEN THE LINE ELEMENTS ds AND $d\ell$

We give here a very short discussion of this subject since a more complete treatment may be found in the literature [2].

Multiplying (34) by the quantity $\gamma_{\mu\alpha}\frac{dx^\alpha}{d\ell}$ we get

$$\gamma_{\mu\alpha}\frac{dx^\alpha}{d\ell}\left\{\frac{\partial S^\beta{}_\lambda}{\partial x^\tau} H^\mu{}_\beta \frac{dx^\tau}{d\ell}\frac{dx^\lambda}{d\ell} + \left(\frac{ds}{d\ell}\right)^{-1}\frac{d^2s}{d\ell^2}\frac{dx^\mu}{d\ell}\right\} = 0$$

since the left hand side of (34) vanishes when multiplyed by the above quantity. At the vicinity of the point O (the origin of the coordinate system used previously) we obtain

$$\left(\frac{\partial S^\beta{}_\lambda}{\partial x^\tau}\right)_o = 0$$

and thus,

$$\left(\frac{ds}{d\ell}\right)_o^{-1}\left(\frac{d^2s}{d\ell^2}\right)_o = 0$$

which gives by integration

$$(ds)_o = C(d\ell)_o$$

where C is a constant. It is always possible to take C = 1, and thus proving that in the neighboroud of O we can set ds = dl, as was used in the text. The relationship between ds and dl outside the point O is more complicate and will not be given here.

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