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DIFFRACTION OF A PULSE BY A HALF-PLANE, TREATED BY THE FOURIER METHOD

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1. Introduction.

The diffraction of an electromagnetic pulse by a perfectly conducting half-plane was first treated by Sommerfeld [1] by the method of branched wave functions, and later by other authors [2], who employed different methods. Apparently, however, the problem has not yet been treated by the Fourier method. Since the solution of the problem for a monochromatic plane wave is well known [3], the latter method is perhaps the most natural one.

Sommerfeld's solution for the monochromatic case will be reproduced in section 2. In section 3, the solution for a delta-type incident pulse will be derived from the monochromatic one by the Fourier method. In section 4, the solution for an arbitrary incident plane pulse is derived, by considering it as a

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superposition of delta-type pulses. The results agree with Bateman's [4] form of the solution for perpendicular incidence.

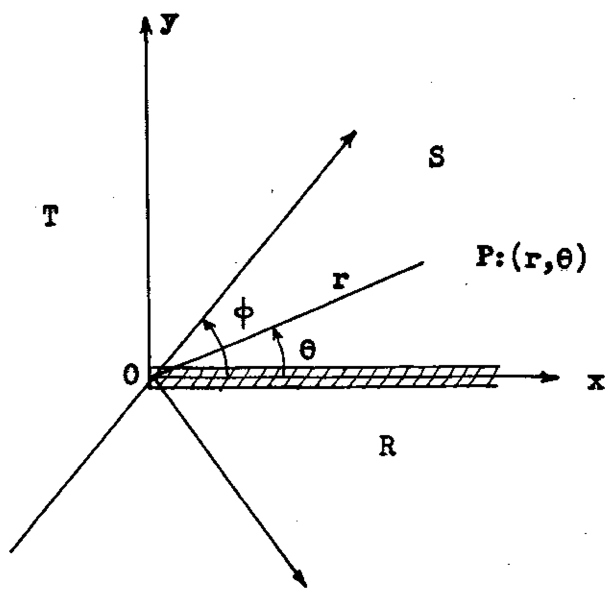
The physical interpretation of the solution will not be considered here; it has already been discussed elsewhere [5].

2. Sommerfeld's solution.

The cartesian co-ordinate system is defined in such a way that the half-plane is represented by $x>0$, $y=0$. S is the geometrical shadow region, T is the region of incidence and transmission and R is the region of incidence and reflexion:

$$\begin{aligned} &\text{region S:} \quad \theta \in (0, \phi);\\ &\text{region T:} \quad \theta \in (\phi, 2\pi-\phi);\\ &\text{region R:} \quad \theta \in (2\pi-\phi, 2\pi); \end{aligned} \tag{1}$$

where r, θ are the polar co-ordinate and ϕ is the angle between the direction of incidence and Ox (fig. 1); ϕ can be restricted to the interval $(0, \pi)$, without loss of generality. The solution is defined on a two-sheeted Riemann surface, as discussed by Sommerfeld [3]. The physical sheet is defined by values of θ belonging to the intervals $(0, 2\pi)$. If we



consider the case of transverse electric waves,

$$\vec{E} = \psi \hat{z}$$
$$\vec{H} = H_x \hat{x} + H_y \hat{y}, \tag{2}$$

the wave function $\psi(r, \theta, \phi, t)$ must satisfy the boundary conditions for a perfectly conducting half-plane,

$$\psi(r, 0, \phi, t) = 0,$$
$$\psi(r, 2\pi, \phi, t) = 0. \tag{3}$$

If the incident wave is a monochromatic plane wave,

$$\psi_{inc}^{(k)}(r, \theta, \phi, t) = e^{ik[r\cos(\theta-\phi)-ct]}, \tag{4}$$

the solution is given by [3]

$$\psi^{(k)}(r, \theta, \phi, t) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \left\{ e^{ik[r\cos(\theta-\phi)-ct]} G(w_-) - \right.$$
$$\left. - e^{ik[r\cos(\theta-\phi)-ct]} G(w_+) \right\}, \tag{5}$$

where

$$w_{\pm}(r, \theta, \phi, k) = \mp\sqrt{2kr} \text{ sen } \frac{\theta \pm \phi}{2} \tag{6}$$

and

$$G(w) = \int_{-\infty}^{w} e^{iq^2} dq. \tag{7}$$

If F(w) denotes Fresnel's integral,

$$F(w) = \int_0^w e^{iq^2}\, dq \;, \tag{8}$$

then, owing to the properties

$$F(\infty) = F(-\infty) = \frac{\pi}{2}\, e^{i\pi/4}, \tag{9}$$

we can write

$$G(w) = \frac{\pi}{2}\, e^{i\pi/4} + F(w)\,. \tag{10}$$

3. Delta-type incident pulse.

The solution to the problem when the incident wave is a delta-type pulse,

$$\psi_{inc}(r, \theta, \phi, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik[r\cos(\theta-\phi)-ct]}\, dk$$

$$= \delta[r\cos(\theta-\phi)-ct], \tag{11}$$

can be obtained from Sommerfeld's solution (5) for a monochromatic plane wave $\psi^{(k)}$ by taking its Fourier transform:

$$\psi(r, \theta, \phi, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi^{(k)}(r, \theta, \phi, t)\, dk. \tag{12}$$

However, for negative values of k, we obtain purely imaginary values for w_+ and w_-. This difficulty can be overcome by separating the integral into two parts, each one from zero to infinity. This procedure is suggested by the fact that, the result being real, (11) can be written:

$$\Psi_{inc}(r, \theta, \phi, t) = \frac{1}{2\pi}\int_0^\infty \psi_{inc}^{(k)}(r, \theta, \phi, t)\, dk + $$

$$+ \frac{1}{2\pi}\int_0^\infty \psi_{inc}^{(k)*}(r, \theta, \phi, t)\, dk. \qquad (13)$$

Hence,

$$\psi(r, \theta, \phi, t) = \frac{1}{2\pi}\int_0^\infty \psi^{(k)}(r, \theta, \phi, t)\, dk +$$

$$+ \frac{1}{2\pi}\int_0^\infty \psi^{(k)*}(r, \theta, \phi, t)\, dk. \qquad (14)$$

Introducing

$$\phi_{\pm}(r, \theta, \phi, t) = \frac{e^{-i\pi/4}}{2\pi^{3/2}}\int_0^\infty \exp(ikY_{\pm})dk \int_{-\infty}^{\sqrt{k}\,W_{\pm}} e^{iq^2}\, dq \qquad (15)$$

where

$$Y_{\pm}(r, \theta, \phi, t) = r\cos(\theta \pm \phi) - ct,$$

$$W_{\pm}(r, \theta, \phi) = \mp\sqrt{2r}\ \mathrm{sen}\,\frac{\theta \pm \phi}{2}, \qquad (16)$$

the result becomes

$$\psi(r, \theta, \phi, t) = \left[\phi_{-}(r, \theta, \phi, t) + \phi_{-}^{*}(r, \theta, \phi, t)\right] -$$

$$- \left[\psi_+(r, \theta, \phi, t) + \psi_+^*(r, \theta, \phi, t)\right]. \quad (17)$$

In order to evaluate

$$I(W,Y) = \int_0^\infty e^{ikY}\, dk \int_{-\infty}^{W\sqrt{k}} e^{iq^2}\, dq , \quad (18)$$

we interchange the order of integration. If $W < 0$, this leads to

$$I(W < 0) = \int_{-\infty}^{0} e^{iq^2}\, dq \int_0^{q^2/W^2} e^{ikY}\, dk \quad (19)$$

or, after integration in k,

$$I(W < 0) = \frac{1}{iY}\left\{ \int_{-\infty}^{0} \exp\left[i\left(1 + \frac{Y}{W^2}\right)q^2\right] dq - \frac{\pi}{2}\, e^{i\pi/4} \right\}.$$

If $W^2 + Y > 0$, we make the change of variable

$$\left(1 + \frac{Y}{W^2}\right) q^2 = q'^2 .$$

Taking into account (9), we find

$$I(W < 0,\ W^2 + Y > 0) = \frac{i}{2Y}\sqrt{\pi}\, e^{i\pi/4} \left(1 + \frac{W}{\sqrt{W^2 + Y}}\right). \quad (20)$$

If $W^2 + Y < 0$, the change of variable is

$$\left(1 + \frac{Y}{W^2}\right)q^2 = - q'^2$$

and the resulting expression can be obtained from (20) by substituting

$$\sqrt{W^2 + Y} \longrightarrow i\sqrt{-(W^2 + Y)}. \tag{21}$$

If $W > 0$, the inversion of the order of integration in (18) leads to

$$I(W > 0) = I(W < 0) + \int_{-\infty}^{\infty} e^{iq^2}\, dq \int_{q^2/W^2}^{\infty} e^{ikY}\, dk, \tag{22}$$

or

$$I(W > 0) = \frac{1}{iY}\left\{\frac{\sqrt{\pi}}{2}\, e^{i\pi/4}\left(2 \lim_{k \to \infty} e^{ikY} - 1\right) - \right.$$

$$\left. - \int_{-\infty}^{0} \exp\left[i\left(1 + \frac{Y}{W^2}\right)q^2\right] dq\right\}. \tag{23}$$

For $W^2 + Y > 0$, we find

$$I(W > 0,\, W^2 + Y > 0) = \frac{i}{2Y}\sqrt{\pi}\, e^{i\pi/4}\left(1 + \frac{W}{\sqrt{W^2 + Y}} - 2 \lim_{k \to \infty} e^{ikY}\right), \tag{24}$$

and for $W^2 + Y < 0$ it is sufficient to make the substitution (21).

By means of (16) we obtain

$$W_{\pm}^2 + Y_{\pm} = r - ct \qquad (25)$$

and so, the four possibilities which occur for (15) are:

$$\Psi_{\pm}\left(r>ct,\theta,\phi,\mp \text{ sen } \frac{\theta\pm\phi}{2}<0\right) = \frac{1}{4\pi\left[r\cos(\theta \pm \phi) - ct\right]} \left[1 \mp \sqrt{2}\sqrt{\frac{r}{r-ct}} \text{ sen } \frac{\theta\pm\phi}{2}\right],$$

$$\Psi_{\pm}\left(r>ct,\theta,\phi,\mp\text{sen } \frac{\theta\pm\phi}{2}>0\right) = \frac{1}{4\pi\left[r\cos(\theta\pm\phi) - ct\right]} \left[1 \mp \sqrt{2}\sqrt{\frac{r}{r-ct}} \text{ sen } \frac{\theta\pm\phi}{2} - 2 \lim_{k\to\infty} e^{ik\left[r\cos(\theta\pm\phi) - ct\right]}\right] \qquad (26)$$

and the two others that result from these by making the substitution (21), which now reads

$$\sqrt{r-ct} \longrightarrow i\sqrt{ct-r} \quad \text{for} \quad r<ct. \qquad (27)$$

From (26) and the analogous expressions for $r<ct$, we can derive the value of the expression within square brackets in (17):

$$\Psi_{\pm}\left(r,\theta,\phi,t,\mp\text{sen } \frac{\theta\pm\phi}{2}<0\right) + \psi_{\pm}^{*}\left(r,\theta,\phi,t,\mp\text{sen } \frac{\theta\pm\phi}{2} < 0\right) = $$

$$= \text{Re}\left[\Phi_{\pm}(r,\theta,\phi,t)\right], \qquad (28)$$

where Re stands for "the real part of",

$$\Phi_{\pm}(r,\theta,\phi,t) = \mp \frac{\sqrt{2}}{2\pi}\sqrt{\frac{r}{ct-r}} \; \frac{\text{sen } \frac{\theta\pm\phi}{2}}{r\cos(\theta\pm\phi) - ct},$$

and

$$\Psi_{\pm}\left(r,\ \theta,\ \phi,\ t \mp \text{sen}\ \frac{\theta \pm \phi}{2} > 0\right) + \Psi_{\pm}^{*}\left(r,\ \theta,\ \phi,\ \mp \text{sen}\ \frac{\theta \pm \phi}{2} > 0\right) =$$

$$= \delta\left[r\ \cos(\theta \pm \phi) - ct\right] - \text{Re}\left[\Phi_{\pm}\ (r,\ \theta,\ \phi,\ t)\right]. \qquad (30)$$

The last result follows from

$$\frac{1}{\pi} \lim_{k \to \infty} \frac{\text{sen}\{k[r\cos(\theta \pm \phi) - ct]\}}{r\cos(\theta \pm \phi) - ct} = \delta\left[r\cos(\theta \pm \phi) - ct\right]. \qquad (31)$$

In the geometrical shadow region,

$$\mp \text{sen}\ \frac{\theta \pm \phi}{2} < 0 \quad (\theta \in S), \qquad (32)$$

so that

$$\psi_S(r,\ \theta,\ \phi,\ t) = \psi_{dif}(r,\ \theta,\ \phi,\ t), \qquad (33)$$

where

$$\psi_{dif}(r,\ \theta,\ \phi,\ t) = \text{Re}\left[\Phi_{-}(r,\ \theta,\ \phi,\ t) - \Phi_{+}(r, \theta, \phi, t)\right]$$

$$= \text{Re}\left\{\frac{\sqrt{2}}{\pi}\sqrt{\frac{r}{ct-r}}\ \text{sen}\ \frac{\theta}{2}\ \cos\frac{\phi}{2}\ \frac{r\left[2(\cos^2\theta/2 - \cos^2\phi/2) + 1\right] - ct}{\left[r^2(\cos^2\theta - \text{sen}^2\phi) - 2r\cos\theta\cos\phi\, ct + c^2t^2\right.}\right\}. \qquad (34)$$

For $r < ct$, the expression within square brackets is real. For $r > ct$, it is purely imaginary, so that

$$\psi_{dif}\ (r > ct,\ \theta,\ \phi,\ t) \equiv 0 \qquad (35)$$

This result also follows directly from the causality principle, according to which there can be no diffracted wave before the arrival of the incident pulse to the half-plane.

In the incidence and transmission region,

$$\operatorname{sen}\frac{\theta-\phi}{2} > 0, \qquad -\operatorname{sen}\frac{\theta+\phi}{2} < 0 \qquad (\theta\in T), \qquad (36)$$

so that the solution becomes

$$\psi_T(r,\ \theta,\ \phi,\ t) = \psi_{inc}(r,\ \theta,\ \phi,\ t) + \psi_{dif}(r,\ \theta,\ \phi,\ t), \quad (37)$$

where ψ_{inc} is the incident pulse (11).

Finally, in the region of incidence and reflexion,

$$\mp\operatorname{sen}\frac{\theta\pm\phi}{2} > 0 \quad (\theta\in R), \qquad (38)$$

so that

$$\psi_R(r,\theta,\phi,t) = \psi_{inc}(r,\theta,\phi,t) + \psi_{ref}(r,\theta,\phi,t) + \psi_{dif}(r,\theta,\phi,t), \ (39)$$

where

$$\psi_{ref}(r,\ \theta,\ \phi,\ t) = -\ \delta[r\cos(\theta+\phi) - ct]. \qquad (40)$$

From the results (33), (37) and (39), we can see that the solution is completely separated into geometrical optics terms and a diffraction term which is common to the three regions. For negative times, we have only the incident wave. For positive times, we have all three waves: incident, reflected and diffracted.

In the particular case of perpendicular incidence, (11), (40) and (34) are reduced to:

$$\psi_{inc}(r,\ \theta,\ \pi/2,\ t) = \delta(y - ct)\ ,$$

$$\psi_{ref}(r,\ \theta,\ \pi/2,\ t) = -\ \delta(y + ct)\ , \qquad (41)$$

$$\psi_{dif}(r,\ \theta,\ \pi/2,\ t) = \mathrm{Re}\left[\frac{1}{\pi}\sqrt{\frac{r}{ct-r}}\ \frac{y\ \cos\theta/2 - ct\ \mathrm{sen}\theta/2}{c^2t^2 - y^2}\right].$$

4. Arbitrary incident plane pulse.

Let

$$\psi_{inc}(r,\ \theta,\ \phi,\ t) = D\left[ct - r\ \cos(\theta - \phi)\right] \qquad (42)$$

be an incident plane pulse of arbitrary form, traveling in the ϕ direction. Such a wave packet can be expressed in terms of delta -type incident pulses by means of

$$\psi_{inc}(r,\ \theta,\ \phi,\ t) = \int_{-\infty}^{\infty} \delta\left[r\cos(\theta - \phi) - ct + \tau\right]\ D(\tau)\ d\tau\ . \qquad (43)$$

If we make $\tau = ct_o$, we can consider (43) as a superposition of delta-type pulses incident on the half-plane at different instants of time t_o, distributed according to a "weight" $D(\tau) = D(ct_o)$. The solution can, therefore, be obtained from that derived in section 3 by substituting t by $t - t_o$ and by integrating over all ct_o.

According to (40), the reflected wave is given by

$$\psi_{ref}(r, \theta, \phi, t) = -D\left[ct - r\cos(\theta+\phi)\right]. \tag{44}$$

The diffracted wave follows from (34):

$$\psi_{dif}(r, \theta, \phi, t) =$$

$$= -\frac{\sqrt{2r}}{\pi}\,\text{sen}\frac{\theta}{2}\,\cos\frac{\phi}{2}\int_{-\infty}^{ct-r}\frac{\left\{ct-\tau-\left[2(\cos^2\theta/2-\text{sen}^2\phi/2)+1\right]r\right\}D(\tau)d\tau}{\sqrt{ct-\tau-r}\left[(ct-\tau)^2-2r\cos\theta\cos\phi(ct-\tau)+r^2(\cos^2\theta-\text{sen}^2\phi)\right]} \tag{45}$$

The upper limit $ct-r$ in the above integral arises from the fact that the diffracted wave is now zero for $r > ct-\tau$.

The solutions in the S, T and R regions are given by (33), (37) and (39), where ψ_{inc}, ψ_{ref} and ψ_{dif} are given by (42), (44) and (45), respectively.

For perpendicular incidence, the results reduce to

$$\psi_{inc}(r, \theta, \pi/2, t) = D(ct-y),$$

$$\psi_{ref}(r, \theta, \pi/2, t) = -D(ct+y)\,, \tag{46}$$

$$\psi_{dif}(r, \theta, \pi/2, t) = -\frac{\sqrt{r}}{\pi}\int_{-\infty}^{ct-r}\frac{\left[(ct-\tau)\,\text{sen}\theta/2 - y\cos\theta/2\right]}{\sqrt{ct-\tau-r}\left[(ct-\tau)^2-y^2\right]}\,D(\tau)d\tau.$$

Bateman's solution [4] of the same problem differs from (46) in the expression for ψ_{dif}, which, according to Bateman, is given by

$$\psi_{dif}(r, \theta, t) = \begin{cases} I_- - I_+ & \text{in region S,} \\ - I_- - I_+ & \text{in region T,} \\ - I_- + I_+ & \text{in region R,} \end{cases} \tag{47}$$

where

$$I_{\pm}(r, \theta, t) = \frac{1}{\pi} \int_0^{\pi/2} D\left[(ct \pm y) - (r \pm y) \sec^2 b\right] db. \tag{48}$$

In order to compare (46) with (47) and (48), let us make the substitution:

$$(ct \pm y) - (r \pm y) \sec^2 b = \tau .$$

Then, (48) becomes

$$I_{\pm}(r, \theta, t) = \frac{\sqrt{r \pm y}}{2\pi} \int_{-\infty}^{ct-r} \frac{D(\tau)\, d\tau}{\sqrt{ct-\tau-r}\left[(ct \pm y) - \tau\right]} \tag{49}$$

so that

$$I_- \pm I_+ = \frac{1}{2\pi} \int_{-\infty}^{ct-r} \frac{(ct-\tau)(\sqrt{r-y} \pm \sqrt{r+y}) + y(\sqrt{r-y} \mp \sqrt{r+y})}{\sqrt{ct-\tau-r}\left[(ct-\tau)^2 - y^2\right]} D(\tau)\, d\tau$$

or, in terms of the polar angle θ,

$$I_- \pm I_+ =$$

$$= \frac{\sqrt{r}}{2\pi} \int_{-\infty}^{ct-r} \frac{(ct-\tau)\left[\sqrt{1-\operatorname{sen}\theta} \pm \sqrt{1+\operatorname{sen}\theta}\right] + y\left[\sqrt{1-\operatorname{sen}\theta} \mp \sqrt{1+\operatorname{sen}\theta}\right]}{\sqrt{ct-\tau-r}\left[(ct-\tau)^2 - y^2\right]} D(\tau)\, d\tau. \tag{50}$$

In region S, $\theta \in (0, \pi/2)$, so that

$$\sqrt{1-\text{sen}\theta} \pm \sqrt{1+\text{sen}\theta} = \sqrt{2}\ \text{sen}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \pm \sqrt{2}\ \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= (1 \pm 1)\cos\frac{\theta}{2} - (1 \pm 1)\ \text{sen}\frac{\theta}{2}$$

The solution in this region, according to (47), is

$$I_- - I_+ = -\frac{\sqrt{r}}{\pi}\int_{-\infty}^{ct-r} \frac{(ct-\tau)\ \text{sen}\theta/2 - y\cos\theta/2}{\sqrt{ct-\tau-r}\left[(ct-\tau)^2 - y^2\right]}\ D(\tau)\,d\tau\,, \tag{51}$$

in complete agreement with (46).

In region T, $\theta \in (\pi/2,\ 3\pi/2)$, so that

$$\sqrt{1-\text{sen}\theta} \pm \sqrt{1+\text{sen}\theta} = -\sqrt{2}\ \text{sen}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \pm \sqrt{2}\ \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= (1 \pm 1)\ \text{sen}\frac{\theta}{2} - (1 \mp 1)\cos\frac{\theta}{2}$$

According to (47), we now have to take $-(I_- + I_+)$, so that we find the same result (51).

Finally, in region R, $\theta \in (3\pi/2, 2\pi)$, so that

$$\sqrt{1-\text{sen}\theta} \pm \sqrt{1+\text{sen}\theta} = -\sqrt{2}\ \text{sen}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \mp \sqrt{2}\ \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$= (1 \mp 1)\ \text{sen}\frac{\theta}{2} - (1 \pm 1)\cos\frac{\theta}{2}$$

and taking $-(I_- \pm I_+)$, according to (47), again yields the same result.

Thus, our solution for perpendicular incidence is identical to Bateman's one.

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