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ALGEBRAS OF FINITE DIFERENTIAL ORDER

AND THE OPERATIONAL CALCULUS

by

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1. INTRODUCTION

The notions of topological spaces and of continuous functions are to be considered as fertilely well established in Mathematics. The related concepts of differentiable spaces and of differentiable functions have not as yet been introduced in a comparable broad and definitive form, as their study is classically confined mostly to locally Euclidean spaces. The present work arose from wishes to extend part of the known results of the theory of differentiable manifolds, due to Whitney, to a larger class of spaces which are not required to be locally Euclidean a priori. We have been then lead naturally to a category of topological algebras which is likely to play a role in the theory of general differentiable spaces. As this category of topological algebras presents itself as the largest one having an operational calculus with ordinary differentiable functions, in a sense made precise below, in order to introduce here such a distinguished category we shall adopt the operational calculus point of view, although it was not our main goal. As a motivation to the question studied here, let us mention than an n - differentiable structure on a topological space X should be definable by a topological algebra A of real continuous functions on X, such an algebra having to satisfy suitable conditions, among them the requirement that $f \in A$ be a local property and that A has an operational calculus with ordinary-differentiable functions, i.e. that $f \in A$ implies $\varphi(f) \in A$ for every n-differentiable real function φ on the real line R and that the map $(\varphi, f) \longrightarrow \varphi(f)$ is continuous: and similarly for functions φ of several variables. The local nature of A (rather its infinitesimal nature) is expressed in a sense by the local convexity assumptions used in this paper. This article is devoted to the proof of the assertion that, if C is a category of pure separated algebras whose radicals are nilpotent and $n \geq 0$ is an integer, in order that every topological algebra which is locally convex with respect to C should have a pre-operational calculus with n-differentiable functions [1], it is necessary and sufficient that C be a subcategory of the category D_{n+1} of all pure separated algebras of differential order n+1. The category of topological algebras, related to n-differentiable spaces, thus distinguished after this statement is that of the topological algebras which are locally convex with respect to D_{n+1} for some $n \geq 0$. The question dealt with in the present paper was influenced by A. Weil's exposition of the infinitesimal calculus on differentiable manifolds [6]. We need, however, use here pure local algebras with nilpotent radicals but free from finite dimensionality or ascending chain restrictions imposed a priori; and, more generally, we have to accept in our considerations semi-local algebras, which are direct sums of finitely many such local algebras.

2. NOTATIONS

We denote by R and Z the systems of the real numbers and of the integers. R^+ and Z^+ refer to the positive real numbers and positive integers. We shall denote by $C^n(R)$ the topological algebra of all real functions f on R having continuous derivatives up to order n included $(n \in Z^+)$, this algebra being endowed with the topology of order n, that is the topology of the uniform convergence of $f, \ldots, f^{(n)}$ on every compact subset of R. We shall also denote by $P(R)$ the subalgebra of all real polynomials on R. δ_i^j will represent the Kronecker delta. All vector spaces considered here will be over R. All algebras are assumed to be commutative and have a unity. Every subalgebra contains the unity with the possible exception of the case of an ideal. A pre-norm on a vector space E is a function $\pi: E \to R^+$ such that $\pi(x+y) \leq \pi(x) + \pi(y)$, $\pi(\lambda x) = |\lambda| . \pi(x)$. If, in addition, $\pi(x) = 0$ implies $x = 0$, then π is said to be a norm. A pre-norm of algebra on an algebra A has to satisfy also $\pi(xy) \leq \pi(x)\,\pi(y)$ and $\pi(I) = 1$ or 0 ($\pi(I) = 0$ implying $\pi = 0$). A topological algebra is assumed to have its topology defined by the continuous pre-norms of algebra. If A is an algebra, $\Pi(A)$ will denote the set of all pre-norms of algebra on A. The natural topology of A is the one defined by all $\pi \in \Pi(A)$. A is said to be separated if its natural topology satisfies the Hausdorff separation axiom, that is, alternatively, for every $x \in A$, $x \neq 0$, there is $\pi \in \Pi(A)$ such that $\pi(x) \neq 0$. The separated algebra associated to A is the quotient of A modulo the ideal on which all $\pi \in \Pi(A)$

vanish. If $p \in P(R)$ and $w > 0$, we define

$$\| p; A, w \| = \sup\{\pi[p(x)] ; x \in A, \pi \in \Pi(A), \pi(x) \leq w\} .$$

More generally, if A is a topological algebra, $\Pi(A)$ will represent the set of all continuous pre-norms of algebras on A. Assuming that $S \subset \Pi(A)$ defines the topology of A, we put

$$\| p; A, S, w \| = \sup\{\pi[p(x)] ; x \in A, \pi \in S, \pi(x) \leq w\}$$

An algebra will be called pure if all its maximal ideals are of codimension one. In dealing with a category C, we shall assume that, if $U \in C$ and U is isomorphic to V, then $V \in C$. Except as stated otherwise, we follow the terminology of Bourbaki[1] and Jacobson[2].

3. LOCAL CONVEXITY

Let E be a real topological vector space. We denote by $\mathbf{S}$ a collection of vector subspaces of E. A subset $X \subset E$ is $\mathbf{S}$-convex[3] when X is convex in the usual sense and $X = \bigcap_S (X + S)$ for $S \in \mathbf{S}$. It amounts to the same to say that X is the inverse image of a product of convex sets in the product space $\prod_S E/S$. The space E is $\mathbf{S}$-locally convex in case the $\mathbf{S}$-convex neighborhoods of 0 form a basis of neighborhoods at 0. This, of course, implies local convexity in the usual sense. When we have two collection $\mathbf{S}_1$, $\mathbf{S}_2$ of vector subspaces of E and $\mathbf{S}_1$ is contained in $\mathbf{S}_2$ or, more generally, every member of $\mathbf{S}_1$ contains some member of $\mathbf{S}_2$, then $\mathbf{S}_1$-convexity and $\mathbf{S}_1$-local convexity imply $\mathbf{S}_2$-convexity and $\mathbf{S}_2$-local convexity. If $\mathbf{S}$ reduces to, or rather contains 0, then $\mathbf{S}$-convexity and $\mathbf{S}$-local convexity reduce to usual convexity and local convexity. For E to be $\mathbf{S}$-locally convex it is necessary and sufficient that the convex neighborhoods V of 0 such that $V = \bigcap_S \overline{V+S}$ form a basis of neighborhoods at 0. Such a neighborhood V is closed and $\mathbb{S}$-convex and, conversely, if a neighborhood V of 0 is closed and $\mathbb{S}$-convex, then $V = \bigcap_S \overline{V+S}$ holds. There results that we may replace $\mathbb{S}$ by the collection of the closures of its members without affecting local convexity; and similarly, that we can omit from $\mathbb{S}$ those vector subspaces which are dense in E. Given $\mathbb{S}$, if we denote by $\mathbb{T}$ the collection of the closed (if we want so) vector subspaces of E of codimension one, each of which contains some member of $\mathbb{S}$, then $\mathbb{S}$-local convexity and $\mathbb{T}$-local convexity are identical. Alternatively, let Φ be the collection of the linear continuous (if we want so) functionals on E each of which vanishes on some member of $\mathbb{S}$. Notice that Φ is not necessarily a vector space, as we can assert only that $R\Phi \subset \subset \Phi$ i.e. $\varphi \in \Phi$ implies $\lambda\varphi \in \Phi$ for any scalar $\lambda \in R$. The $\mathbb{S}$-convex symmetric closed neighborhoods of 0 are the polar sets of the equicontinuous subsets of Φ. Hence E is $\mathbb{S}$-locally convex if and only if its topology is that of uniform convergence on the equicontinuous subsets of Φ. If π is a pre-norm on E, π is said to be S-convex if its unit closed ball $\{x \in E;\ \pi(x) \leq 1\}$ is S-convex. π induces on every E/S a pre-norm which, after being pulled back to E, has the expression $\pi_S(x) = \inf\{\pi(x-s);\ s \in S\}$. Clearly $\pi_S \leq \pi$. Then $\pi = \sup_S \pi_S$ for $S \in S$ amounts to π being S-convex. Also E is S-locally convex if and only if the S-convex continuous pre-norms determine the topology.

Let us now consider a commutative algebra A of operators on the real topological vector space E. We always assume that A contains the unity. Corresponding to every ideal $I \subset A$, represent by IE the vector subspace spanned by the $T(x)$, with $T \in I$ and $x \in E$. Clearly IE is invariant under A. Letting $\mathbb{I}$ be a collection of ideals in A and $\mathbf{S}(\mathbb{I}) = \{IE;\ I \in \mathbb{I}\}$, we shall say that a set $X \subset E$ is convex with respect to $\mathbb{I}$ when X is $\mathbf{S}(\mathbb{I})$-convex; and that E is locally convex with respect to $\mathbb{I}$ in case E is $\mathbf{S}(\mathbb{I})$-locally convex. In further applications, we shall make use of this definition in the following form. Calling $\mathbf{C}$ a category of commutative real algebras with unity, then $X \subset E$ and E are said to be convex and locally convex, respectivelly, under A with respect to $\mathbf{C}$ in case X and E are convex and locally convex with respect to the collection $\mathbb{I}(\mathbf{C})$ of ideals $I \subset A$ such that $A/I \in \mathbf{C}$. When A is a topological algebra, we may take $E = A$, consider A as an algebra of operators on itself and apply the above considerations. If I is a collection of ideals in A, I-convexity and I-local convexity have each two different meanings, which agree in each case. A is I-locally convex if and only if its topology is defined by the continuous pre-norms of algebra on A which are I-convex. We shall simplify slightly the terminology by saying "convex (locally convex) with respect to C" when we should say "convex (locally convex) under A with respect to C" in the topological algebra case. We notice that, if a topological algebra A is locally convex with respect to a category C, then A is locally convex with respect to the category of the separated algebras associated to the algebras of C.

As two simple examples, we mention the following ones. Consider the topological algebra $C^n(R^m)$ of n-differentiable real functions of m real variables. For every compact subset $K \subset R^m$ and $f \in C^n(R^m)$, we define

$$\pi_K(f) = \sup\left\{ \sum_{|i|\leq n} \frac{1}{i!} \mid D^i f(x) \mid ; x \in K \right\}$$

(where $i = (i_1,\ldots,i_m)$, $|i| = i_1+\ldots+i_m$, $D^i = \partial^{|i|}/\partial x_1^{i_1} \ldots \partial x_m^{i_m}$) to get pre-norms of algebra which define the topology. Consideration of the collection of ideals of all $f \in C^n(R^m)$ such that $D^i f(x) = 0$ for $|i| \leq n$ at a given point $x \in R^m$ shows that $C^n(R^m)$ is locally convex with respect to the category of algebras isomorphic to the local algebra R^n_m of real polynomials in m variables of degree $\leq n$; and similarly for differentiable manifolds. Analogously, let $L^1(R^m)$ be the topological algebra of Lipschitz real functions on R^m of order 1. For every compact subset $K \subset R^m$ and $f \in L^1(R^m)$, we introduce

$$\pi_K(f) = \sup\left\{ \sup[f(x),f(y)] + \frac{|f(x)-f(y)|}{\|x - y\|}; x,y \in K, x \neq y \right\}$$

(where $x \longrightarrow \|x\|$ is a norm on R^m) to get pre-norms of algebra which define the topology. $L^1(R^m)$ is then locally convex with respect to the category of algebras isomorphic to R^2, as it is sufficient to consider the collection of ideals of all $f \in L^1(R^m)$ vanishing at two points $x, y \in R^m$, $x \neq y$; and analogously for Lipschitz functions of arbitrary order m on general spaces, with R^{m+1} in place of R^2.

4. MEAN VALUE LEMMAS

A divisor on the real line R is a function $f: R \longrightarrow Z$ whose support $\{x;\ f(x) \neq 0\}$ is finite. The divisors on R form an additive group, the "free abelian group" generated by R. Put $\sum f = \sum_{x \in R} f(x)$. We write $f \leq g$ for two divisors if $f(x) < g(x)$ $(x \in R)$ and use the usual conventions about ordered sets. In particular, the divisor f is positive if $f \geq 0$. If $a \in R$, we denote by a too the divisor equal to 1 at a and to 0 everywhere else. The points of R thus correspond to certain positive divisors. Notice that $a \leq f$ means that a belongs to the support of f. We say that the divisor f is covered by the divisor g if $g - f = a$ for some $a \in R$. A maximal chain of divisors is a finite set C of divisors which can be indexed as $f_1, \ldots, f_n$ in a unique way so that f_i is covered by f_{i+1} if $1 \leq i \leq n-1$. To every positive divisor f we associate the polynomial $p(f)$ or degree $\sum f$ given by $p(f)(x) = \prod_{t \in R} (x - t)^{f(t)}$. This establishes a one-to-one correspondence between positive divisors and the monic non-zero polynomials on R. To every positive divisor f on R, every $a \in R$ and $i \in Z^+$, we associate polynomials and numbers as follows. In case $0 \leq i \leq f(a)-1$ (which requires $f(a) \geq 1$), we denote by $p_i(f, a)$ the polynomial whose exact degree is $\sum f - 1$, satisfying the conditions: at the point $x = a$, all derivatives of order j, $0 \leq j \leq f(a)-1$, of the polynomial vanish, with the exception of the i-th derivative whose value divided by $i!$ should equal 1; and, at the points $x \neq a$ where $f(x) \geq 1$, all derivatives of order j, $0 \leq j \leq f(x)-1$, of the polynomial vanish. Such a polynomial exists and is unique. If $i \geq f(a)$, we put $p_i(f, a) = 0$. For any $i \geq 0$, we can alternatively define $p_i(f,a)$, as the polynomial p of degree $\sum f - 1$ characterized by the conditions that $p(x) - (x-a)^i$ is divisible by $(x-a)^{f(a)}$ and $p(x)$ is divisible by $(x-t)^{f(t)}$ for all $t \neq a$. We denote by $\omega_i(f,a)$ the leading coefficient of $p_i(f,a)$ (with the understanding that the leading coefficient of the polynomial 0 is 0). Therefore $\omega_i(f,a) \neq 0$ provided $0 \leq i \leq f(a)-1$ and $\omega_i(f,a) = 0$ if $i \geq f(a)$. It is easily seen that, when $0 \leq i \leq f(a)-1$, we have

$$\omega_i(f,a) = \sum \prod_{t \neq a} \binom{-f(t)}{g(t)} (a-t)^{-f(t)-g(t)} ,$$

where the summation is extended over all divisors $g \geq 0$ for which $g(a) = 0$, $\sum g = f(a) - i - 1$; and $\binom{x}{i} = x \dots (x-i+1)/i!$ for $x \in R$, $i \in Z^+$. We shall not, however, use this explicit expression for $\omega_i(f,a)$ and an analogue for $p_i(f,a)$, but rather prove directly the elementary properties we shall need.

LEMMA 1 (First Lagrange's mean value theorem). _Let $f > 0$ be a divisor on R and φ be a real function defined and having derivatives up to order $N-1$, $N = \sum f$, in the least closed interval containing the support of f. There exists a point ζ belonging to this interval such that_

$$\sum_{t \leq f} \sum_{0 \leq i \leq f(t)-1} \omega_i(f,t) \frac{\varphi^{(i)}(t)}{i!} = \frac{\varphi^{(N-1)}(\zeta)}{(N-1)!} . \tag{1}$$

Proof. Let us introduce the polynomial

$$P = \sum_{t \leq f} \sum_{0 \leq i \leq f(t)-1} \frac{\varphi^{(i)}(t)}{i!} p_i(f,t) . \tag{2}$$

It is immediate that $\varphi^{(j)}(t) = P^{(j)}(t)$ for $0 \leq j \leq f(t)-1$, $t \leq f$.

The general form of Rolle's theorem implies the existence of ζ, in the least closed interval containing the support of f, such that $\varphi^{(N-1)}(\zeta) = P^{(N-1)}(\zeta)$. Using the fact that the degree of P is $N-1$ and (2), we see that the leading coefficient of P is both equal to $P^{(N-1)}(\zeta)/(N-1)!$ and to the left-hand side of (1). This proves (1).

Motivated by this lemma, we introduce the notation

$$\Delta(\varphi,f) = \sum_{t\in R,\, i\in Z^+} \omega_i(f,t)\ \frac{\varphi^{(i)}(t)}{i!} . \tag{3}$$

Then (1) can be written as $\Delta(\varphi,f) = \varphi^{(N-1)}(\zeta)/(N-1)!$

LEMMA 2 - <u>If</u> $f \geq 0$ <u>is a divisor, then</u> $\sum_{t\in R} \omega_0(f,t) = 0$, <u>with the exception of the case in which</u> $f = a$ <u>for some</u> $a \in R$.

Proof. The lemma is true if $f = 0$. Assume $f > 0$ and exclude the case $f = a$, $a \in R$. Then $\Sigma f \geq 2$. Take $\varphi = 1$ in Lemma 1 to get Lemma 2.

LEMMA 3 - <u>Let</u> $f_0 = 0,\ldots,f_N = f$ $(N \geq 1)$ <u>be a maximal chain of divisors on</u> R <u>and</u> φ <u>a real function defined and having derivatives up to order</u> $\sup_{t\in R} f(t)-1$ <u>in the least closed interval containing the support of</u> f. <u>The coefficients in the polynomial</u>

$$P = \sum_{0\leq i\leq N-1} A_i p(f_i) \tag{4}$$

<u>can be uniquely determined so that</u>

$$\varphi^{(j)}(t) = P^{(j)}(t) \quad \text{for} \quad 0\leq j\leq f(t)-1,\ t\leq f , \tag{5}$$

<u>and then</u>

$$A_i = \Delta(\varphi,f_{i+1}) \quad (0 \leq i \leq N-1) . \tag{6}$$

Proof. There is a unique polynomial P of degree $N-1$ satisfying (5). Since the exact degree of $p(f_i)$ is i, P has a unique expression (4). To prove (6), we proceed by induction. For $N = 1$ the lemma is true. Assume $N \geq 2$ and the lemma true for $N-1$. Let $Q = \sum_{0 \leq i \leq N-2} A_i p(f_i) = P - A_{N-1} p(f_{N-1})$. Then (5) implies that Q satisfies $\varphi^{(j)}(t) = Q^{(j)}(t)$ for $0 \leq j \leq f_{N-1}(t)-1$, $t \in f_{N-1}$. By the induction assumption, we get (6) for $0 \leq i \leq N-2$. There remains to consider A_{N-1}. We know that P is given by (2). By comparison of the leading coefficients in (2) and (4), the proof is completed.

COROLLARY - $p_j(f,a) = \sum_{0 \leq i \leq N-1} A_i p(f_i)$ _where_ $A_i = \omega_j(f_{i+1},a)$ _for_ $0 \leq i \leq N-1$, $j \geq 0$ _and_ $a \in R$.

Proof. If we take $\varphi = p_j(f,a)$, then $P = p_j(f,a)$. Also $A_i = \Delta(p_j(f,a),f_{i+1}) = \omega_j(f_{i+1},a)$, since $\Delta(p_j(f,a),g) = \omega_j(g,a)$ whenever $0 \leq g \leq f$.

LEMMA 4 - (Second Lagrange's mean value theorem). _Let_ $f_o = 0, \ldots, f_N = f$ $(N \geq 1)$ _be a maximal chain of divisors on_ R _and_ φ _a real function defined and having derivatives up to order_ $N-1$ _in the least closed interval containing the support of_ f. _There are_ ξ_i _belonging to the least closed interval containing the support of_ f_{i+1} $(0 \leq i \leq N-1)$ _such that the polynomial_

$$P = \sum_{0 \leq i \leq N-1} \frac{\varphi^{(i)}(\xi_i)}{i!} p(f_i)$$

will satisfy $\varphi^{(j)}(t) = P^{(j)}(t)$ _for_ $0 \leq j \leq f(t)-1$, $t \in f$.

<u>Proof</u>. Apply lemma 1 to the coefficients of P in lemma 3.

LEMMA 5 - (Orthogonality relations). <u>Let</u> $f_o = 0,\ldots,f_N = f$ $(N\geq 1)$ <u>be a maximal chain of divisors on</u> R. <u>Then, for</u> $a_1,a_2 \in R$ <u>and</u> $r,s \in Z^+$, <u>we have</u>

$$\sum_{1\leq i\leq N} \omega_r(f_i,a_1)\,\omega_s(f-f_{i-1},a_2) = \begin{cases} \omega_{r+s}(f,a) & \text{if } a_1 = a_2 = a, \\ 0 & \text{if } a_1 \neq a_2 . \end{cases}$$

<u>Proof</u>. We put $g_j = f - f_{N-j}$ $(o \leq j \leq N)$ to get a maximal chain which satisfies $g_o = 0$, $g_N = f$, $f_i + g_j = f$ for $1 \leq i,j$ and $i+j = N$. By the corollary to lemma 3, we can write

$$p_r(f,a_1) = \sum_{0\leq i\leq N-1} A_i p(f_i) \text{ where } A_i = \omega_r(f_{i+1},a_1)\ (0\leq i\leq N-1),$$

$$p_s(f,a_2) = \sum_{0\leq j\leq N-1} B_j p(g_j) \text{ where } B_j = \omega_s(g_{j+1},a_2)\ (0\leq j\leq N-1).$$

We form the product $p_r(f,a_1)\cdot p_s(f,a_2)$, which will be a sum of terms $A_iB_jp(f_i+g_j)$. We denote by U the sum of these terms for $0\leq i,j$ and $i+j\leq N-1$; and by V the sum extended over $1\leq i,j$ and $i+j\geq N$. Therefore

$$p_r(f,a_1)\cdot p_s(f,a_2) = U + V. \tag{7}$$

If i,j contribute to V, there are h,k so that $1\leq h\leq i$, $1\leq k\leq j$, $h+k = N$. Hence $p(f_i+g_j)$ is divisible by $p(f_h+g_k) = p(f)$. This implies that V is divisible by $p(h)$. We now distinguish two cases. If $a_1 = a_2 = a$, we know that $p_r(f,a)(x) - (x-a)^r$ and $p_s(f,a)(x) - (x-a)^s$ are divisible by $(x-a)^{f(a)}$. Therefore $p_r(f,a)(x)\cdot p_s(f,a)(x) - (x-a)^{r+s}$ is also divisible by $(x-a)^{f(a)}$. This fact together with (7) show us that $U(x) - (x-a)^{r+s}$

is divisible by $(x-a)^{f(a)}$. Moreover, (7) implies that $U(x)$ is divisible by $(x-t)^{f(t)}$ for all $t \neq a$. Since U has degree $N-1$, we conclude that $U = p_{r+s}(f,a)$. By comparison of the leading coefficients in this equality, we get the lemma in this case. Assume now $a_1 \neq a_2$. Since $p_r(f,a_1)(x)$ is divisible by $(x-t)^{f(t)}$ for all $t \neq a_1$ and $p_s(f,a_2)(x)$ is divisible by $(x-t)^{f(t)}$ for all $t \neq$ $\neq a_2$, we see that $p_r(f,a_1)(x).p_s(f,a_2)(x)$ is divisible by $(x-t)^{f(t)}$ for all t, that is by $p(f)(x)$. Then (7) implies that U is divisible by $p(h)$. Since U is of degree $N-1$ and $p(h)$ is of exact degree N, we conclude that $U = 0$. Writting down that the leading coefficient of U is 0, we get the lemma in this case.

LEMMA 6 - *Let* $0 < f_1 < \ldots < f_N$ *be divisors. If* $c_1, \ldots, c_N \in$ $\in R$ *and*

(8) $$\sum_{1 \leq i \leq N} c_i \, \omega_j(f_i, t) = 0 \quad \textit{for all} \quad t \in R, \; j \in Z^+,$$

then $c_1 = \ldots = c_N = 0$.

Proof. If $N>1$, we choose t so that $f_N(t) > f_{N-1}(t)$ and put $j = f_N(t)-1$. Then (8) gives $c_N = 0$. Similarly, if $N>2$, (8) will give $c_{N-1} = 0$ and so on until we get $c_1 \omega_j(f_1,t) = 0$. Choosing t so that $f_1(t) > 0$ and $j = 0$, we get $c_1 = 0$.

5. DOMINATION LEMMAS

In this section, we shall consider pre-normed vector spaces whose prenorms will be denoted by the standard notations $\|.\|$. At the same time, we shall consider auxiliary pre-norms. It is to be tacitly understood that the topological concepts will be taken in the sense of the main pre-norms $\|.\|$.

LEMMA 1 - Let V, W be pre-normed vector spaces,

(1) $H\subset V$, $M_o = V\supset M_1\supset\ldots\supset M_r$, $K\subset W$, $N_o = W\supset N_1\supset\ldots\supset N_r$,

be vector subspaces, φ_i a linear continuous functional on $H + M_i$, ψ_i a linear continuous functional on $K + N_i$ $(1\leq i\leq r)$. Assume that the φ_i are linearly independent on H and the ψ_i are linearly independent on K, that $\varphi_i(M_i) = 0$ and $\psi_i(N_i) = 0$ and that M_i, N_i are of co-dimension i respectively in $H + M_i$, $K + N_i$ $(1\leq i\leq r)$. Letting $h_1,\ldots,h_r, k_1,\ldots,k_r\geq 0$ be real numbers, call $a_1,\ldots,a_r$, $b_1,\ldots,b_r$ the largest pre-norms on V, W such that

(2) $$\begin{matrix} a_i(x)\leq|\varphi_i(x)| & (x\in H + M_i), & a_i(x)\leq h_i\ \|x\| & (x\in V), \\ b_i(y)\leq|\psi_i(y)| & (y\in K + N_i), & b_i(y)\leq k_i\ \|y\| & (y\in W), \end{matrix} \qquad (1\leq i\leq r).$$

Let $F: V\times W\to R$ be a function such that $F(x_1+x_2,y)\leq F(x_1,y) + F(x_2,y)$, $F(\lambda x,y) = |\lambda|F(x,y)$, $F(x,y_1+ y_2)\leq F(x,y_1) + F(x,y_2)$, $F(x,\mu y) = |\mu|F(x,y)$.

Then

(3) $$F(x,y)\leq \sum_{p+q=r+1} a_p(x)b_q(y) \qquad (x\in V, y\in W)$$

holds for $h_1,\ldots,h_r, k_1,\ldots,k_r$ large enough if and only if

(4.1) $$F(x,y)\leq \sum_{p+q=r+1} |\varphi_p(x)|\cdot|\psi_q(y)| \qquad (x\in H, y\in K),$$

(4.2) $F(M_i,N_j)\leq 0$ for $i + j = r$,

(4.3) $F(x,y)\leq L\ \|x\|\cdot\|y\|$ $(x\in V, y\in W)$ for some constant $L\geq 0$.

Proof. Necessity of (4.1) follows directly from (3) and (2). Necessity of (4.2) results from the fact that if $x\in M_i$, $y\in N_j$, $i + j\geq r$, then in every term of the summation in (3) we shall have either $i\geq p$ or $j\geq q$, hence either $a_p(x) = 0$ or $b_q(y)=0$, hence $F(x,y)\leq 0$.

Necessity of (4.3) follows from (3) and (2) with $L = \sum_{p+q=r+1} h_p k_q$.

For future reference , we notice that (4.3) allows us to write (4.2) in the more general form

(5) $$F(\bar{M}_i, \bar{N}_j) \leq 0 \quad \text{for} \quad i + j \geq r ,$$

where the bars denote closures.

We now prove sufficiency and assume all (4). Since the φ_i are linearly independent on H , we can find $\xi_1, \ldots, \xi_r \in H$ such that $\varphi_i(\xi_j) = \delta_i^j$. Clearly $M_i = \varphi_1^{-1}(0) \cap \ldots \cap \varphi_i^{-1}(0)$, since the inclusion of M_i in the right-hand side is obvious and equality follows from a co-dimension argument. Therefore

(6) $$\xi_i \notin \bar{M}_i , \quad \xi_i \in M_{i-1} .$$

Clearly $H + M_i$ is generated by $\xi_1, \ldots, \xi_i$ and M_i . Repeating the same with the ψ_i on K , we find $\zeta_1, \ldots, \zeta_r \in K$ having analogous properties.

Then (4.1) gives

(7) $$F(\xi_i, \zeta_j) \leq \delta_{i+j}^{r+1} .$$

In fact, if $i + j \neq r + 1$, then either $p \neq i$ or $q \neq j$ if $p + q = r + 1$, so we get $F(\xi_i, \zeta_j) \leq 0$ from (4.1). If $i + j = r + 1$, then $p \neq i$ and $q \neq j$, except when $p = i$, $q = j$ and so $F(\xi_i, \zeta_j) \leq 1$. This proves (7).

In the proof of sufficiency, we proceed in three steps. Firstly, we indicate an equivalent form for (3) to be used in the third step of the proof. It is immediate from (2) that

$$a_i(x) = \inf \left\{ |\lambda_i| + h_i \, \| x - (\lambda_1 \xi_1 + \ldots + \lambda_i \xi_i + u) \| ; \lambda_1, \ldots, \lambda_i \in R, u \in M_i \right\} ,$$

$$b_i(y) = \inf \left\{ |\mu_i| + k_i \| y - (\mu_1 \zeta_1 + \ldots + \mu_i \zeta_i + v) \| ; \mu_1, \ldots, \mu_i \in R, v \in N_i \right\} .$$

Therefore (3) amounts to

$$(9)\quad F(x,y)\leq \sum_{p+q=r+1}\left\{|\lambda_{pp}|+h_p\|x-(\lambda_{1p}\zeta_1+\dots+\lambda_{pp}\zeta_p+u_p)\|\right\}\cdot \cdot\left\{|\mu_{qq}|+k_q\|y-(\mu_{1q}\zeta_1+\dots+\mu_{qq}\zeta_q+v_q)\|\right\}$$

for all $x\in V$, $y\in W$, λ_{ij}, $\mu_{ij}\in R$ $(1\leq i,j\leq r)$, $u_i\in M_i$, $v_i\in N_i (1\leq i\leq r)$.

Secondly, we establish sufficiency in case V, W are finite dimensional, with emphasis on the nature of the lower bound for the h_i, k_i which will assure (3). Assume that the dimensions of V, W are less than a given integer s. Since the ξ_i and the ζ_i form two sets of linearly independent vectors, we have $r\leq s$. We also assume that the φ_i, ψ_i are linearly extended over the whole V, W with preservation of their pre-norms, by the Hahn-Banach theorem. We also let $\|F\| = \sup\{F(x,y);\ \|x\|\leq 1,\ \|y\|\leq 1\}$. We shall prove that there is a lower bound

$$(10)\qquad \ell=\mathcal{L}\{s,\|F\|,\ \|\varphi_i\|,\ \|\psi_i\|,\ \|\xi_i\|,\ \|\zeta_i\|\ (1\leq i\leq r)\}$$

depending exclusively on the numbers enclosed within patenthesis, increasing with respect to $\|F\|$, $\|\varphi_i\|$, $\|\psi_i\|$, such that (3) will hold provided h_i, $k_i\geq \ell\,(1\leq i\leq r)$.

To this end, we shall construct a topological direct decomposition of V and W, depending on dim V, dim $W\leq s$, as follows. We shall deal with V and consider W as an analogous case. We have $\overline{M}_i\subset \overline{M}_{i-1}\cap\varphi_i^{-1}(0)$. The left-hand side in this inclusion is closed and has co-dimension less than s in the right hand-side. By a known elementary fact, there is a projection ρ_i of $\overline{M}_{i-1}\cap\varphi_i^{-1}(0)$ onto a vector subspace S_i of it, whose kernel is $\overline{M}_i$ and such that

$$(11)\qquad \|\rho_i\|\leq\omega_s,$$

where ω_s is an absolute constant depending on s (2). We have

(12) $$\bar{M}_{i-1} \cap \varphi_i^{-1}(0) = S_i \oplus M_i$$

as a topological direct sum. Since

(13) $$\bar{M}_i = R\zeta_{i+1} \oplus \left\{ \bar{M}_i \cap \varphi_{i+1}^{-1}(0) \right\} ,$$

as a topological direct sum, by induction (12) and (13) give a topological direct sum

(14) $$V = R\zeta_1 \oplus S_1 \oplus R\zeta_2 \oplus \dots \oplus S_r \oplus \bar{M}_r$$

and, more generally,

(15) $$\bar{M}_i = R\zeta_{i+1} \oplus S_{i+1} \oplus \dots \oplus S_r \oplus \bar{M}_r \quad (0 \le i \le r-1).$$

Every $x \in V$ can be uniquely written as

(16) $x = \lambda_1 \zeta_1 + w_1 + \dots + w_r + w_{r+1}$, $\lambda_i \in R$, $w_i \in S_i$ $(1 \le i \le r)$, $w_{r+1} \in \bar{M}_r$.

Therefore

(17) $$w_i \in \bar{M}_{i-1} \quad (1 \le i \le r+1) .$$

To determine expressions for the projections giving the components in (16) as functions of x, we define $\alpha_i : V \longrightarrow S_i$ and $\alpha_i' : V \longrightarrow M_i$ by induction as follows

(18)
$$\begin{aligned} \alpha_0'(x) &= x \\ \alpha_i(x) &= \rho_i \left\{ \alpha_{i-1}'(x) - \varphi_i \left[\alpha_{i-1}'(x) \right] \zeta_i \right\} , \qquad (1 \le i \le r), \\ \alpha_i'(x) &= (I - \rho_i) \left\{ \alpha_{i-1}'(x) - \varphi_i \left[\alpha_{i-1}'(x) \right] \zeta_i \right\} , \end{aligned}$$

where I denotes the identity map of the vector space involved in each case. We remark that

$$\alpha_{i-1}'(x) - \varphi_i \left[\alpha_{i-1}'(x) \right] \zeta_i \in \bar{M}_{i-1} \cap \varphi_i^{-1}(0) ,$$

so that the induction makes sense. Then it is immediate that

(19)
$$\begin{aligned} \lambda_i &= \varphi_i \left[\alpha_{i-1}'(x) \right], \quad w_i = \alpha_i(x) \qquad (1 \le i \le r) , \\ w_{r+1} &= \alpha_r'(x) . \end{aligned}$$

For future reference, we determine now bounds for the pre-norms of the projections α_i, α'_i. Clearly (18) give

$$\|\alpha_i\| < \|\rho_i\|(1 + \|\varphi_i\| \cdot \|\zeta_i\|)\, \|\alpha'_{i-1}\| ,$$
$$\|\alpha'_i\| \leq (1 + \|\rho_i\|)(1 + \|\varphi_i\| \cdot \|\zeta_i\|)\, \|\alpha'_{i-1}\| ,$$

Since $\|\alpha'_0\| \leq 1$, using (11) we get

(20) $\|\alpha_i\| < A$, $\|\alpha'_i\| < A$, where $A = (1+\omega_s)^s \prod_{1 \leq j \leq r} (1+\|\varphi_j\| \cdot \|\zeta_j\|)$.

We proceed similarly with W and introduce $\sigma_i, T_i, \beta_i, \beta'_i$, B analogous to $\rho_i, S_i, \alpha_i, \alpha'_i$, A for V, so that we will have properties analogous to those described explicitly for V, in particular every $y \in W$ will uniquely be

(21) $y = \mu_1 \zeta_1 + z_1 + \ldots + z_r + z_{r+1}$, $\mu_i \in R$, $z_j \in T_i$ $(1 \leq i \leq r)$, $z_{r+1} \in \bar{N}_r$.

To prove (3) for large h_i, k_i we shall now prove an inequality analogous to (3), but involving auxiliary pre-norms $a'_1, \ldots, a'_r, b'_1, \ldots, b'_r$. Let $h'_1, \ldots, h'_r, k'_1, \ldots, k'_r > 0$ and introduce the pre-norms on V, W as follows

$$(22) \qquad \begin{aligned} a'_i(x) &= |\lambda_i| + h'_i \sum_{1 \leq j \leq i} \|w_j\| , \\ b'_i(y) &= |\mu_i| + k'_i \sum_{1 \leq j \leq i} \|z_j\| , \end{aligned} \qquad (1 \leq i \leq r),$$

provided $x \in V$, $y \in W$ have the expressions (16), (21). We shall prove the following properties similar to (2)

$$(23) \qquad \begin{aligned} a'_i(x) &\leq |\varphi_i(x)| \quad (x \in H+M_i), \quad a'_i(x) \leq h''_i \|x\| \quad (x \in V), \\ b'_i(y) &\leq |\psi_i(y)| \quad (y \in K+N_i), \quad b'_i(y) \leq k''_i \|y\| \quad (y \in W), \end{aligned} \qquad (1 \leq i \leq r),$$

where

$$h''_i = A(\|\varphi_i\| + s h'_i), \quad k''_i = B(\|\psi_i\| + s k'_i), \quad (1 \leq i \leq r).$$

The first inequality for a_i^0 follows from (15) for $0 \leq i \leq r-1$ and is clear for $i = r$ too. The second inequality for a_i^0 follows (23), (19) and (20) by

$$a_i^0(x) \leq \|\varphi_i\| \cdot \|\alpha_{i-1}^0\| \cdot \|x\| + h_i^0 \sum_{1 \leq j \leq i} \|\alpha_i\| \cdot \|x\| \leq h_i^{"} \|x\| .$$

Similarly for b_i^0. From (23) and the definition of a_i, b_i, it results that

(24) $\quad a_i^0 \leq a_i \; , \quad b_i^0 \leq b_i \quad$ provided $\; h_i \geq h_i^{"} \; , \; k_i \geq k_i^{"} \quad (1 \leq i \leq r)$.

We now prove the following relation analogous to (3)

(25) $$F(x,y) \leq \sum_{p+q=r+1} a_p^0(x) b_q^0(y) \qquad (x \in V, \quad y \in W)$$

provided $h_1^0, \ldots, h_r^0$, $k_1^0, \ldots, k_r^0$ are large enough. Taking (22) into account and by simple substitution in (25), we remark that (25) is equivalent to

(26) $$F(x,y) \leq \sum_{i+j=r+1} |\lambda_i| \cdot |\mu_j| + \sum_{i+n \leq r+1} k_{r+1-i}^0 |\lambda_i| \cdot \|z_n\| + \sum_{j+m \leq r+1} h_{r+1-j}^0 |\mu_j| \cdot \|w_m\| + \sum_{m+n \leq r+1} \left\{ \sum_{\substack{p+q=r+1 \\ p \geq m, q \geq n}} h_p^0 k_q^0 \right\} \|w_m\| \cdot \|z_n\| ,$$

where $1 \leq i, j \leq r$ and $1 \leq m, n \leq r+1$, x and y having the expressions (16) and (21). Also clearly

(27) $$F(x,y) = F(\Sigma \lambda_i \xi_i + \Sigma w_m, \Sigma \mu_j \zeta_j + \Sigma z_n) \leq \Sigma |\lambda_i| \cdot |\mu_j| F(\xi_i, \zeta_j) + \Sigma |\lambda_i| F(\xi_i, z_n) + \Sigma |\mu_j| F(w_m, \zeta_j) + \Sigma F(w_m, z_n) .$$

Using (7) to dominate the first summation in the last side of (27) and making use of (17) and its analogue in W, (5), (6) and its analogue in W, (1) and the definition of $\|F\|$ to dominate the re maining three summations, we get

$$(28)\quad F(x,y)\leq \sum_{i+j=r+1}|\lambda_i|\cdot|\mu_j| + \sum_{i+n\leq r+1}\|F\|\cdot|\lambda_i|\cdot\|\zeta_i\|\cdot\|z_n\| + \sum_{j+n\leq r+1}\|F\|\cdot|\mu_j|\cdot\|w_m\|\cdot\|\zeta_j\| + \sum_{m+n\leq r+1}\|F\|\cdot\|w_m\|\cdot\|z_n\| .$$

Comparing (26) and (28), we conclude that we shall have (26), that is (25), provided

$$(29)\quad \begin{aligned} &\|F\|\cdot\|\zeta_i\|\leq k^0_{r+1-i}, \quad \|F\|\cdot\|\zeta_j\|\leq h^0_{r+1-j} \quad (1\leq i,j\leq r),\\ &\|F\|\leq \sum_{\substack{p+q=r+1\\ p\geq m,q\geq n}} h^0_p k^0_q \quad (1\leq m,n;\ m+n\leq r+1) . \end{aligned}$$

Since (29) is compatible for large $h^0_1,\ldots,h^0_r,\ k^0_1,\ldots,k^0_r$, we are done. To be specific, let $C\geq 0$ be the least number such that (29) holds if $h^0_1,\ldots,h^0_r,\ k^0_1,\ldots,k^0_r\geq C$. Notice that C depends on $\|F\|,\ \|\zeta_i\|,\ \|\zeta_i\|$ $(1\leq i\leq r)$. Put then all $h^0_1=\ldots=h^0_r=k^0_1=$ $=\ldots=k^0_r=C$ and define

$$(30)\quad \ell = \sup\left\{ A(\|\varphi_i\|+sC),\ B(\|\psi_i\|+sC);\ 1\leq i\leq r\right\},$$

which has the nature indicated in (10), by the definition of $A,B,$ C, and is increasing in its arguments. Then we have (25) and (24), hence a fortiori (3) for $h_1,\ldots,h_r,\ k_1,\ldots,k_r\geq \ell$, as we wanted in the finite dimensional case.

Finally we prove (3) for large h_i,k_i in the general case. As we already know, this amounts to (9) for large h_i,k_i. Let $x\in V,\ y\in W,\ u_i\in M_i,\ v_i\in N_i\ (1\leq i\leq r)$ be given arbitrarily. Call V^* the vector subspace of V spanned by x and the ζ_i,u_i $(1\leq i\leq r)$;

and W^* the vector subspace of W spanned by y and the ζ_i, v_i $(1 \leq i \leq r)$. Then $\dim V^* \leq s$, $\dim W^* \leq s$, where $s = 2r+1$ by definition. Define $H^* = H \cap V^*$, $M_i^* = M_i \cap V^*$, $K^* = K \cap W^*$, $N_i^* = N_i \cap W^*$. Define φ_i^* as the restriction of φ_i to $H^* + M_i^*$ and ψ_i^* as the restriction of ψ_i to $K^* + N_i^*$. Call F^* the restriction of F to $V^* \times W^*$. The analogous of the assumptions for V, W hold for V^*, W^*. If we denote by $a_1^*, \ldots, a_r^*, b_1^*, \ldots, b_r^*$ the largest pre-norms on V^*, W^* such that

$$a_i^*(x^*) \leq |\varphi_i^*(x^*)| \quad (x^* \in H^*+M_i^*), \qquad a_i^*(x^*) \leq h_i \|x^*\| \quad (x^* \in V^*),$$
$$b_i^*(y^*) \leq |\psi_i^*(y^*)| \quad (y^* \in K^*+N_i^*), \qquad b_i^*(y^*) \leq k_i \|y^*\| \quad (y^* \in W^*),$$

we shall have the analogue of (3)

$$F^*(x^*, y^*) \leq \sum_{p+q=r+1} a_p^*(x^*) b_q^*(y^*) \qquad (x^* \in V^*, y^* \in W^*)$$

provided $h_1, \ldots, h_r, k_1, \ldots, k_r \geq \ell^*$ and

$$\ell^* = \mathcal{L}\left\{2r+1, \|F^*\|, \|\varphi_i^*\|, \|\psi_i^*\|, \|\xi_i\|, \|\zeta_i\| \quad (1 \leq i \leq r)\right\}$$

according to (10). This shows, by the first step in the proof, that the analogue of (9)

$$F^*(x^*, y^*) \leq \sum_{p+q=r+1} \left\{|\lambda_{pp}| + h_p \|x^* - (\lambda_{1p}\xi_1 + \ldots + \lambda_{pp}\xi_p + u_p^*)\|\right\} \cdot$$
$$\cdot \left\{|\mu_{qq}| + k_q \|y^* - (\mu_{1q}\zeta_1 + \ldots + \mu_{qq}\zeta_q + v_q^*)\|\right\}$$

holds for all $x^* \in V^*$, $y^* \subset W^*$, $\lambda_{ij}, \mu_{ij} \in R$ $(1 \leq i, j \leq r)$, $u_i^* \in M_i^*$, $v_i^* \in N_i^*$ $(1 \leq i \leq r)$ provided $h_1, \ldots, h_r, k_1, \ldots, k_r \geq \ell^*$. Putting $x^* = x$, $y^* = y$, $u_i^* = u_i$, $v_i^* = v_i$ in this inequality, we get (9) provided $h_1, \ldots, h_r, k_1, \ldots, k_r \geq \ell$, where

$$\ell = \mathcal{L}\left\{2r+1, \|F\|, \|\varphi_i\|, \|\psi_i\|, \|\xi_i\|, \|\zeta_i\| \quad (1 \leq i \leq r)\right\},$$

since $\ell \geq \ell^*$. The bound ℓ does not depend on the x, y, λ's, μ's, u's,

v's, hence we have proved (9), that is (3) for large $h_i, k_i (1 \leq i \leq r)$, QED.

LEMMA 2 - Let E be a pre-normed vector space $E' \subset E$ and $S \subset E$ vector subspaces, ω a prenorm on S. Assume that E' is closed and of finite codimension in $E' + S$. If a' is a given pre-norm on E' and, for $h \geq 0$, a is the largest pre-norm on E such that $a(x) \leq \omega(x)$ $(x \in S)$ and $a(x) \leq h\|x\|$ $(x \in E)$, then $a'(x) \leq a(x)$ $(x \in E')$ will hold for h large enough if and only if $a'(x) \leq \omega(x)$ $(x \in E' \cap S)$ and $a'(x) \leq L\|x\|$ $(x \in E')$ for some constant $L \geq 0$.

Proof. Necessity is clear. To prove sufficiency, we notice that $a(x) = \inf\{\omega(s) + h\|x - s\|;\ s \in S\}$ $(x \in E)$. If $S \subset E'$, then $a'(x) \leq a(x)$ $(x \in E')$ if $h \geq L$. Assume then $S \not\subset E'$. Let $s_1, \ldots, s_n$ be a basis for S modulo $E' \cap S$. We shall prove that $a'(x) \leq \omega(s + \sum \lambda_i s_i) + h\|x - (s + \sum \lambda_i s_i)\|$ for $x \in E'$, $s \in E' \cap S$, $\lambda_i \in R$ provided h is large enough, as this inequality will imply $a'(x) \leq a(x)$ $(x \in E')$ for h large enough. To this end we write $a'(x) \leq a'(s) + a'(x-s) \leq \omega(s) + L\|x-s\| \leq \omega(s + \sum \lambda_i s_i) + \omega(\sum \lambda_i s_i) + L\|x-s\|$. It is then sufficient to assure that $\omega(\sum \lambda_i s_i) + L\|x-s\| \leq h\|x - (s + \sum \lambda_i s_i)\|$. Define $\pi(\lambda_1, \ldots, \lambda_n) = \inf\{\|y - \sum \lambda_i s_i\|;\ y \in E'\}$. Then, assuming $h \geq L$, it is sufficient to assure that $\omega(\sum \lambda_i s_i) + L\|\sum \lambda_i s_i\| \leq (h-L)\,\pi(\lambda_1, \ldots, \lambda_n)$. Since E' is closed in $E' + S$, π is a norm on R^n. Hence there are $\alpha, \beta \geq 0$ such that $\omega(\sum \lambda_i s_i) \leq \alpha\pi(\lambda_1, \ldots, \lambda_n)$, $\|\sum \lambda_i s_i\| \leq \beta\pi(\lambda_1, \ldots, \lambda_n)$. It is then sufficient to take $h \geq \alpha + (1+\beta)L$.

LEMMA 3 - Let A be a pre-normed algebra, B a subalgebra, $M_0 \supset \dots \supset M_r \supset M_{r+1}$, $N_0 \supset \dots N_r \supset N_{r+1}$ $(r \geq 0)$ *be vector subspaces of* A, *where* $B+M_0 = A$, $B+N_0 = A$ *and* $M_{r+1} = N_{r+1}$ (*call* P *this vector subspace*). *Let* φ_i *be a linear continuous functional on* $B+M_i$ *and* ψ_i *be a linear continuous functional on* $B+N_i$ $(0 \leq i \leq r+1)$, *where* $\varphi_{r+1} = \psi_{r+1}$ (*call* χ *this functional*). *We assume that* M_i *is of codimension* $i+1$ *in* $B+M_i$ *and* N_i *is of codimension* $i+1$ *in* $B+N_i$ $(0 \leq i \leq r)$; *that* $\varphi_i(M_i) = 0$ *and* $\psi_i(N_i) = 0$ $(0 \leq i \leq r+1)$; *that* $\varphi_0(I) = 1$, $\psi_0(I) = 1$, $\varphi_i(I) = 0$ *and* $\psi_i(I) = 0$ $(1 \leq i \leq r+1)$; *that the* φ_i $(0 \leq i \leq r)$ *are linearly independent on* B *and that the* ψ_i $(0 \leq i \leq r)$ *are linearly independent on* B. *We also assume that*

$$(1) \qquad \chi(xy) = \sum_{\substack{p+q=r+1 \\ p,q \geq 0}} \varphi_p(x)\, \psi_q(y) \qquad (x,y \in B),$$

$$(2) \qquad M_i N_j \subset P \quad (0 \leq i,j;\ i+j = r).$$

If $h_0,\dots,h_{r+1}, k_0,\dots,k_{r+1} \geq 0$, *where* $h_{r+1} = k_{r+1}$ (*call* ℓ *this number*), *let* $a_0,\dots,a_{r+1}, b_0,\dots,b_{r+1}$ *be the largest prenorms on* A *such that*

$$\left.\begin{array}{ll} a_i(x) \leq |\varphi_i(x)| \ (x \in B+M_i), & a_i(x) \leq h_i \|x\| \ (x \in A), \\ b_i(y) \leq |\psi_i(y)| \ (y \in B+N_i), & b_i(y) \leq k_i \|y\| \ (y \in A), \end{array}\right. \quad (0 \leq i \leq r+1),$$

(*since* $a_{r+1} = b_{r+1}$, *call* c *this pre-norm*). *Then, given* ℓ, *the inequality*

$$(3) \qquad c(xy) \leq \sum_{\substack{p+q=r+1 \\ p,q \geq 0}} a_p(x) b_q(y) \qquad (x,y \in A)$$

will hold for $h_0,\dots,h_r, k_0,\dots,k_r$ *large enough*.

Proof. Put $V = M_0$, $W = N_0$ and notice that $M_0 = \varphi_0^{-1}(0)$,

$N_o = \varphi_o^{-1}(0)$ since M_o, N_o are of codimension 1 in $B+M_o = B+N_o = A$. The φ_i $(1 \leq i \leq r)$ are linearly independent on $H = B \cap V$ and the ψ_i $(1 \leq i \leq r)$ are linearly independent on $K = B \cap W$. Also $M_i \subset H+M_i = (B+M_i) \cap \varphi_o^{-1}(0)$ shows that M_i is of codimension i in $H+M_i$ $(0 \leq i \leq r)$. Similarly N_i is of codimension i in $K+N_i$. Putting $F(x,y) = c(xy)$ $(x \in V, y \in W)$, the conditions (4) of Lemma 1 are satisfied, in view of the above conditions (1), (2). Hence

$$(4) \qquad c(xy) \leq \sum_{\substack{p+q=r+1 \\ p,q \geq 1}} a_p^0(x) b_q^0(y) \qquad (x \in V, y \in W)$$

holds if $h_1^0, \ldots, h_r^0, k_1^0, \ldots, k_r^0 \geq 0$ are large enough, where $a_1^0, \ldots, a_r^0$, $b_1^0, \ldots, b_r^0$ are largest prenorms on V, W such that

$$\begin{matrix} a_i^0(x) \leq |\varphi_i(x)| & (x \in H+M_i), & a_i^0(x) \leq h_i^0 \|x\| & (x \in V), \\ b_i^0(y) \leq |\psi_i(y)| & (y \in K+N_i), & b_i^0(y) \leq k_i^0 \|y\| & (y \in W), \end{matrix} \quad (1 \leq i \leq r).$$

We fix the values of the h_i^0, k_i^0 $(1 \leq i \leq r)$ so that (4) holds. Since V, W are closed and of codimension 1 in A, we may apply Lemma 2 and say that $a_i^0 \leq a_i, b_i^0 \leq b_i$ $(1 \leq i \leq r)$ on V, W, provided the h_i, k_i $(1 \leq i \leq r)$ are large enough. Therefore

$$(5) \qquad c(xy) \leq \sum_{\substack{p+q=r+1 \\ p,q \geq 1}} a_p(x) b_q(y) \qquad (x \in V, y \in W)$$

holds provided the h_i, k_i $(1 \leq i \leq r)$ are large enough. Finally, if $x, y \in A$, we remark that $a_o(x) = |\varphi_o(x)|$ $(x \in A)$, $b_o(y) = |\psi_o(y)|$ $(y \in A)$ provided $h_o \geq \|\varphi_o\|$, $k_o \geq \|\psi_o\|$ and that $xy = (x-\varphi_o(x)I)(y-\psi_o(y)I) + \varphi_o(x)y + \psi_o(y)x + \varphi_o(x)\psi_o(y)I$. Since $x - \varphi_o(x)I \in V$, $y - \psi_o(y)I \in W$, we conclude that (5) implies (3), as wanted.

6. ALGEBRAS OF FINITE DIFFERENTIAL ORDER

A local algebra is an algebra with a unique maximal ideal [5]. An algebra is said to be of finite differential order whenever A is a direct sum of a finite number of ideals $A_1,\dots,A_s$ which are local algebras whose maximal ideals $M_1,\dots,M_s$ satisfy $M_1^{n_1} = \dots = M_s^{n_s} = 0$ for some integers $n_1,\dots,n_s \geq 1$. Such a direct sum decomposition is unique, apart from the order in which the components are indexed. Assuming that $n_1,\dots,n_s$ are the least integers satisfing the above conditions, the sum $n = \sum n_i$ is called the differential order of A (3). An alternative way of defining an algebra of finite differential order consists in requesting that the algebra should have a finite number of maximal ideals and a nilpotent radical [3].

We call attention to the fact that, in applications to algebraic geometry, a local algebra A is usually called separated in case the intersection of the powers M^n $(n = 1,2,\dots)$ of its maximal ideal M is 0. Such a meaning for the concept of separated local algebra should not be confused with the one needed in this paper (cfr. §2 for the concept of separated algebra), as it is possible to give an example of a pure local algebra A which fails to be separated in the analytical sense used here, but has a maximal ideal M that satisfies $M^3 = 0$, hence is separated in the other sense.

LEMMA 1 - _Let_ A _be a pure separated algebra of differential order_ $n+1$ $(n > 0)$. _For_ $p \in P(R)$ _and_ $w \geq 0$, _we have_

(1) $$\sup\left\{\sum_{0\leq i\leq n}\frac{\lambda^i}{i!}\,|p^{(i)}(t)|\,;\ \lambda\geq 0,\ |t|+\lambda\leq w\right\}\leq \|p;\,A,\,w\|\leq$$
$$\leq\sum_{0\leq i\leq n}\frac{(2w)^i}{i!}\ \sup\left\{|p^{(i)}(t)|\,;\ \ |t|\leq w\right\}.$$

Proof. Let $A = A_1 \oplus \dots \oplus A_s$ be the direct decomposition of A into a sum of ideals $A_1,\dots,A_s$ which are pure local algebras whose maximal ideals $M_1,\dots,M_s$ satisfy $M_1^{n_1} = \dots = M_s^{n_s} = 0$, where $n_1,\dots,n_s \geq 1$ are the least integers satisfying such conditions and $\sum n_i = n+1$. We denote by $I = I_1 + \dots + I_s$ the decomposition of the unity of A. We also denote by $\sigma_1,\dots,\sigma_s$ the suitably indexed homomorphisms of A onto R. We shall firstly establish the final part of (1). Let $\pi \in \Pi(A)$ and ϕ be a linear functional on A whose pre-norm with respect to π satisfies $\pi(\phi) \leq 1$ that is $|\phi(x)| \leq \pi(x)$ for all $x \in A$. By the Hahn-Banach theorem, the final part of (1) will follow from

(2) $$|\phi(p(x))|\leq \sum_{0\leq i\leq n}\frac{(2\pi(x))^i}{i!}\sup\left\{|p^{(i)}(t)|\,;\ \ |t|\leq \pi(x)\right\}.$$

To prove (2), we write $\phi = \sum \phi_i$, where ϕ_i is the linear functional given by $\phi_i(x) = \phi(I_i x)$ $(x \in A)$. Then $\pi(\phi_i) \leq \pi(I_i)$, showing that ϕ_i is continuous with respect to π. We now show that, for every i such that $\phi_i \neq 0$, the homomorphism σ_i is continuous with respect to π. In fact, for every $x,y \in A$ we have
$\phi_i\left\{(x - \sigma_i(x)I)^{n_i}y\right\} = 0$, that is

(3) $$\sum_{0\leq j\leq n}\binom{n_i}{j}(-\sigma_i(x))^{n_i-j}\,\phi_i(x^j y) = 0.$$

Choosing $y \in A$ so that $\phi_i(y) = (-1)^{n_i}$, we see that $\sigma_i(x)$ is a root of an algebraic equation of degree n_i whose leading coefficient is 1 and whose remaining coefficients tend to 0 as $\pi(x)$ tends to 0. Hence $\sigma_i(x)$ tends to 0 as $\pi(x)$ tends to 0, that is σ_i is continuous with respect to π. It then follows from an elementary remark in the theory of normed algebras that $|\sigma_i(x)| \leq \pi(x)$ $(x \in A)$.

In proving (2), we may assume that $\phi \neq 0$. Let $x \in A$ be fixed and consider the finite set T formed by the distinct values of $\sigma_i(x)$ for which $\phi_i \neq 0$. We define a divisor f on the real line R by letting $f(t) = \sup\{ n_i ; \sigma_i(x) = t , \phi_i \neq 0\}$ if $t \in T$ and $f(t) = 0$ if $t \notin T$. We then apply Lemma 4 of §4 by choosing a maximal chain $f_o = 0, \ldots, f_N = f$ of divisors. Take the function φ of the lemma to be p and determine P, ζ_i $(0 \leq i \leq N-1)$ as indicated. We shall prove that

$$\phi(p(x)) = \phi(P(x)) . \tag{4}$$

To this end, we shall establish that $\phi_i(p(x)) = \phi_i(P(x))$ for every i, $1 \leq i \leq s$. This is clear if $\phi_i = 0$. Assume then $\phi_i \neq 0$. It is then sufficient to prove that $p(x) - P(x) = (x - \sigma_i(x)I)^{n_i} y_i$ for some $y_i \in A$. To obtain such a relation, we remark that the polynomials p and P have at the point $\sigma_i(x)$ the same derivatives up to order $n_i - 1$ at least, hence $p(t) - P(t) = (t - \sigma_i(x))^{n_i} \cdot Q_i(t)$ $(t \in R)$ for some polyhomial Q_i and then it suffices to take $y_i = Q_i(x)$. Having proved (4), from it we get

$$|\phi(p(x))| = |\phi(P(x))| \leq \pi(P(x)) \leq \sum_{0 \leq j \leq N-1} \frac{|p^{(j)}(\zeta_j)|}{j!} \pi[p(f_j)(x)] . \tag{5}$$

If $\phi_i \neq 0$, then $|\sigma_i(x)| \leq \pi(x)$, hence $\pi(x - \sigma_i(x)I) \leq 2\pi(x)$. It follows that $\pi[p(f_j)(x)] \leq (2\pi(x))^j$. Moreover all the ζ_j belong to the least closed interval containing the support of f and, a fortiori, satisfy $|\zeta_i| \leq \pi(x)$. Since clearly $N = \Sigma f \leq \Sigma n_i = n+1$, (5) implies (2), as we wanted in proving the final part of (1).

We now prove the initial part of (1). Consider real numbers $t_1,\ldots,t_s$ which are assumed to be pairwise different. We shall keep them fixed until almost the end of the proof, when we shall use their arbitrariness. We then call f the divisor on R equal to n_i at t_i $(1 \leq i \leq s)$ and to 0 everywhere else. Clearly $f > 0$.

We assume given a correspondence assigning to every divisor g on R , $0 < g \leq f$, a prenorm a_g on A . We shall choose such a correspondence later. Define

$$\alpha_i(x) = \sup\left\{ a_g(x);\ 0 < g \leq f,\ \Sigma g = i \right\} \quad \text{for} \quad 1 \leq i \leq n+1,\ x \in A ,$$

$$\pi(x) = \sum_{1 \leq i \leq n+1} \alpha_i(x) \quad \text{for} \quad x \in A .$$

Then the α_i $(1 \leq i \leq n+1)$ and π are pre-norms on A . Notice that, if $a_g(I) = 1$ for $\Sigma g = 1$ and $a_g(I) = 0$ for $\Sigma g \geq 2$, as we shall assume, then $\pi(I) = 1$.

We also assume that to every divisor g , $0 < g \leq f$, we have associated a maximal chain $C(g)$ of divisors, whose first term covers 0 and whose last term is g itself, this choice being made once for all but arbitrarily. For every $d \in C(g)$, we denote by d^* the predecessor of d in $C(g)$, with the exception of the case in which d is the first element of $C(g)$, as then we put $d^* = 0$.

We find it convenient to divide this part of the proof into few steps for proper reference.

(i) If every a_g , $0 < g \leq f$, satisfies

(6) $$a_g(xy) \leq \sum_{d \in C(g)} a_d(x) a_{g-d^*}(y) \qquad (x, y \in A),$$

then $\pi : x \to \pi(x)$ is a prenorm of algebra on A. In fact, the correspondence $d \in C(g) \longrightarrow i = \Sigma d$ is one-to-one between $C(g)$ and the integers from 1 to $j = \Sigma g$. Moreover $\Sigma(g-d^*) = j - (i-1)$. So $a_d(x)a_{g-d^*}(y) \leq \alpha_i(x)\alpha_{j-i+1}(y)$. Therefore

$$\alpha_j(xy) \leq \sum_{1 \leq i \leq j} \alpha_i(x)\alpha_{j-i+1}(y) \qquad (x, y \in A,\ 1 \leq j \leq n+1)$$

which suffices to prove the statement.

(ii) There are $z_i \in M_i$ such that $z_i^{n_i - 1} \neq 0$ $(1 \leq i \leq s)$. Therefore, since A is separated, there is a pre-norm of algebra $x \to \|x\|$ on A such that $\|z_i^{n_i-1}\| \neq 0$ $(1 \leq i \leq s)$. The topological notions on A will be understood in terms of thi. pre-norm. Without loss of generality, we may assume that A is topologically the direct sum of $A_1, \ldots, A_s$ as we may replace the pre-norm of algebra $x \to \|x\|$ on A by $x \to \sup_i \{ |\lambda_i| + \|u_i\| \}$ if $x = \Sigma_i(\lambda_i I_i + u_i)$, where $u_i \in M_i$ $(1 \leq i \leq s)$. Notice that $\sigma_1, \ldots, \sigma_s$ are continuous.

(iii) If, for some i, $1 \leq i \leq s$, and some n, $1 \leq n \leq n_i$, we have

(7) $$\sum_{j \geq 0} c_j (z_j)^j \in \overline{M_i^n},$$

where $(z_i)^0 = I_i$, the bar denotes closure in A and $c_j = 0$ for almost all $j \geq 0$, then $c_j = 0$ for $0 \leq j \leq n-1$. In fact, multiplying (7) by $(z_i)^{n_i-1}$ we get $c_0 = 0$. If $n \geq 2$, multiplying (7) by $(z_i)^{n_i-2}$ we get $c_1 = 0$, and so on until $c_{n-1} = 0$. There results, in particular, that the vectors $(z_i)^j$ for $0 \leq j \leq n_i - 1$, $1 \leq i \leq s$, are linearly independent.

(iv) We shall denote by B the vector subspace of A generated by all $(z_i)^j$, $1 \leq i \leq s$, $j \geq 0$. B is a subalgebra of A. For every divisor g, $0 < g \leq f$, we denote by B_g the vector subspace of A generated by the $(z_i)^j$, $0 \leq j \leq g(t_i)-1$, $1 \leq i \leq s$. By the remark at the end of (iii), these generators of B_g are linearly independent, so $\dim B_g = \Sigma g$. We also introduce the following vector subspace of A

$$A_g = \sum_{1 \leq i \leq s} M_i^{g(t_i)} \qquad (M_i^0 = A_i) .$$

We notice that $B + A_g = B_g + A_g$. Also $B_g \cap \bar{A}_g = 0$, by (iii), where the bar denotes closure in A. In particular, $B_g \cap A_g = 0$, hence A_g is of codimension Σg in $B + A_g$. This allows us also to define a linear functional τ_g on $B + A_g$ by requiring that $\tau_g(A_g) = 0$, $\tau_g[(z_i)^j] = \omega_j(g, t_i)$ for $0 \leq j \leq g(t_i)-1$, $1 \leq i \leq s$ (where $\omega_j(g, t_i)$ is meant in the sense introduced in §4). Since $B_g \cap \bar{A}_g = 0$, this linear functional τ_g is well defined and continuous. We note that, more generally

(8) $$\tau_g[(z_i)^j] = \omega_j(g, t_i) \quad \text{for} \quad 1 \leq i \leq s, \quad j \geq 0 .$$

Remark that, if $\Sigma g = 1$ i.e. $g = t_i$ for some i, then we shall have $\tau_g = \sigma_i$, $B + A_g = A$. In this case $\tau_g(I) = 1$. We also remark that $\Sigma g \geq 2$ implies $\tau_g(I) = 0$, by the Lemma 2, §4.

(v) We now indicate how to choose the a_g, $0 < g \leq f$. We assume that to every divisor g, $0 < g \leq f$, there is assigned a number $h_g \geq 0$ and that a_g is the largest pre-norm on A such that

(9) $$a_g(x) \leq \lambda^{\Sigma g - 1} . |\tau_g(x)| \quad (x \in B+A_g), \quad a_g(x) \leq h_g \|x\| \quad (x \in A),$$

where $\lambda \geq 0$ is fixed.

If $\Sigma g = 1$, that is if $g = t_i$ for some i, since $|\sigma_i(x)| \leq \|x\|$ for all $x \in A$, we see that we shall have $a_g(x) = |\sigma_i(x)|$ $(x \in A)$ as soon as $h_g \geq 1$. In particular, $a_g(I) = 1$. We note also that $\Sigma g \geq 2$ implies that $a_g(I) = 0$.

(vi) Since all τ_g, $0 < g \leq f$, are continuous and their number is finite, by the Hahn-Banach theorem there is a number $h' \geq 1$ such that

(10) $$a_g(x) = \lambda^{\Sigma g - 1} \cdot |\tau_g(x)| \quad (x \in B + A_g)$$

hold for $0 < g \leq f$, provided all $h_g \geq h'$, as we shall assume.

(vii) Given g, $0 < g \leq f$ and h_g, the expression for a_g when $\Sigma g = 1$ indicated in (iv) shows that (6) holds if $h_g \geq 1$, as we assumed. Let us then consider the case $\Sigma g \geq 2$. Then there is a number $Z_g(h_g) \geq 0$ depending on h_g such that, if all h_d, $d \in C(g)$, $d \neq g$ satisfy $h_d \geq Z_g(h_g)$, then (6) holds. To show this, we shall apply Lemma 3 of §5 as follows. Denote by $d_1, \ldots, d_m = g$ the elements of $C(g)$ in their natural order, hence $m = \Sigma g \geq 2$. Then, in Lemma 3, take A and B to be the present algebra A and its subalgebra B, A being endowed with the pre-norm of algebra mentioned in (ii). Let $r = m-2$, $M_i = A_{d_{i+1}}$, $N_i = A_{g-d^*_{m-i}}$, $\varphi_i = \tau_{d_{i+1}}$, $\psi_i = \tau_{g-d^*_{m-i}}$. We then have $M_i N_j = A_{d_{i+1}} A_{g-d^*_{m-i}} = A_g$ if $0 \leq i, j$ and $i+j = r$. We also have

$$\tau_g(xy) = \sum_{d \in C(g)} \tau_d(x) \tau_{g-d^*}(y) = \sum_{\substack{p+q=r+1 \\ p,q \geq 0}} \varphi_p(x) \psi_q(y) \quad (x, y \in B).$$

In fact, it is sufficient to verify this for $x = (\zeta_i)^j$, $y = (\zeta_\lambda)^u$

and make use of the orthogonality relations given by Lemma 5 of §4. The fact that the φ_i $(0\le i\le r)$ are linearly independent on B follows from Lemma 6 of §4. Similarly for the ψ_i $(0\le i\le r)$. The application of Lemma 3 of § 5 is then legitime as all the remaining conditions are satisfied.

Having established these remarks, we start from a_f and determine h_f and succesively all the h_g, $0<g\le f$, in the backward order so that all (6) and (10) hold. π is then a pre-norm of algebra on A. We consider $x=\Sigma(t_i I_i + \varepsilon_i)$. For any polynomial $p\in P(R)$, we have

$$p(x) = \sum_{1\le i\le s} p(t_i I_i + \varepsilon_i) = \sum_{1\le i\le s,\, j\ge 0} \frac{p^{(j)}(t_i)}{j!}(\varepsilon_i)^j$$

where the summation with respect to j is finite. There results from (10) and (8) that

$$a_g(p(x)) = \lambda^{\Sigma g-1}\cdot|\Delta(p,g)|$$

according to the notation introduce in (3) or §4, hence

$$\text{(11)}\qquad \pi[p(x)] = \sum_{1\le i\le n+1} \lambda^{i-1}\sup\{|\Delta(p,g)|;\ 0<g\le f,\ \Sigma g = i\}.$$

In particular $\pi(x) = \sup(|t_i|;\ 1\le i\le s)+\lambda$. Now, if $w>0$ and $|t|+\lambda\le w$, we may choose the t_i, $1\le i\le s$, pairwise distinct satisfying $|t_i|+\lambda\le w$ $(1\le i\le s)$ and let $x=\Sigma(t_i I_i+\varepsilon_i)$. Since $\pi(x)\le w$, hence $\pi[p(x)]\le\|p\|_w$, if we apply Lemma 1 of §4 to each $\Delta(p,g)$, then (11) will give us, as every $t_i\to t$, that

$$\sum_{1\le i\le n+1}\frac{\lambda^i}{i!}|p^{(i)}(t)|\le\|p;\ A,\ w\|$$

from which the initial part of (1) follows in the case $w>0$. If $w=0$, what we have to prove is $|p(0)|\le\|p\|_0$, which is clear.

So the proof is completed.

7. OPERATIONAL CALCULUS WITH DIFFERENTIABLE FUNCTIONS

A topological algebra A is said to have an operational calculus with $C^n(R)$ whenever A satisfies the Hausdorff separation axiom and there is a continuous mapping $C^n(R) \times A \longrightarrow A$ denoted by $(f,x) \longrightarrow f(x)$ such that, in case $f \in P(R)$, $f(t) = a_o + \ldots + a_m t^m$ $(t \in R)$, then $f(x) = a_o + \ldots + a_m x^m$ for $x \in A$. The Weierstrass approximation theorem implies that such a mapping is necessarily unique. More generally, we shall say that A has a pre-operational calculus with $C^n(R)$ provided the natural mapping $P(R) \times A \longrightarrow A$ given by $(f,x) \longrightarrow f(x)$ is continuous as soon as $P(R)$ is endowed with the topology of order n induced on it by $C^n(R)$. Of course, in case A satisfies the Hausdorff axiom and has a pre-operational calculus, then A will have an operational calculus if A is complete in the sense of Cauchy-Weil, or even in a weaker sense.

In case, for all $w \geq 0$, the pre-norms of algebra on $P(R)$ given by $p \longrightarrow \|p; A,S,w\|$ are continuous with respect to the topology of order n, where S is a collection of continuous pre-norms of algebra on A determining its topology, then the topological algebra A will have a pre-operational calculus with $C^n(R)$.

THEOREM - Let C be a category of pure separated algebras whose radicals are nilpotent and $n \geq 0$ an integer. In order that every topological algebra A which is locally convex with respect to C should have a pre-operational calculus with $C^n(R)$ it is necessary and sufficient that C be a subcategory of the category D_{n+1} of all pure separated algebra of differential order $\leq n+1$.

Proof. In proving sufficiency, it is enough to consider a topological algebra A which is locally convex with respect to D_{n+1} and then show that A has a pre-operational calculus with $C^n(R)$, since $C \subset D_{n+1}$ by assumption. Let S be the set of all continuous pre-norms of algebra π on A which are convex with respect to the collection I_{n+1} of ideals $I \subset A$ such that $A/I \in D_{n+1}$. For $\pi \in S$, we have (1) $\pi(x) = \sup\{\pi_I(x); I \in I_{n+1}\}$, where $\pi_I(x) = \inf\{\pi(x-y); y \in I\}$, $x \in A$. Letting $I \in I_{n+1}$, we consider the natural homomorphism $A \longrightarrow A/I$, denoted by $x \longrightarrow \bar{x}_I$, and the quotient pre-norm of algebra $\bar{\pi}_I$ induced by π on A/I , where $\bar{\pi}_I(\bar{x}_I) = \pi_I(x)$ $(x \in A)$. By applying the final part of (1) of Lemma 1, §6, to A/I , we see that

$$\bar{\pi}_I[p(\bar{x}_I)] \leq \sum_{0 \leq i \leq n} \frac{[2\bar{\pi}_I(\bar{x}_I)]^i}{i!} \sup\left\{ |p^{(i)}(t)| ; |t| \leq \bar{\pi}_I(\bar{x}_I)\right\}$$

for $p \in P(R)$. Since $\bar{\pi}_I(\bar{x}_I) = \pi_I(x) \leq \pi(x)$ and $\bar{\pi}_I[p(\bar{x}_I)] = \pi_I[p(x)]$, we get, in view of (1) and using the arbitrariness of I,

$$\|p ; A, S, w\| \leq \sum_{0 \leq i \leq n} \frac{(2w)^i}{i!} \sup\left\{|p^{(i)}(t)| ; |t| \leq w\right\}$$

for $w > 0$. This shows that the pre-norms of algebra $p \longrightarrow \|p; A, S, w\|$ on $P(R)$ are continuous for the topology of order n , hence A has a pre-operational calculus with $C^n(R)$, since S determines the topology of A .

Conversely, let C be a category of pure separated algebras with nilpotent radicals having the property that local convexity with respect to C implies a pre-operational calculus with $C^n(R)$. Take $A \in C$ and call $C(A)$ the category of all algebras isomorphic to A. Since $C(A) \subset C$, we see that, a fortiori, $C(A)$ has the property that local convexity with respect to $C(A)$ implies a pre-operational calculus with $C^n(R)$.

We shall prove now that every pre-norm of algebra $p \longrightarrow \|p; A, w\|$ on $P(R)$, for $w > 0$, is continuous for the topology of order n. In fact, let A^* be the algebra of all functions f defined on $A \times \Pi(A)$, with values in A, such that

$$\|f\| = \sup\left\{\pi[f(x,\pi)]\,;\ x \in A,\ \pi \in \Pi(A)\right\} < +\infty .$$

On this algebra A^*, the function $f \longrightarrow \|f\|$ is a pre-norm of algebra. For every $x_0 \in A$, $\pi_0 \in \Pi(A)$, the functions $f \in A^*$ such that $f(x_0, \pi_0) = 0$ form an ideal $I(x_0, \pi_0)$ in A^*. The homomorphism $f \longrightarrow f(x_0, \pi_0)$ of A^* onto A has $I(x_0, \pi_0)$ as its kernel, hence $A^*/I(x_0, \pi_0) \in C(A)$. Since the pre-norm of A^* is convex with respect to the collection of these ideals, a fortiori A^* is locally convex with respect to $C(A)$. There results, by our assumption, that A^* has a pre-operational calculus with $C^n(R)$. For every $w \geq 0$, we introduce the subset $L(w)$ of $A \times \Pi(A)$ defined by $L(w) = \left\{(x, \pi);\ \pi \in \Pi(A),\ \pi(x) \leq w\right\}$ and consider the function f_w on $A \times \Pi(A)$ with values in A such that $f_w(x, \pi)$ is equal to x if $(x, \pi) \in L(w)$ and to 0 if $(x, \pi) \notin L(w)$. Clearly $f_w \in A^*$. Since the mapping $(p, f) \longrightarrow p(f)$ from $P(R) \times A^*$ into A^* is continuous, $P(R)$ being endowed with the topology of order n, we see in particular that the mapping $p \longrightarrow p(f_w)$ from $P(R)$ into A^* is continuous for every $w \geq 0$. Hence the mapping $p \longrightarrow \|p(f_w)\|$ is continuous too. It is then sufficient to notice that $\|p(f_w)\| = \|p;\ A,\ w\|$ to get our assertion.

We now prove that A has at most $n+1$ maximal ideals. In fact, assume that A has at least $m \geq 1$ maximal ideals. A being pure, it has at least m homomorphisms onto R. The homomorphisms of A onto R are linearly independent. Therefore there exists a homomorphism of A onto R^m. The existence of such a homomorphism implies that $\|p;\ R^m,\ w\| \leq \|p;\ A,\ w\|$ for $p \in P(R)$, $w \geq 0$. Since R^m has finite differential order equal to m, we may apply the initial part of (1) in Lemma 1, §6, by replacing A by R^m and assuming $w \geq 1$, $\lambda = 1$, as the arbitrariness of λ is not essential here. Then we get

$$(2) \qquad \sup \left\{ \sum_{0 \leq i \leq m-1} \frac{|p^{(i)}(t)|}{i!};\ |t| \leq w-1 \right\} \leq \|p;\ A, w\| .$$

This inequality together with the continuity property of $p \longrightarrow \|p;\ A,\ w\|$ with respect to the topology of order n implies that $m-1 < n$, hence A has at most $n+1$ maximal ideals. The finiteness of the number of maximal ideals of A jointly with the nilpotency of its radical show us that A has finite differential order M. If we apply again the initial part of (1) of Lemma 1, §6, this time to A itself with $w \geq 1$, $\lambda = 1$, we get again the inequality (2) with m replaced by M. The same continuity argument allows us to say that $M-1 \leq n$, that is $M < n+1$, hence $A \in D_{n+1}$. We thus conclude that $C \subset D_{n+1}$, as wanted.

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FOOTNOTES

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(1) We shall restrict ourselves to the operational calculus with the topological algebra $C^n(R)$ of all n-differentiable real functions on the real line R. Since the topological algebra $C^n(R^\Lambda)$ of n-differentiable real functions of several (finitely or infinitely many) variables indexed by Λ is the topological tensor product of $C^n(R)$ by itself Λ-times in one of the two extreme senses considered by Grothendieck [2], existence of the pre-operational calculus with $C^n(R)$ under the local convexity assumptions employed here implies existence of the pre-operational calculus with $C^n(R^\Lambda)$ for all Λ.

(2) For every integer $s \geq 0$, there is an absolute constant $\omega_s \geq 0$ depending on s such that, if E is a pre-normed vector space of dimension $\leq s$ and $F \subset E$ is a closed vector subspace, there exists a projection of E into itself whose kernel is F and whose pre-norm is less than ω_s.

(3) In view of the statement of the theorem of §7 and the terminology adopted by Weil [6], it might seem more adequate to call $\Sigma n_i - 1$ the differential order of A. We prefer the terminology adopted here as then the differential order of a direct sum is the sum of the differential orders of the components.