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Classification of irreps and invariants of the N-extended Supersymmetric Quantum Mechanics.

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Abstract

We present an algorithmic classification of the irreps of the N-extended one-dimensional supersymmetry algebra linearly realized on a finite number of fields. Based on the 1-to-1 [1] correspondence between Weyl-type Clifford algebras (whose irreps are fully classified) and classes of irreps of the N-extended $1D$ supersymmetry, we are able to prove that, for $N = 3, 5 \quad mod \quad 8$, two classes of irreps, real and quaternionic, are found.

The complete classification of irreps is presented up to $N \leq 10$. The fields of an irrep are accommodated in l different spin states. $N = 10$ is the minimal value admitting length $l > 4$ irreps. The classification of length-4 irreps of the $N = 12$ and *real* $N = 11$ extended supersymmetries is also explicitly presented.

Tensoring irreps allows us to systematically construct manifestly (N-extended) supersymmetric multi-linear invariants *without* introducing a superspace formalism. Multi-linear invariants can be constructed both for *unconstrained* and *multi-linearly constrained* fields. Examples are given.

Tensoring zero-energy irreps leads us to the notion of the *fusion algebra* of the $1D$ N-extended supersymmetric vacua.

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1 Introduction

Supersymmetric Quantum Mechanics is a twenty-five years old topic [2] which is still under intensive development and even received in the last few years a considerable renewed attention. Part of the reason is due to the wide range of applicability of one-dimensional supersymmetric theories and especially superconformal quantum mechanics [3] for extremal black holes [4], in the AdS-CFT correspondence [5] (when setting AdS_2), for investigating partial breaking of extended supersymmetries [6, 7] and so on. Another very important motivation is due to the fact that considerable advances in understanding the structure of large-N extended supersymmetry itself have been made in the recent years. It is well known that large N (up to $N = 32$, starting from the maximal, eleven-dimensional supergravity) one-dimensional supersymmetric quantum mechanical models are automatically derived [8] from the dimensional reduction of higher-dimensional supersymmetric field theories. Large N one-dimensional supersymmetry on the other hand (possibly in the $N \to \infty$ limit) even emerges in condensed matter phenomena, as described by the BCS model, see e.g. [9].

Controlling one-dimensional N-extended supersymmetry for arbitrary values of N (that is, the nature of its representation theory, how to construct manifestly supersymmetric invariants, etc.) is a technical, but challenging program with important consequences in many areas of physics, see e.g. the discussion in [10] concerning the nature of *on-shell versus off-shell* representations, for its implications in the context of the supersymmetric unification of interactions.

Along the years, progresses came from two lines of attack. In the pivotal work of [11] irreducible representations were investigated to analyze supersymmetric quantum mechanics. A special role played by Clifford algebra was pointed out [12]. Clifford algebras were also used in [13] to construct representations of the extended one-dimensional supersymmetry algebra for arbitrarily large values of N. Another line of attack, consisted in using the superspace, so that manifest invariants could be constructed through superfields. For low values of N this is indeed the most convenient approach. However, with increasing N, the associated superfields are getting highly reducible and require the introduction of constraints to extract the irreducible representations. This approach gets soon unpractical for large N. Indeed, only very recently a manifestly $N = 8$ superfield formalism for one-dimensional theory has been introduced, see [14] and references therein. A manifest superfield formalism is however still not available for larger values of N.

In [1] a contribution in understanding the nature of the linearly realized irreducible representations of arbitrary N-extended supersymmetries was made, proving that *any* irrep can be classified and recovered from an associated Clifford algebra. This is the starting point for the present work.

In this paper we furnish a systematic classification of the irreps of the N-extended one-dimensional supersymmetry algebra. In [1] it was shown that all such irreps fall into classes of equivalence in 1-to-1 correspondence with a certain subclass (the Weyl subclass) of Clifford algebras, the dimensionality of the Clifford algebra being linked to the integer N labeling the extension of the supersymmetry. It was further proven that any given irrep can be constructed by applying a so-called *dressing transformation* to the length-2 irrep (the length of an irrep expresses the number of its different spin states) belonging to its equivalence class. The classification of length-2 irreps is immediately available, being borrowed from the known [15, 16, 17] classification of Clifford algebras. On the other hand, the classification of irreps of general length $l > 2$ requires the investigation of the properties (mostly the *locality* property, discussed in Section **2**) of the dressing transformations. In [1] length-3 irreps were easily classified, but no general attempt was made to classify irreps with length $l \geq 4$ (only one specific example, the unique length-4 irrep of the $N = 3$ supersymmetry which, btw, coincides with the $N = 3$ adjoint representation, as discussed in Section **5**, was explicitly constructed). The full classification of general length irreps can be achieved and systematically organized by using specific properties of the Clifford irreps (encoded, e.g., in the so-called "block-symbols" of Section **4**) which can be algorithmically computed according to the method presented in [18]. In our approach an important role is also played by the notion of *oxidized* Clifford algebras (corresponding to, essentially, the maximal number of Clifford generators which can be accommodated into irreducible representations of a given matrix size, see [19]), together with their associated *oxidized* extended supersymmetries discussed in Section **3**. The fact that oxidized Clifford algebras are either real or quaternionic [19] can be used, e.g., to prove that for $N = 3, 5 \quad mod \quad 8$ two separated classes of irreps, real (denoted with "$^{(*)}$") and quaternionic (denoted with "$^{(**)}$"), are found (the $mod \quad 8$ property is in consequence of the Bott's periodicity of Clifford Gamma matrices; the remaining values of N admit a unique type of irreps).

The algorithmic [18] presentation of Clifford irreps allows us to classify, for any given N, the irreducible representations of the one-dimensional supersymmetry algebra (2.1) realized on a *finite* number of fields and to explicitly construct a representative in each irreducible class. In this paper we limit ourselves to explicitly present the complete classification of irreps for $N \leq 10$ (and furnish the classification of length-4 irreps for the oxidized $N = 11^{(*)}, 12$ supersymmetries).[1]

It deserves to be mentioned at this point that the inequivalent irreducible representations have been used in the literature to produce super-particle models moving in one and higher dimensional target manifolds. The three length-3 irreps of the $N = 4$ supersymmetry (namely $(1, 4, 3)$, $(2, 4, 2)$ and $(3, 4, 1)$, see Appendix **A**) were respectively used, e.g., to construct one-dimensional [6, 20, 21], two-dimensional [22] and three-dimensional [23] Supersymmetric Quantum Mechanics. An updated list of references concerning the Supersymmetric Quantum Mechanics constructed via the length-3 $N = 8$ irreps can be found in [14].

Besides classifying irreps, in this paper we also point out that tensoring irreps allows us to systematically construct manifestly (multi)-linear invariants of the N-extended supersymmetry algebra (2.1). The reason lies in the fact that the component fields of highest spin in the tensored multiplets transform, under supersymmetry, as time-derivatives. They can therefore be used as lagrangian terms entering a mani-

[1] It is worth pointing out that our method can be applied to arbitrarily large values of N. It should be taken into account however that, while some properties of the irreps, like their dimensionality (2.2) or the fact that the class of irreps is closed under the *high* $\Leftrightarrow$ *low* spin duality (4.3), can be easily stated, at increasing N not only the actual computations, but also the presentations of the complete lists of results are getting more and more cumbersome.

fest invariant action. It is worth mentioning that in this framework these invariants are constructed *without* introducing the notion of superspace and of their associated superfields. As already recalled, for large values of N, superfields are of limited use.

In our framework two big classes of invariants, *constrained* and *unconstrained*, can be constructed. Indeed, the tensor product of irreps produces, in general, reducible representations. A basic illustrative example can be considered the $N = 4$ self-tensoring of the $(1, 4, 3)$ multiplet, see Section **6**, producing at the leading order the $N = 4$ adjoint representation (the adjoint representations are reducible for $N \geq 4$). Therefore, either we extract the invariants in terms of unconstrained fields from the highest spin component(s) of the tensored reducible representations or we implement bilinear (in general, multilinear for multiple tensorings) constraints to extract an irreducible representation realized with bilinear (multilinear) combinations of the original fields. The highest spin components of the bilinearly (multilinearly) realized irrep generate invariants, once the bilinear (multinear) constraints are taking into account.

We further discuss the possibility to accommodate, within our framework, manifest N-extended invariants for σ-model types of terms [24].

Another concept introduced in this work, is that of the *fusion algebra* of the zero-energy irreps (i.e. of the supersymmetric vacua) of the N-extended one-dimensional supersymmetry. This concept is mimicked after the corresponding notion for RCFT's, see [25]. It allows us to encode, in integer-valued fusion matrices, the decomposition into irreps of the tensor products of the supersymmetry irreps at zero energy.

The plan of the paper is the following. In Section **2** the [1] 1-to-1 connection between Weyl-type Clifford algebras and classes of irreps of the N-extended supersymmetry algebra is reviewed. In Section **3** the classification of Clifford algebras is recalled. On the light of the (2.7) connection, this allows us to construct inequivalent irreps, respectively real and quaternionic, for the $N = 3, 5 \quad mod \quad 8$ one-dimensional extended supersymmetries. The notion of *oxidized* Clifford algebras, leading to *oxidized* and *reduced* N-extended supersymmetries, is introduced. In Section **4** we use the knowledge of Clifford algebras (encoded in the so-called "block-symbols") to classify (and explicitly construct representatives for each irreducible multiplet) arbitrary length-l irreps of the N-extended supersymmetry. The complete classification of irreps is explicitly reported up to $N \leq 10$. In Section **5** some comments are made on the nature of the adjoint representations. In Section **6**, the tensorings of irreps and their decomposition into irreps is used to construct multi-linear invariants of the N-extended supersymmetries. These invariants can be used as potential, constant kinetic, σ-model type, etc., terms entering a manifest N-extended invariant action, *without* introducing a superspace and the associated superfield formalism. The decomposition of tensored-products into irreps lead us to introduce the two big classes of the *unconstrained* invariants and the bilinearly (in general, multilinearly) *constrained* invariants, see the $(1, 4, 3) \otimes (1, 4, 3)$ $N = 4$ example. In Section **7** the notion of the fusion algebra of the supersymmetric vacua is introduced. Non-negative integral-valued fusion matrices encode the tensoring of the zero-energy supersymmetry irreps. The associativity of the tensoring implies the commutativity of the fusion matrices. In the Conclusions we present a more detailed analysis of the results here achieved and discuss future perspectives. The work is further integrated with four appendices. Appendix **A** is devoted to explicitly present representatives of each supersymmetry irrep for all N-extended supersymmetries with $N \leq 8$. In Appendix **B** we furnish the complete classification of the irreps for the $N = 9, 10$ extended supersymmetries and explicitly present the length-4 classification of irreps for the oxidized $N = 11^{(*)}, 12$ extended supersymmetries. In Appendix **C** the tensoring of the $N = 2$ irreps and of some selected examples of the $N = 4$ irreps, is explicitly presented. In Appendix **D** we produce the fusion matrices (for both cases, either disregarding or taking into account the statistics of the multiplets) of the $N = 2$ supersymmetric vacua.

2 Irreps of the N-extended $d = 1$ supersymmetry and Clifford algebras: the connection revisited

In this section we review the main results of ref. [1] concerning the classification of irreps of the N-extended one-dimensional supersymmetry algebra.

The N extended $D = 1$ supersymmetry algebra is given by

$$\{Q_i, Q_j\} = \eta_{ij} H \tag{2.1}$$

where the Q_i's are the supersymmetry generators (for $i, j = 1, \ldots, N$) and $H \equiv -i\frac{\partial}{\partial t}$ is a hamiltonian operator (t is the time coordinate). If the diagonal matrix η_{ij} is pseudo-Euclidean (with signature (p, q), $N = p+q$) we can speak of generalized supersymmetries. The analysis of [1] was done for this general case. For convenience in the present paper (despite the fact that our results can be straightforwardly generalized to pseudo-Euclidean supersymmetries, having applicability, e.g., to supersymmetric spinning particles moving in pseudo-Euclidean manifolds) we work exclusively with ordinary N-extended supersymmetries. Therefore for our purposes here $\eta_{ij} \equiv \delta_{ij}$.

The (D-modules) representations of the (2.1) supersymmetry algebra realized in terms of *linear* transformations acting on *finite* multiplets of fields satisfy the following properties. The total number of bosonic fields equal the total number of fermionic fields. For irreps of the N-extended supersymmetry the number of bosonic (fermionic) fields is given by d, with N and d linked through

$$\begin{aligned} N &= 8l + n, \\ d &= 2^{4l} G(n), \end{aligned} \tag{2.2}$$

where $l = 0, 1, 2, \ldots$ and $n = 1, 2, 3, 4, 5, 6, 7, 8$. $G(n)$ appearing in (2.2) is the Radon-Hurwitz function [1]

n	1	2	3	4	5	6	7	8
$G(n)$	1	2	4	4	8	8	8	8

(2.3)

The modulo 8 property of the irreps of the N-extended supersymmetry is in consequence of the famous modulo 8 property of Clifford algebras. The connection between supersymmetry irreps and Clifford algebras is specified later.

Due to the fact that the $D = 1$ dimensional reduction of the maximal $N = 8$ supergravity produces a supersymmetric quantum mechanical system with $N = 32$ extended number of supersymmetries, it is convenient to explicitly report the number of bosonic/fermionic component fields in any given irrep of (2.1) for any N up to $N = 32$. We get the table

$N=1$	1	$N=9$	16	$N=17$	256	$N=25$	4096
$N=2$	2	$N=10$	32	$N=18$	512	$N=26$	8192
$N=3$	4	$N=11$	64	$N=19$	1024	$N=27$	16384
$N=4$	4	$N=12$	64	$N=20$	1024	$N=28$	16384
$N=5$	8	$N=13$	128	$N=21$	2048	$N=29$	32768
$N=6$	8	$N=14$	128	$N=22$	2048	$N=30$	32768
$N=7$	8	$N=15$	128	$N=23$	2048	$N=31$	32768
$N=8$	8	$N=16$	128	$N=24$	2048	$N=32$	32768

(2.4)

The bosonic (fermionic) fields entering an irreducible multiplet can be grouped together according to their dimensionality. Throughout this paper we use, interchangeably, the words "dimension" or "spin" to refer to the dimensionality of the component fields. It is in fact useful, especially when discussing the $D = 1$ dimensional reduction of higher-dimensional supersymmetric theories, to refer at the dimensionality of the $D = 1$ fields as their "spin". The number (equal to l) of different dimensions (i.e. the number of different spin states) of a given irrep, will be referred to as the *length* l of the irrep. Since there are at least two different spin states (one for bosons, the other for fermions), obtained when all bosons (fermions) are grouped together within the same spin, the minimal length of an irrep is $l = 2$.

A general property of (linear) supersymmetry in any dimension is the fact that the states of highest spin in a given multiplet are auxiliary fields, whose supersymmetry transformations are given by total derivatives. Just for $D = 1$ total derivatives coincide with the (unique) time derivative. Using this specific property of the one-dimensional supersymmetry it was proven in [1] that all finite linear irreps of the (2.1) supersymmetry algebra fall into classes of equivalence, each class of equivalence being singled out by an associated minimal length ($l = 2$) irreducible multiplet. It was further proven that the minimal length irreducible multiplets are in 1-to-1 correspondence with a subclass of Clifford algebras (the ones which satisfy a Weyl property). The connection goes as follows. The supersymmetry generators acting on a length-2 irreducible multiplet can be expressed as

$$Q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i \cdot H & 0 \end{pmatrix} \tag{2.5}$$

where the σ_i and $\widetilde{\sigma}_i$ are matrices entering a Weyl type (i.e. block antidiagonal) irreducible representation of the Clifford algebra relation

$$\Gamma_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_i \\ \widetilde{\sigma}_i & 0 \end{pmatrix} \quad , \quad \{\Gamma_i, \Gamma_j\} = 2\eta_{ij} \tag{2.6}$$

The Q_i's in (2.5) are supermatrices with vanishing bosonic and non-vanishing fermionic blocks, acting on an irreducible multiplet m (thought as a column vector) which can be either bosonic or fermionic[2]. The connection between Clifford algebra irreps of Weyl type and minimal length irreps of the N-extended one-dimensional supersymmetry is such that D, the dimensionality of the (Euclidean, in the present case) space-time of the Clifford algebra (2.6) coincides with the number N of the extended supersymmetries, according to

♯ of space-time dim. (Weyl-Clifford)	$\Leftrightarrow$	♯ of extended su.sies (in 1-dim.)
D	$=$	N

(2.7)

The matrix size of the associated Clifford algebra (equal to $2d$, with d given in (2.2)) corresponds to the number of (bosonic plus fermionic) fields entering the one-dimensional N-extended supersymmetry irrep.

The classification of Weyl-type Clifford irreps, furnished in [1], can be easily recovered from the well-known classification of Clifford irreps, given in [15] (see also [16] and [17]).

The (2.5) Q_i's matrices realizing the N-extended supersymmetry algebra (2.1) on length-2 irreps have entries which are either c-numbers or are proportional to the hamiltonian H. Irreducible representations of higher length ($l \geq 3$) are systematically produced [1] through repeated applications of the dressing transformations

$$Q_i \quad \mapsto \quad \widehat{Q}_i^{(k)} = S^{(k)} Q_i {S^{(k)}}^{-1} \tag{2.8}$$

realized by diagonal matrices $S^{(k)}$'s ($k = 1, \ldots, 2d$) with entries ${s^{(k)}}_{ij}$ given by

$${s^{(k)}}_{ij} \quad = \quad \delta_{ij}(1 - \delta_{jk} + \delta_{jk} H) \tag{2.9}$$

Some remarks are in order [1]

i) the dressed supersymmetry operators ${Q_i}'$ (for a given set of dressing transformations) have entries which are integral powers of H. A subclass of the ${Q_i}'$ s dressed operators is given by the local dressed operators, whose entries are *non-negative* integral powers of H (their entries have no $\frac{1}{H}$ poles). A local representation (irreps fall into this class) of an extended supersymmetry is realized by local dressed operators. The number of the extension, given by N' ($N' \leq N$), corresponds to the number of local dressed operators.

ii) The local dressed representation is not necessarily an irrep. Since the total number of fields (d bosons and d fermions) is unchanged under dressing, the local dressed representation is an irrep iff d and N' satisfy the (2.2) requirement (with N' in place of N).

iii) The dressing changes the dimension (spin) of the fields of the original multiplet m. Under the $S^{(k)}$ dressing transformation (2.8), $m \mapsto S^{(k)} m$, all fields entering m are unchanged apart the k-th one (denoted, e.g., as φ_k and mapped to $\dot{\varphi}_k$). Its dimension is

[2]We conventionally consider a length-2 irreducible multiplet as bosonic if its upper half part of component fields is bosonic and its lower half is fermionic. It is fermionic in the converse case.

changed from $[k] \mapsto [k]+1$. This is why the dressing changes the length of a multiplet. As an example, if the original length-2 multiplet m is a bosonic multiplet with d spin-0 bosonic fields and d spin-$\frac{1}{2}$ fermionic fields (in the following such a multiplet will be denoted as $(x_i;\psi_j) \equiv (d,d)_{s=0}$, for $i,j=1,\ldots,d$), then $S^{(k)}m$, for $k \leq d$, corresponds to a length-3 multiplet with $d-1$ bosonic spin-0 fields, d spin-$\frac{1}{2}$ fermionic fields and a single spin-1 bosonic field (in the following we employ the notation $(d-1,d,1)_{s=0}$ for such a multiplet).

Let us fix now the overall conventions. The most general multiplet is of the form $(d_1,d_2,\ldots,d_l)$, where d_i for $i=1,2,\ldots,l$ specify the number of fields of a given spin $s+\frac{i-1}{2}$. The spin s, i.e. the spin of the lowest component fields in the multiplet, will also be referred to as the "spin of the multiplet". When looking purely at the representation properties of a given multiplet the assignment of an overall spin s is arbitrary, since the supersymmetry transformations of the fields are not affected by s. Introducing a spin is useful for tensoring multiplets and becomes essential for physical applications, e.g. in the construction of supersymmetric invariant terms entering an action.

In the above multiplet l denotes its length, d_l the number of auxiliary fields of highest spins transforming as time-derivatives. The total number of odd-indiced equal the total number of even-indiced fields, i.e. $d_1+d_3+\ldots=d_2+d_4+\ldots=d$. The multiplet is bosonic if the odd-indiced fields are bosonic and the even-indiced are fermionic (the multiplet is fermionic in the converse case). For a bosonic multiplet the auxiliary fields are bosonic (fermionic) if the length l is an odd (even) number.

Just like the overall spin assignment, the assignment of a bosonic (fermionic) character to a multiplet is arbitrary since the mutual transformation properties of the fields inside a multiplet are not affected by its statistics. Therefore, multiplets always appear in dually related pairs s.t. to any bosonic multiplet there exists its fermionic counterpart with the same transformation properties (see also [26]).

Throughout this paper we assign integer valued spins to bosonic multiplets and half-integer valued spins to fermionic multiplets.

As recalled before, the most general $(d_1,d_2,\ldots,d_l)$ multiplet is recovered as a dressing of its corresponding N-extended length-2 (d,d) multiplet. In [1] it was shown that all dressed supersymmetry operators producing any length-3 multiplet (of the form $(d-p,d,p)$ for $p=1,\ldots,d-1$) are of local type. Therefore, for length-3 multiplets, we have $N'=N$. This implies, in particular, that the $(d-p,d,p)$ multiplets are inequivalent irreps of the N-extended one-dimensional supersymmetry. For what concerns length $l\geq 4$ multiplets, the general problem of finding irreps was not addressed in [1]. It was shown, as a specific example, that the dressing of the length-2 $(4,4)$ irrep of $N=4$, realized through the series of mappings $(4,4)\mapsto(1,4,3)\mapsto(1,3,3,1)$, produces at the end a length-4 multiplet $(1,3,3,1)$ carrying only three local supersymmetries ($N'=3$). Since the relation (2.2) is satisfied when setting equal to three the number of extended supersymmetries and equal to 4 the total number of bosonic (fermionic) fields entering a multiplet, as a consequence, the $(1,3,3,1)$ multiplet corresponds to an irreducible representation of the $N=3$ extended supersymmetry.

In the next two sections, based on an algorithmic construction of representatives of Clifford irreps, we present an iterative method to classify all irreducible representations of higher length for arbitrary N values of the extended supersymmetry (the complete results up to $N \leq 10$, plus some further selected cases, are explicitly presented in the appendices **A** and **B**).

3 Oxidized supersymmetries

In order to proceed at the classification of the length $l > 3$ irreducible multiplets and attack the problem of classifying the one-dimensional N extended supersymmetries irreps, we need to use specific properties of the associated Clifford irreps.

We report here the needed mathematical background. We recall at first, see [17], that the Clifford algebras generated by the Γ-matrices Γ_i, $i = 1, \ldots, p+q$, satisfying $\{\Gamma_i, \Gamma_j\} = 2\eta_{ij}$ for a (p,q) signature (s.t. the η_{ij} matrix is diagonal with p positive, $+1$, and q negative, -1, entries), can be classified according to the most general matrix S commuting with all Γ's (i.e. $[S, \Gamma_i] = 0$ for any i). If the most general S is a multiple of the identity we get the normal (**R**) case. This situation occurs for $p-q = 0,1,2 \; mod \; 8$. Otherwise, for $p-q = 3,7 \; mod \; 8$, the most general S is the sum of two matrices, the second one multiple of the square root of -1 (this case is named the "almost complex", **C**, case) and, finally, for $p-q = 4,5,6 \; mod \; 8$, the most general S is a linear combination of four matrices closing the quaternionic algebra (this case is referred to as the quaternionic, **H**, case)[3].

A real irreducible representation of the Clifford algebra is always unique [17] unless the relation

$$p-q \;=\; 1,5 \quad mod \quad 8 \tag{3.1}$$

is verified. For the above space-time signatures two inequivalent irreducible real representations are present, the second one recovered by flipping the sign of all Γ's ($\Gamma_i \mapsto -\Gamma_i$ for any i).

In the following, the Clifford irreps corresponding to the (p,q) signatures are denoted as $Cl(p,q)$ (for our purposes there is no need to discriminate the two inequivalent irreps related with the (3.1) signatures).

A concept that will be applied later is that of *maximal Clifford algebra* [19]. It corresponds to the maximal number of Gamma matrices of (p,q) signature which can be accommodated in a Clifford irrep of a given matrix size. Non-maximal Clifford irreps are recovered from the maximal ones, after deleting a certain number of Clifford Gamma matrices thought as external generators (see [19] for details). Maximal Clifford algebras can also be referred to as the *oxidized* forms of a Clifford algebra, using a pun introduced in the superstrings/M-theory literature, where *oxidation* denotes the inverse operation w.r.t. the dimensional reduction [27].

Some remarks are in order.

i) An *oxidized* form of a Clifford algebra is encountered if and only if the associated signature satisfies the (3.1) $p-q = 1,5 \; mod \; 8$ condition.

[3] Throughout this paper we work with irreducible representations realized as real matrices. The **R**, **C** and **H** cases, however, can also be described by matrices whose entries are valued in the corresponding division algebra.

ii) Oxidized Clifford irreps are not of Weyl-type (see the previous section discussion). It is indeed always present, among their Clifford generators, the block-diagonal space-like Gamma matrix[4] $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ which, on the light of the (2.7) *Weyl-Clifford* $\Leftrightarrow$ 1-*dim. N-extended su.sies irreps* correspondence, plays the role of a fermion number operator. All the remaining generators of the oxidized Clifford irreps can be assumed of block-antidiagonal (Weyl type) form.

We can define as *oxidized* N-extended supersymmetries the ones constructed in terms of the supersymmetry generators associated, according to (2.5) and (2.6), with the whole set of block-antidiagonal space-like gamma matrices of a corresponding oxidized Clifford irrep.

The concept of *reduced* N-extended one-dimensional supersymmetries can be introduced in full analogy with the concept of non-maximal Clifford algebras. The reduced N-extended supersymmetries are such that their length-2 irreps *do not* accommodate the whole maximal number of supersymmetry generators at disposal. Stated otherwise, a reduced extended supersymmetry is always obtained from an associated oxidized N-extended supersymmetry after deleting a certain number of supersymmetry generators. Please notice that the irreducibility requirement is ensured if $\widetilde{N}$ (where $\widetilde{N} < N$ is the number of reduced supersymmetry generators picked up from the whole set of generators of the oxidized extended supersymmetry) satisfies a constraint related with the (2.2) condition.

¿From the results of [19] we can construct a first table, expressing the oxidized (and respectively the reduced) N-extended one-dimensional supersymmetries in terms of their associated oxidized Clifford irreps. We get

oxidized Clifford irreps	oxidized su.sies	reduced su.sies
$Cl(2+8m,1)_{\mathbf{R}}$	$N=1+8m$	$-$
$Cl(3+8m,2)_{\mathbf{R}}$	$N=2+8m$	$-$
$Cl(4+8m,3)_{\mathbf{R}}$	$N=3+8m^{(*)}$	$-$
$Cl(5+8m,0)_{\mathbf{H}}$	$N=4+8m$	$N-1=3+8m^{(**)}$
$Cl(6+8m,1)_{\mathbf{H}}$	$N=5+8m^{(**)}$	$-$
$Cl(9+8m,0)_{\mathbf{R}}$	$N=8+8m$	$N-1=7+8m$ $N-2=6+8m$ $N-3=5+8m^{(*)}$

(3.2)

In the above table $m=0,1,2,\ldots$ is a non-negative integer. The oxidized Clifford irreps are of real, **R**, type (and, respectively, quaternionic, **H**, type) if $p-q=1\ mod\ 8$ ($p-q=5\ mod\ 8$).

It is worth mentioning that the oxidized and reduced extended supersymmetries are not affected by which one of the two inequivalent choices for the (3.1) Clifford irreps is made. As a consequence, the length-2 irreps of the $N=1,2,4,6,7,8\ mod\ 8$ extended

[4]Space-like gamma matrices γ are those whose square is the identity; conversely, a time-like gamma matrix γ is such that $\gamma^2=-\mathbf{1}_{2d}$, where $\mathbf{1}_{2d}$ is the $2d\times 2d$ identity operator.

supersymmetries are unique. With respect to the $N=3,5$ *mod* 8 extended supersymmetries the situation is as follows. For each such value of N two inequivalent irreps are found (labeled, respectively, as $N=3^{(*)},3^{(**)}$ *mod* 8 and $N=5^{(*)},5^{(**)}$ *mod* 8) acting on multiplets with the same number of bosonic/fermionic fields. The first class of "$^{(*)}$" irreps corresponds to real-type supersymmetries, the second one (the "$^{(**)}$" irreps) to quaternionic-type supersymmetries.

Oxidized extended supersymmetries are found in the $N=1,2,3^{(*)},4,5^{(**)},8$ *mod* 8 cases. Reduced extended supersymmetries are found for $N=3^{(**)}$ *mod* 8 (obtained from the $N=4$ *mod* 8 oxidized form) and $N=5^{(*)},6,7$ *mod* 8 (these three cases are recovered from the $N=8$ *mod* 8 class of oxidized extended supersymmetries). The whole picture is summarized in the following two tables

Clifford irreps		Extended su.sies (mod 8)
$Cl(2+8m,1)_{\mathbf{R}}$	$\rightarrow$	$N=1$
$Cl(3+8m,2)_{\mathbf{R}}$	$\rightarrow$	$N=2$
$Cl(4+8m,3)_{\mathbf{R}}$	$\rightarrow$	$N=3^{(*)}$
$Cl(5+8m,0)_{\mathbf{H}}$	$\rightarrow$	$N=3^{(**)},4$
$Cl(6+8m,1)_{\mathbf{H}}$	$\rightarrow$	$N=5^{(**)}$
$Cl(9+8m,0)_{\mathbf{R}}$	$\rightarrow$	$N=5^{(*)},6,7,8$

(3.3)

and

$N=3^{(*)}$	mod 8	real	oxidized
$N=3^{(**)}$	mod 8	quaternionic	reduced
$N=5^{(*)}$	mod 8	real	reduced
$N=5^{(**)}$	mod 8	quaternionic	oxidized

(3.4)

The fundamental $N=3^{(*)},3^{(**)},5^{(*)},5^{(**)}$ extended supersymmetries, whose complete list of irreps is explicitly presented in Appendix **A**, are obtained through

$Cl(4,3)\longrightarrow N=3^{(*)}$,
$Cl(5,0)\longrightarrow N=4\longrightarrow N=3^{(**)}$,
$Cl(9,0)\longrightarrow N=8\longrightarrow N=5^{(*)}$,
$Cl(6,1)\longrightarrow N=5^{(**)}$.

It is worth noticing that the $N=3^{(*)}$ oxidized supersymmetry (not admitting another Euclidean supersymmetry generator) can however be extended to an oxidized pseudo-Euclidean generalized supersymmetry (confront the discussion in Section **2**) with maximal number of six (with $(3,3)$ signature) pseudo-Euclidean supersymmetry generators.

We recall, finally, that the dimensionality of the irreps of the N-extended supersymmetries can be read, for any N, from (2.2) (see also, for $N\leq 32$, the (2.4) table).

We conclude this section with some necessary remarks on the nature of the Clifford irreps. A convenient way of systematically constructing a representative of each class of $Cl(p,q)$ irreducible Clifford representations for any (p,q) signature is through the algorithmic procedures, see [18], *i)* $\gamma_i \mapsto \Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, mapping a (p,q) Clifford irrep spanned by the $p+q$ γ_i's matrices ($i = 1,\ldots,p+q$) into a $(p+1,q+1)$ Clifford irrep and

ii) $\gamma_i \mapsto \Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, mapping the (p,q) Clifford irrep into a $(q+2,p)$ Clifford irrep.

With the help of the two algorithms above, applied to the set of fundamental Clifford irreps $1 \equiv Cl(1,0)$, $Cl(0,3+8m)$ and $Cl(0,7+8m)$ (for $m = 0,1,2,\ldots$) we can construct [18] a representative of any Clifford irrep for arbitrary values of p and q. The set of $Cl(0,3+8m)$ and $Cl(0,7+8m)$ Clifford irreps were explicitly presented in [18] as repeated tensor products of the set of the three real 2×2 matrices τ_1, τ_2 and τ_A, given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

plus the 2×2 identity $\mathbf{1}_2$.

$Cl(0,3)$ is given by the three matrices $\tau_1\otimes\tau_A$, $\tau_2\otimes\tau_A$, $\tau_A\otimes\mathbf{1}_2$, while $Cl(0,7)$ is realized through the seven matrices $\tau_1\otimes\tau_A\otimes\mathbf{1}_2$, $\tau_2\otimes\tau_A\otimes\mathbf{1}_2$, $\tau_A\otimes\mathbf{1}_2\otimes\tau_1$, $\tau_A\otimes\mathbf{1}_2\otimes\tau_2$, $\mathbf{1}_2\otimes\tau_1\otimes\tau_A$, $\mathbf{1}_2\otimes\tau_2\otimes\tau_A$ and $\tau_A\otimes\tau_A\otimes\tau_A$.

$Cl(0,3+8m)$ (and, respectively, $Cl(0,7+8m)$), for $m = 1,2,\ldots$, are recursively obtained as repeated tensor products of $Cl(0,3)$ ($Cl(0,7)$) with m sets of matrices from $Cl(1,8)$ and the $\mathbf{1}_{16}$ identity (see [18] for details).

For our purposes it is sufficient to recall here that, as a consequence of the above algorithmic constructions, the $Cl(p,q)$ real Clifford irreps present gamma matrices admitting one and only one non-vanishing entry (given by ± 1) in each column and in each row. Moreover, space-like gamma matrices do not share the same non-vanishing entries; stated otherwise, there is no overlap, an $e_{ij}\neq 0$ entry belongs to one, and only one, space-like gamma matrix. For any oxidized Clifford irrep we can further compute the associated "block-symbol", introduced in the next section, which allows us to systematically and efficiently count the the number N' of local supersymmetry dressed operators (see (2.8) and the discussion thereafter).

4 Classification of the irreps

In this section we present a systematic procedure to produce and classify length $l > 3$ irreps of the (2.1) supersymmetry algebra for arbitrary values of N. We apply it to fully classify all irreps up to $N \leq 10$ and, for the next cases of the oxidized $N = 11^{(*)}$ and $N = 12$ supersymmetries, the length-4 irreps (the results are explicitly presented in the Appendices **A** and **B**).

Our approach is based on the following points (names and conventions here employed have been introduced in the previous two sections):

i) the (2.5) and (2.6) connection between oxidized Clifford irreps and (oxidized and reduced) length-2 irreps of the (2.1) supersymmetry algebra, *ii*) the (2.8) dressing transformation of length-2 irreps, producing length $l > 2$ local type representations of the (2.1) supersymmetry algebra,

iii) the matching condition (2.2) between the number of the extended supersymmetries and the dimension of the representation. It is satisfied if and only if the representation is irreducible and, finally,

iv) the algorithmic properties of the real Clifford irreps discussed at the end of Section **3**.

As explained in Section **2**, the dressing can produce $\frac{1}{H}$ poles in the dressed supersymmetry operators. An $S^{(k)}$ dressing (2.8, 2.9) of a given supersymmetry operator Q has the total effect of multiplying by $\frac{1}{H}$ all Q's entries belonging to the k-th column and by H all Q's entries belonging to the k-th row, leaving unchanged all remaining entries. In order to count (and remove) dressed operators with $\frac{1}{H}$ poles one has to determine how non-vanishing entries are distributed in the whole set of supersymmetry operators (since the Q's are 2×2 block-antidiagonal matrices, we can focus on the upper-right block, the lower-left block presenting the same structure). Up to $N \leq 8$, all non-vanishing entries of an oxidized supersymmetry fill the whole upper-right block (for $N = 8$, e.g., we have eight supersymmetry operators with 8 non-overlapping non-vanishing entries each, s.t. $8 \times 8 = 64$, filling the 8×8 upper block chessboard of the $N = 8$ supersymmetry). Starting from $N \geq 9$ this is no longer the case. The 16×16 right upper block "chessboard" of the $N = 9$ supersymmetry is filled with a total number of $9 \times 16 = 144 < 16^2$ non-overlapping non-vanishing entries.

In the $N = 9$ example each column and each row of the upper-right (bottom-left) block intercepts the same amount of 9 non-vanishing entries belonging to the whole set of 9 gamma matrices; the remaining $16 - 9 = 7$ entries are zero.

Not only the total number, but also the distribution of the non vanishing-entries inside the block matrices matters when computing the locality condition of the dressed supersymmetry operators. The structure of the non-vanishing entries filling the large-N oxidized supersymmetries can be recovered from the algorithmic construction of the Clifford irreps discussed in Section **3**. For $N \geq 8$, the filling of the upper-right block can be symbolically presented (the block-symbol diagrams below) in terms of the three fundamental fillings of an 8×8 matrix. The three fundamental fillings, denoted as O, I, X, represent, respectively,

i) O $\equiv$ only vanishing entries,

ii) I $\equiv$ non-vanishing entries filling the diagonal,

iii) X $\equiv$ non-vanishing entries filling the whole 8×8 matrix.

The block-symbols, explicitly presented here for the oxidized supersymmetries with

$8 \leq N \leq 12$, are given by

$$
\begin{array}{lcc}
N=8 & : & \begin{pmatrix} \mathrm{X} \end{pmatrix} \\
N=9 & : & \begin{pmatrix} \mathrm{I} & \mathrm{X} \\ \mathrm{X} & \mathrm{I} \end{pmatrix} \\
N=10 & : & \begin{pmatrix} \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} \\ \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} \\ \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} \\ \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} \end{pmatrix} \\
N=11^* & : & \begin{pmatrix} \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} \\ \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} \\ \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{O} \\ \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} & \mathrm{O} \\ \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{I} \end{pmatrix} \\
N=12 & : & \begin{pmatrix} \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} \\ \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} \\ \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{O} & \mathrm{I} & \mathrm{X} & \mathrm{I} \end{pmatrix}
\end{array}
\tag{4.1}
$$

Block-symbols can be straightforwardly computed for arbitrary large-N values of the oxidized supersymmetries.

For reduced supersymmetries extra holes appear in the block-symbols, corresponding to the non-vanishing entries belonging to the $N - N'$ supersymmetry operators that have been "removed" from the whole set of oxidized operators in order to produce the reduced N'-extended supersymmetry.

Concerning multiplets, it is worth reminding that the diagonal dressing operator

$$
S = \begin{pmatrix} H \cdot \mathbf{1}_d & 0 \\ 0 & \mathbf{1}_d \end{pmatrix} \tag{4.2}
$$

applied on a (d, d) length-2 multiplet reverses its statistics (the same transformation reverses the statistics of fields in any given multiplet).

Length-3 multiplets are obtained by applying, on a (d, d) length-2 multiplet, diagonal dressing operators S with a total number of k (with $1 \leq k \leq d-1$) single powers of H in the first d diagonal entries, while the $2d - k$ remaining diagonal entries are 1.

Length-4 multiplets require dressing operators with $\tilde{k}$ (for $1 \leq \tilde{k} \leq d-1$) single powers of H diagonal entries in the positions $d+1, \ldots, 2d$.

Length-5 (length-6) multiplets require a dressing operator S with at least one H^2 second power diagonal entry in the position $1, \ldots, d$ (and, respectively, $d+1, \ldots, 2d$).

Length-7 and length-8 multiplets require dressing operators with at least a third power, H^3, diagonal entry and so on.

We are now in the position to compute the length $l \geq 4$ irreducible representations of the oxidized supersymmetries. Let us illustrate at first an $N = 9$ example. An $N = 9$ length-3 irrep with 15 auxiliary fields (i.e. $(1, 16, 15)$) is such that the original $(16, 16)$ upper-right block $\mathcal{B}$ is mapped into a new block, $\mathcal{B} \mapsto \mathcal{B}'$, by multiplying 15 columns by H, while leaving the remaining column unchanged. The lengthening $3 \mapsto 4$, obtained by leaving unchanged the number of fields, 15, in the third position, produces a block-mapping $\mathcal{B}' \mapsto \mathcal{B}''$, where the new block is obtained from $\mathcal{B}'$ by multiplying a certain number of rows by $\frac{1}{H}$, while the remaining ones are left unchanged. The condition that no $\frac{1}{H}$ poles appear in $\mathcal{B}''$ implies that, at most, seven rows can be picked up. They have to be chosen among the ones corresponding to the zeroes of the single, unchanged, column of $\mathcal{B}'$. It turns out that $N = 9$ admits seven inequivalent length-4 irreps of the type $(1, 16-k, 15, k)$, for $k = 1, 2, \ldots, 7$.

The same strategy can be applied starting from $(2, 16, 14)$, $(3, 16, 13)$ and so on. At the end we produce the complete list of length-4 irreps of $N = 9$ (listed in Appendix **B**). This procedure straightforwardly works for computing length-4 irreps of any oxidized value of N, once the corresponding block-symbols are known.

For what concerns $l > 4$, let us illustrate the $N = 10$ length-5 example, since 10 is the least value of an extended supersymmetry admitting irreps with $l > 4$. Let us check, at first, whether we can produce a single auxiliary field in the fifth position. This amounts to multiply by $\frac{1}{H^2}$ a single row of the original $(32, 32)$ bottom-left block. Since all its entries, see (2.5), are already multiplied by H, this implies that the new bottom-left block admits a single $\frac{1}{H}$ pole in correspondence with the non-vanishing entries of the transformed row, while it is regular anywhere else. We get on the transformed row ten poles. In order to kill them we need to multiply (at least) the 10 corresponding columns of the bottom-left block by H. This multiplication corresponds to the transformation which maps (at least) 10 fields from the second to the fourth position. This transformation acts on the upper-right block by multiplying the corresponding rows by $\frac{1}{H}$. In its turn, these extra-poles have to be cancelled by multiplying a convenient number of columns by H (in correspondence with the transformation mapping fields from the first to the third position). The extra $\frac{1}{H}$ poles produced by this new compensating transformation on the corresponding rows of the bottom-left block do not produce any further singularity, due to the presence of the overall H factor mentioned above.

The same procedure can be later applied to verify whether there is enough room to have two, three or more fields in the fifth position.

Length $l \geq 6$ irreps can be analyzed along the same lines. The complete result for the $N = 10$ irreps is furnished in Appendix **B**.

For what concerns the reduced extended supersymmetries, the computation of their irreps can be carried on just like the oxidized supersymmetries, but taking into account that their block-symbols admit extra holes. We concentrate on $N = 8$ reductions. The eight gamma matrices generating $N = 8$ under the (2.7) correspondence are all on equal footing. We can single out any one of them (let's say the one with a diagonal upper-right block) in order that the remaining ones generate $N = 7$. The diagonal holes in the $N = 7$ block-symbol imply that, just like the first $N = 9$ example discussed above, we can lengthen the $N = 8$ $(1, 8, 7)$ irrep into an $N = 7$ $(1, 7, 7, 1)$ irrep. The analysis of the $N = 5^{(*)}, 6$ (and $N = 3^{(**)}$ derived from $N = 4$) cases is done in the same way.

Let us now make some necessary remarks on the irreducible representations. Two types of dualities act on them. We have at first the *fermion* $\Leftrightarrow$ *boson* duality, obtained by exchanging, via the (4.2) dressing, the statistics of the component fields in the multiplet. A second type of duality can be referred to as the *high* $\Leftrightarrow$ *low* spin duality. This new duality involves the mapping of a $(d_1, d_2, \ldots, d_l)$ irreducible multiplet into its irreducible dual multiplet

$$(d_1, d_2, \ldots, d_l) \quad \Leftrightarrow \quad (d_l, d_{l-1}, \ldots, d_1) \tag{4.3}$$

obtained by turning the highest-spin fields into the lowest spin fields. Therefore this duality relates two opposite statistics multiplets if l is even and two multiplets with the same statistics if l is odd.

Let us denote with $({}^1x_{j_1}; {}^2x_{j_2}; \ldots; {}^lx_{j_l})$ the set of fields entering $(d_1, d_2, \ldots, d_l)$ (here $j_i = 1, \ldots, d_i$). The dual irreducible $(d_l, d_{l-1}, \ldots, d_1)$ multiplet can be realized with the fields $({}^lx_{j_l}; {}^{l-1}\dot{x}_{j_{l-1}}; \ldots; {}^1x_{j_l}{}^{(l-1)})$, where $x^{(k)}$ here denotes the application of the time derivative k-times. Applying the same transformation on the latter multiplet we obtain a new multiplet, $({}^1x_{j_1}{}^{(l-1)}; {}^2x_{j_2}^{(l-1)}; \ldots; {}^lx_{j_l}^{(l-1)})$, whose supersymmetry transformations are nevertheless the same as the original ones. As a corollary, the class of the irreducible representations is closed under the (4.3) *high* $\Leftrightarrow$ *low* spin duality.

The *high* $\Leftrightarrow$ *low spin* duality (4.3) concides with the *fermion* $\Leftrightarrow$ *boson* (4.2) duality only when applied to self-dual (under (4.3)) multiplets of even length. It is a distinct duality transformation in the remaining cases.

For what concerns the total number $\overline{\kappa}$ of inequivalent irreps of the N-extended supersymmetry, it is given by the sum of the $\overline{\kappa}_l$ inequivalent irreps of length-l, namely,

$$\overline{\kappa} = \sum_{l=2}^{L} \overline{\kappa}_l \tag{4.4}$$

where L is the maximal length for an N-extended supersymmetry irrep.

$\overline{k}$ is the counting of inequivalent irreps irrispectively of the overall statistics of the multiplets. A factor 2 can be introduced if we want to discriminate the statistics of the multiplets (bosonic or fermionic). In this case the number of inequivalent irreps is κ, with

$$\kappa = 2\overline{\kappa} \tag{4.5}$$

Let us present now a series of results concerning the irreducible irreps (a more detailed list can be found in the Appendices **A** and **B**).

Up to $N \leq 8$, length-4 irreps are present only for reduced supersymmetries. The complete list of length-4 irreps up to $N = 8$ is given by

$N = 1$	NO
$N = 2$	NO
$N = 3^{(*)}$	NO
$N = 3^{(**)}$	$(1,3,3,1)$
$N = 4$	NO
$N = 5^{(*)}$	$(1,5,7,3),\ (3,7,5,1),\ (1,6,7,2),\ (2,7,6,1),\ (2,6,6,2),\ (1,7,7,1)$
$N = 5^{(**)}$	NO
$N = 6$	$(1,6,7,2),\ (2,7,6,1),\ (2,6,6,2),\ (1,7,7,1)$
$N = 7$	$(1,7,7,1)$
$N = 8$	NO

(4.6)

Since there are no length-l irreps with $l \geq 5$ for $N \leq 9$, the above list, together with the already known length-2 and length-3 irreps, provides the complete classification of inequivalent irreps for $N \leq 8$.

Please notice that the length-4 irrep of $N = 3^{(**)}$, $(1,3,3,1)$, is self-dual under the (4.3) *high* $\Leftrightarrow$ *low* spin duality, while two of the inequivalent length-4 $N = 5^{(*)}$ irreps are self-dual, $(2,6,6,2)$ and $(1,7,7,1)$. The remaining ones are pair-wise dually related ($(1,5,7,3) \Leftrightarrow (3,7,5,1)$ and $(1,6,7,2) \Leftrightarrow (2,7,6,1)$).

Due to the reduction chain from the $N = 8$ oxidized supersimmetry

$$N = 8 \rightarrow N = 7 \rightarrow N = 6 \rightarrow N = 5^{(*)} \tag{4.7}$$

it turns out that the $(1,7,7,1)$ irrep of $N = 5^{(*)}$ can be *oxidized* as an $N = 6$ and $N = 7$ irrep. The $(1,6,7,2) \Leftrightarrow (2,7,6,1)$ and $(2,6,6,2)$ multiplets, thought as $N = 5^{(*)}$ irreps, can be oxidized and promoted to be $N = 6$ irreps.

In the Appendix **B** the complete classification of inequivalent irreps for $N = 9, 10$ is presented. Therefore, we are able to produce here another table, expressing the maximal length L and the total number $\overline{\kappa}$ of inequivalent irreps for the N-extended supersymmetries with $N \leq 10$. We have

su.sies	L	$\overline{\kappa}_2 + \ldots + \overline{\kappa}_L = \overline{\kappa}$
$N = 1$	2	1
$N = 2$	3	$1 + 1 = 2$
$N = 3^{(*)}$	3	$1 + 3 = 4$
$N = 3^{(**)}$	4	$1 + 3 + 1 = 5$
$N = 4$	3	$1 + 3 = 4$
$N = 5^{(*)}$	4	$1 + 7 + 6 = 14$
$N = 5^{(**)}$	3	$1 + 7 = 8$
$N = 6$	4	$1 + 7 + 4 = 12$
$N = 7$	4	$1 + 7 + 1 = 9$
$N = 8$	3	$1 + 7 = 8$
$N = 9$	4	$1 + 15 + 28 = 44$
$N = 10$	5	$1 + 31 + 176 + 140 = 348$

(4.8)

We conclude this section pointing out that the procedure here outlined can be systematically carried on to fully classify inequivalent irreps for arbitrarily large values of N; the limitations are only due to the increasing of the required computational work.

5 On the adjoint representations

For each N the adjoint representation of the N-extended one-dimensional supersymmetry is given by 2^{N-1} bosonic and 2^{N-1} fermionic states spanned by the monomials

$$\prod_{i=1}^{N} {Q_i}^{\alpha_i},$$

where the α_i's take the values 0 and 1.

The state with $\alpha_i = 0$ for any i is a bosonic state of spin $s = 0$ and corresponds to the identity operator $\mathbf{1}$. The $\left(\begin{array}{c} N \\ k \end{array}\right)$ states given by $Q_{i_1} \cdot Q_{i_2} \cdot \ldots Q_{i_k} \cdot \mathbf{1}$ (all i_j's are different) belong to spin $s = \frac{k}{2}$.

For $N = 1, 2, 3$, the adjoint representation is irreducible. It can be identified with the bosonic irreducible multiplets $(1, 1)$, $(1, 2, 1)$ and $(1, 3, 3, 1)$, respectively. The last multiplet is the unique length-4 multiplet of $N = 3$. It belongs, as discussed in Section **3**, to the class of quaternionic ($N = 3^{(**)}$) irreps.

Starting from $N \geq 4$, the adjoint representation is no longer irreducible. The $N = 4$ adjoint representation corresponds to the bosonic multiplet $(1, 4, 6, 4, 1)$, which admits the following decomposition into its irreducible components

$$(1, 4, 6, 4, 1)_{s=0} \quad \equiv \quad (1, 4, 3)_{s=0} + (3, 4, 1)_{s=1} \tag{5.1}$$

The explicit decomposition of the $N = 4$ adjoint representation into its irrep constituents (also discussed in Appendix **C**, see the *iia*) case) requires the knowledge of the (5.2) constraints below.

All $N = 4$ irreps satisfy

$$\begin{array}{rcl} Q_1 Q_2 &=& Q_3 Q_4 \Gamma^5, \\ Q_2 Q_3 &=& Q_1 Q_4 \Gamma^5, \\ Q_3 Q_1 &=& Q_2 Q_4 \Gamma^5, \end{array} \tag{5.2}$$

where $\Gamma^5 = \left(\begin{array}{cc} \mathbf{1}_4 & 0 \\ 0 & -\mathbf{1}_4 \end{array}\right)$ plays the role of the fermion number operator

$$\begin{array}{rcl} \Gamma^5(boson) &=& +(boson), \\ \Gamma^5(fermion) &=& -(fermion) \end{array} \tag{5.3}$$

for any bosonic (fermionic) state in the multiplet.

The set of (5.2) equations can be easily verified on the length 2 $(4, 4)$ irrep. It holds also for any length 3 irrep, since in each case the dressing transformations (2.8) $Q_i \mapsto SQ_iS^{-1}$ discussed in Section **2** are realized by diagonal matrices S which commute with Γ^5.

6 Construction of invariants for potential and kinetic terms

The knowledge of all finite, linear multiplets of irreducible representations of the N-extended supersymmetries allows us to systematically construct invariants for any N-extended supersymmetry. In this section we point out how this can be done. We recall that those component fields in any given multiplet corresponding to the highest spin can be regarded as auxiliary fields. They transform, under each one of the N supersymmetries, as time-derivatives. They can therefore be picked up us lagrangian terms which, inside an action, provide a manifest invariant for the N-extended supersymmetry.

The linearity of the supersymmetry transformations implies that the tensored multiplets, obtained as bilinear, trilinear or in general k-linear products[5] of the original irreps, keep the same structure as the original irreps. In particular they can be decomposed into their irreducible component multiplets, which provide the associated multilinear invariants. Specific kinds of such invariants include the kinetic terms, as well as the potential terms (their multilinear invariants are functions of the original component fields alone and do not involve their time derivatives). For illustrative purposes we present here the construction of the invariants in some selected examples. The procedure here outlined can be carried out sistematically, without any conceptual problem, just increasing of computational work, for arbitrarily large values of N. The method here discussed allows to construct manifest invariants of the N-extended supersymmetries, *without* introducing a superspace and a superfield formalism, recovering the same results of the superspace construction for small values of N, but allowing to extend it when no superspace formalism is available (for $N > 8$, see [14]).

Let us discuss at first the multi-linear invariants associated with the two inequivalent irreps of the $N = 2$ supersymmetry (see Appendix **C**). k products of the same $(1,2,1)_{s=0} = (x;\psi_1,\psi_2;g)$ irrep produce the spin $s = 0$ k-linear $(1,2,1)$ irrep given by

$$(x_k;\psi_{1k},\psi_{2k};g_k) \;\; = \;\; (x^k;k\psi_1x^{k-1},k\psi_2x^{k-1};kgx^{k-1}-k(k-1)x^{k-2}\psi_1\psi_2) \quad (6.1)$$

A single $N = 2$ invariant is produced at spin $s = 1$. It is given by

$$I \;\; = \;\; \int dt\left(\sum_{k=1}^{\infty}c_kg_k\right) \quad (6.2)$$

with arbitrary constants c_k's. This $N = 2$ invariant corresponds to the most general self-interacting potential of a single $(1,2,1)$ real superfield.

Multilinear invariants for the spin $s = 0$ $(2,2)_{s=0} \equiv (x_1,x_2;\psi_1,\psi_2)$ chiral superfield in our framework can be constructed as follows. The unique bilinear invariant at $s = 1$

[5] We remark that, due to the renormalizability condition, the supersymmetric actions of the dimensional reduction to $D = 1$ of the renormalizable four-dimensional supersymmetric field theories admit terms which are at most quartic in the fields. As for the total number of extended supersymmetries of the one-dimensional reduced models, it is four times the number of supersymmetries in $D = 4$. Therefore, the maximal $N = 4$ four-dimensional SuperYang-Mills theories are reduced to $N = 16$ supersymmetric $1D$ systems, while the $1D$ dimensional reduction of the (non-renormalizable) maximal $N = 8$ supergravity gives an $N = 32$ system.

for the $(2,2)_{s=0}$ irrep is obtained (see Appendix **C**, case *ic*)) from the spin $s=1$ term in the $(1,2,1)_{\parallel s=0}$ entering the r.h.s. decomposition of $(2,2)_{s=0}\times(2,2)_{s=0}$ after identifying the left and right tensored multiplets. We get for the corresponding auxiliary field

$$\hat{g} \;=\; 2x_1\dot{x}_2 - 2x_2\dot{x}_1 + 4\psi_2\psi_1 \tag{6.3}$$

Two spin $s=1$ trilinear invariants are obtained in two equivalent ways, either from the $(1,2,1)^{(3)}_{\parallel s=0}$ and $(1,2,1)^{(3)}_{\perp s=0}$ multiplets in the irrep decompositions of the tensor product $(2,2)_{s=0}\times{(2,2)_{\parallel s=0}}^{(2)}$, or from the $2\times{(1,2,1)_{s=0}}^{a,b(3)}$ multiplets in the irrep decompositions of the $(2,2)_{s=0}\times{(1,2,1)_{\parallel s=0}}^{(2)}$ product (here $(\ldots)^{(k)}$ specifies an irrep which is k-linear w.r.t. the original fields and ${(2,2)_{\parallel s=0}}^{(2)}$, ${(1,2,1)_{\parallel s=0}}^{(2)}$ denote the corresponding irreps in the $(2,2)_{s=0}\times(2,2)_{s=0}$ decomposition).

In both cases we get the auxiliary fields

$$\begin{aligned} g_I &= -8x_1\psi_1\psi_2 - 2\dot{x}_1x_1x_2 + \dot{x}_2(3{x_1}^2+{x_2}^2) \\ g_{II} &= -8x_2\psi_1\psi_2 + 2\dot{x}_2x_1x_2 - \dot{x}_1(3{x_2}^2+{x_1}^2) \end{aligned} \tag{6.4}$$

Please notice that g_I, g_{II} are mutually recovered by exchanging $x_1 \leftrightarrow -x_2$, $\psi_1\leftrightarrow\psi_2$.

Spin $s=1$ k-linear self-invariants of the $N=2$ chiral superfield can be recursively constructed by applying the same scheme. Please notice that such invariants do not fall into the class of potential invariants since they involve time-derivatives of the original fields.

It is convenient to illustrate now the next simplest example of invariant, given by the self-interaction of the spin $s=0$ $(1,4,3)$ irrep of the $N=4$ extended supersymmetry. According to the *iia*) case of Appendix **C** the tensor product of two $(1,4,3)$ irreps (which, for our purposes here, are identified) gives rise to an adjoint representation of $N=4$. It contains a single auxiliary field at $s=2$ which generates the $N=4$ invariant. In our case the auxiliary field is given by

$$K \;=\; -\ddot{x}x - \psi_1\dot{\psi}_1 - \psi_2\dot{\psi}_2 - \psi_3\dot{\psi}_3 - \psi_4\dot{\psi}_4 + {g_1}^2 + {g_2}^2 + {g_3}^2 \tag{6.5}$$

and coincides with the kinetic term for the $(1,4,3)$ irrep.

An alternative construction is however available, due to the fact that the adjoint $N=4$ representation is reducible and can be decomposed into its irreps according to (5.1). The eight bilinear terms entering the $(3,4,1)_{s=1}$ multiplet in the irrep decomposition of the $N=4$ adjoint representation can be consistently set all equal to zero. The surviving bilinear terms enter, see(5.1), the $(1,4,3)_{s=0}$ irrep admitting three auxiliary fields (and, therefore, three associated invariants) at spin $s=1$. In terms of *bilinearly constrained* $(1,4,3)$ component fields we obtain three invariants, associated with the three auxiliary fields a_1, a_2, a_3 given by

$$\begin{aligned} a_1 &= \psi_2\psi_4 - g_2x \\ a_2 &= \psi_2\psi_3 + g_3x \\ a_3 &= \psi_2\psi_1 - g_1x \end{aligned} \tag{6.6}$$

together with eight bilinear constraints given by

$$
\begin{aligned}
C_1 &= \psi_4\psi_2 + \psi_3\psi_1 = 0\\
C_2 &= \psi_4\psi_3 + \psi_1\psi_2 = 0\\
C_3 &= \psi_4\psi_1 + \psi_2\psi_3 = 0\\
C_4 &= \dot{\psi}_1 x - g_2\psi_3 - g_3\psi_4 - g_1\psi_2 = 0\\
C_5 &= \dot{\psi}_3 x + g_2\psi_1 - g_1\psi_4 + g_3\psi_2 = 0\\
C_6 &= \dot{\psi}_4 x + g_3\psi_1 + g_1\psi_3 - g_2\psi_2 = 0\\
C_7 &= \dot{\psi}_2 x - g_1\psi_1 + g_3\psi_3 - g_2\psi_4 = 0\\
C_8 &= -\ddot{x}x - \psi_1\dot{\psi}_1 - \psi_2\dot{\psi}_2 - \psi_3\dot{\psi}_3 - \psi_4\dot{\psi}_4 + {g_1}^2 + {g_2}^2 + {g_3}^2 = 0
\end{aligned}
\tag{6.7}
$$

This procedure can be straightforwardly iterated to produce three spin $s = 1$ multilinear invariants and eight multilinear constraints.

We can discuss along the same lines the multilinear invariants for the self-interacting $N = 4$ spin $s = 0$ $(2,4,2)_{s=0} \equiv (x_1, x_2; \psi_1, \psi_2, \psi_3, \psi_4; g_1, g_2)$ irrep. Due to the *iic*) decomposition of Appendix **C** we obtain, in terms of *unconstrained* component fields, two bilinear invariants at spin $s = 1$, associated with the auxiliary fields

$$
\begin{aligned}
\tilde{g}_I &= g_1x_1 - g_2x_2 - \psi_2\psi_4 + \psi_1\psi_3,\\
\tilde{g}_{II} &= g_2x_1 + g_1x_2 + \psi_1\psi_4 + \psi_2\psi_3
\end{aligned}
\tag{6.8}
$$

plus the kinetic invariant (with $\tilde{K}$ as kinetic density) at spin $s = 2$ arising from the adjoint representation

$$
\tilde{K} = \dot{x}_1\dot{x}_1 + \dot{x}_2\dot{x}_2 - \psi_1\dot{\psi}_1 - \psi_2\dot{\psi}_2 - \psi_3\dot{\psi}_3 - \psi_4\dot{\psi}_4 + {g_1}^2 + {g_2}^2 \tag{6.9}
$$

As in the previous case, we can consistently introduce eight bilinear constraints in relation with the decomposition into irreps of the $N = 4$ adjoint representation. As a result, three extra bilinear invariants for the bilinearly *constrained* fields are obtained at spin $s = 1$. The three invariants are associated with the auxiliary fields

$$
\begin{aligned}
\tilde{a}_1 &= \psi_2\psi_4 - g_1x_1 - g_2x_2 + \psi_1\psi_3\\
\tilde{a}_2 &= \psi_2\psi_3 + +g_2x_1 - \psi_1\psi_4 - g_1x_2\\
\tilde{a}_3 &= 2\psi_2\psi_1 - \dot{x}_2x_1 + \dot{x}_1x_2
\end{aligned}
\tag{6.10}
$$

while the eight bilinear constraints are explicitly given by

$$
\begin{aligned}
\widetilde{C}_1 &= \psi_2\psi_4 + \psi_1\psi_3 = 0\\
\widetilde{C}_2 &= \psi_4\psi_3 + \psi_1\psi_2 = 0\\
\widetilde{C}_3 &= \psi_1\psi_4 + \psi_3\psi_2 = 0\\
\widetilde{C}_4 &= x_1\dot{\psi}_1 - \dot{x}_1\psi_1 + x_2\dot{\psi}_2 - \dot{x}_2\psi_2 - 2g_1\psi_3 - 2g_2\psi_4 = 0\\
\widetilde{C}_5 &= x_1\dot{\psi}_3 - \dot{x}_1\psi_3 + x_2\dot{\psi}_4 - \dot{x}_2\psi_4 + 2g_1\psi_1 + 2g_2\psi_2 = 0\\
\widetilde{C}_6 &= x_1\dot{\psi}_4 - \dot{x}_1\psi_4 - x_2\dot{\psi}_3 + \dot{x}_2\psi_3 - 2g_1\psi_2 + 2g_2\psi_1 = 0\\
\widetilde{C}_7 &= x_1\dot{\psi}_2 - \dot{x}_1\psi_2 - x_2\dot{\psi}_1 + \dot{x}_2\psi_1 + 2g_1\psi_4 - 2g_2\psi_3 = 0\\
\widetilde{C}_8 &= \dot{x}_1^2 + \dot{x}_2^2 - \ddot{x}_1x_1 - \ddot{x}_2x_2 - 2\psi_1\dot{\psi}_1 - 2\psi_2\dot{\psi}_2 - 2\psi_3\dot{\psi}_3 - 2\psi_4\dot{\psi}_4 + 2{g_1}^2 + 2{g_2}^2 = 0
\end{aligned}
$$

(6.11)

Just like in the self-interacting $(1,4,3)$ case, the last bilinear constraint coincides with the kinetic density.

Summarizing, either we work with unconstrained fields and obtain two spin $s=1$ invariants plus an invariant kinetic term, or we work with constrained fields, obtaining $5=2+3$ invariants at spin $s=1$ associated with 8 bilinear constraints.

Let us discuss now a general construction of the (constant) invariant kinetic term for arbitrary N (the cases previously discussed were specific of the $N=4$ case, since "accidentally" the $N=4$ adjoint representation admits a spin $s=2$ auxiliary field).

Let us consider a generic length $l\leq 4$ spin $s=0$ irrep of the N-extended supersymmetry, given by (see Appendix **A**) $(d-p,d-q,p,q)\equiv(x_i;\psi_j;g_k;\omega_l)$, where the x's and g's are bosonic spin $s=0$ and respectively $s=1$ component fields, while the ψ's and ω's are fermionic spin $s=\frac{1}{2}$ (respectively $s=\frac{3}{2}$) component fields (for length 2 and 3 irreps there are no ω's fields, namely $q=0$).

The kinetic density has dimension 2 (we recall, see the discussion in Section **2**, that we use, interchangeably, the words "spin" and "dimension"). It can be symbolically written, dropping the field indices, as $\dot{x}^2+\psi\dot{\psi}+g^2+\omega\psi$. In order to produce a spin $s=2$ auxiliary field which can be interpreted as a kinetic density we proceed as follows. At first we transform the $(x_i;\psi_j;g_k;\omega_l)_{s=0}$ irrep into

i) a bosonic spin $s=1$ length-2 irrep $(d,d)_{B,s=1}$, through the mapping $(x_i;\psi_j;g_k;\omega_l)\mapsto(\dot{x}_i,g_k;\dot{\psi}_j,\omega_l)\equiv(d,d)_{B,s=1}$ and

ii) a fermionic spin $s=\frac{1}{2}$ length-3 irrep $(d-q,d,q)_{F,s=\frac{1}{2}}$, through the mapping $(x_i;\psi_j;g_k;\omega_l)\mapsto(\psi_j;\dot{x}_i,g_k;\omega_l)\equiv(d-q,d,q)_{F,s=\frac{1}{2}}$.

Next, we consider the tensor product $(d,d)_{B,s=0}\times(d-q,d,q)_{F,s=\frac{1}{2}}$ and look whether, in its irrep decomposition, a leading term of the form $(d,d)_{s=\frac{3}{2}}$ appears, namely whether we get

$$(d,d)_{B,s=0}\times(d-q,d,q)_{F,s=\frac{1}{2}} \equiv (d,d)_{s=\frac{3}{2}}+\ldots \tag{6.12}$$

In this case, the auxiliary fields entering the $(d,d)_{s=\frac{3}{2}}$ irrep on the r.h.s. can be associated with the (invariant) kinetic density.

Let us verify how this construction works by computing explicitly the invariant kinetic term for the chiral spin $s=0$ $N=2$ length-2 irrep $(x_1,x_2;\psi_1,\psi_2)$ (in this case $p=q=0$). The needed formulae can be directly read from the *ic*) tensor products of (bosonic) multiplets of Appendix **C** (we use the trick of introducing an ϵ Clifford parameter to change the statistics of the $(d,d)_{F,s=\frac{3}{2}}$ fermionic multiplet; this Clifford parameter will be reabsorbed at the end of the computation). An $N=2$, $(2,2)_{s=\frac{3}{2}}\equiv(\tau_1,\tau_2;w_1,w_2)$ irrep appears on the r.h.s. Its component fields are given by

$$\begin{aligned}\tau_1 &= \dot{x}_1\psi_1-\dot{x}_2\psi_2\\ \tau_2 &= \dot{x}_1\psi_2+\dot{x}_2\psi_1\\ w_1 &= \dot{\psi}_1\psi_1+\dot{x}_1\dot{x}_1+\dot{\psi}_2\psi_2+\dot{x}_2\dot{x}_2\\ w_2 &= \dot{\psi}_2\psi_1+\psi_2\dot{\psi}_1\end{aligned} \tag{6.13}$$

Please notice that w_2 is a total derivative and therefore does not produce any $N = 2$ invariant. On the other hand, w_1 is the required kinetic density, whose time integration provides an $N = 2$ invariant action.

It should be remarked that the arising of the $(d, d)_{s=\frac{3}{2}}$ term in the r.h.s. of the irrep decomposition of the $(d, d)_{B,s=0} \times (d - q, d, q)_{F,s=\frac{1}{2}}$ tensor product is not guaranteed and has to be checked case by case. In particular there is no $N = 3$ invariant kinetic term associated with the $N = 3$ length-4 irrep $(1, 3, 3, 1)$ that we discussed at length in the previous two sections.

So far we have focused ourselves on *constant* kinetic terms. σ-model kinetic terms are of more general type (see e.g. [24]), allowing field-dependent metric tensors. An N-extended supersymmetric σ-model invariant action has the form

$S = \int dt\, (g^{ij}(x_k)\dot{x}_i\dot{x}_j + \ldots)$, where the x_i's are spin 0 component fields and the dots denote the contribution from component fields of spin $s \geq \frac{1}{2}$. Invariant sigma-models actions can be easily accommodated in our framework. For illustrative purposes we explicitly discuss the $N = 2$ case realized in terms of the previously introduced spin $s = 0$ length-2 chiral irrep $(x_1, x_2; \psi_1, \psi_2)$.

By tensoring k times the original $(2, 2)_{s=0}$ multiplet we get, from the *ic*) case of Appendix **C**, a k-linear ${(2, 2)_{s=0}}^{(k)}$ multiplet with component fields $({x_1}^{(k)}, {x_2}^{(k)}; {\psi_1}^{(k)}, {\psi_2}^{(k)})$ given by

$$
\begin{aligned}
{x_1}^{(k)} &= x_1{x_1}^{(k-1)} - x_2{x_2}^{(k-1)}\\
{x_2}^{(k)} &= x_1{x_2}^{(k-1)} + x_2{x_1}^{(k-1)}\\
{\psi_1}^{(k)} &= k\psi_1{x_1}^{(k-1)} + k\psi_2{x_2}^{(k-1)}\\
{\psi_2}^{(k)} &= k\psi_2{x_1}^{(k-1)} - k\psi_1{x_2}^{(k-1)}
\end{aligned}
\tag{6.14}
$$

A k-linear (in the original coordinates) σ-model type of term is recovered from the ${(2, 2)_{s=\frac{3}{2}}}^{(k)}$ irrep entering the

$$
(2, 2)_{s=\frac{3}{2}} \times {(2, 2)_{s=0}}^{(k)} = {(2, 2)_{s=\frac{3}{2}}}^{(k)} + \ldots \tag{6.15}
$$

decomposition.

Up to bilinear products of the original coordinates we get, e.g., the following σ-model type of $N = 2$ invariant kinetic term

$$
\begin{aligned}
K \;=\;& (\dot{x}_1^2 + \dot{x}_2^2 - \psi_1\dot{\psi}_1 - \psi_2\dot{\psi}_2)\left(C_1(1 + \alpha_1 x_1 + \alpha_2({x_1}^2 - {x_2}^2)) - C_2(\alpha_1 x_2 + \alpha_2 x_1 x_2)\right) +\\
& \psi_1\psi_2\left(C_1(3\alpha_1\dot{x}_2 + 6\alpha_2(\dot{x}_1 x_2 + x_1\dot{x}_2)) + C_2(3\alpha_1\dot{x}_1 - 6\alpha_2(\dot{x}_1 x_1 - x_2\dot{x}_2))\right) + \ldots
\end{aligned}
\tag{6.16}
$$

where α_1, α_2 are arbitrary constants associated to the k-linear terms $k = 1, 2$ respectively, while C_1, C_2 are constants related to the two auxiliary fields entering the ${(2, 2)_{s=\frac{3}{2}}}^{(k)}$ irrep.

7 The fusion algebra of the supersymmetric vacua

The supersymmetry transformations of the component fields in a multiplet involve time derivatives. When tensoring, e.g., two irreps, the resulting representation can be decomposed into its irreducible constituents. This, in general, will produce infinite towers of irreps of increasing spin, in terms of bilinear products of the original component fields and their (higher-order) derivatives. However, a drastic simplification arises when we consistently set all time derivatives equal to zero. Since the hamiltonian operator acts, up to a factor, as a time derivative, this is tantamount to analyze the decomposition of the tensored products of irreps at the zero-energy level, i.e. for the unbroken supersymmetric vacua.

As an example, when tensoring two real $N = 2$ spin $s = 0$ irreps (we remember that each one is given by two spin $s = 0$ and two spin $s = \frac{1}{2}$ component fields), we obtain 4 spin $s = 0$, 8 spin $s = \frac{1}{2}$ and 4 spin $s = 1$ bilinear fields entering a reducible, vacuum (0-energy) supersymmetric multiplet.

By tensoring supersymmetric vacua irreps we always produce a finite number of fields. As a consequence, we are allowed to introduce a notion of fusion algebra for the supersymmetric vacua, which can be mimicked after the notion of fusion algebra for the rational conformal field theories (RCFT), see e.g. [25].

The fusion algebra can be defined as follows. Let us denote with $[i]$, $[j]$, $[k]$, ... the inequivalent irreps of an N-extended supersymmetry, where i, j, k takes the values $1, 2, \ldots, \overline{\kappa}$ or $1, 2, \ldots, \kappa = 2\overline{\kappa}$ (see table (4.8) and the discussion thereafter), according to whether we discriminate or not irreps according to their bosonic/fermionic character.

The tensoring of two zero-energy vacuum-state irreps can be symbolically written as

$$[i] \times [j] \quad = \quad {N_{ij}}^k [k] \tag{7.1}$$

where ${N_{ij}}^k$ are non-negative integers specifying the decomposition of the tensored-products irreps into its irreducible constituents. The ${N_{ij}}^k$ integers satisfy a fusion algebra with the following properties

1) Constraint on the total number of component fields,

$$\forall\ i, j \quad \sum_k {N_{ij}}^k \quad = \quad 2d \tag{7.2}$$

where d (see (2.2)) is the number of bosonic (fermionic) fields in the given irreps. Please notice that, for a fixed N-extended supersymmetry, d is the same for any irrep.

2) The symmetry property

$${N_{ij}}^k \quad = \quad {N_{ji}}^k \tag{7.3}$$

3) The associativity condition. This property can be expressed on the ${N_{ij}}^k$ integers as follows. Let us set ${(N_i)_j}^k =_{def} {N_{ij}}^k$, then the relation

$$[i] \times ([j] \times [k]) \quad = \quad ([i] \times [j]) \times [k] \tag{7.4}$$

implies that the r.h.s. ${N_{ij}}^r[r] \times [k] = {N_{ij}}^r {N_{rk}}^t[t] = {(N_i N_k)_j}^t[t]$ coincides with the l.h.s. $[i] \times {N_{jk}}^r[r] = {N_{jk}}^r {N_{ir}}^t[t] = {(N_k N_i)_j}^t[t]$. Namely, the associativity condition (7.4) implies the commutativity of the fusion matrices

$$[N_i, N_k] = 0 \tag{7.5}$$

The notion of fusion algebra of the supersymmetric vacua can be usefully applied even when tensoring fundamental irreps that do not satisfy the vacuum condition. For instance, the decomposition into irreps of a leading tensored multiplet (see appendix **C** for a discussion) can be directly read from the associated vacuum fusion algebra.

It is worth mentioning that the above properties of the ${N_{ij}}^k$ fusion matrices can, in some cases, help determining the decomposition into irreps without explicitly computing them. The simplest such kind of application is discussed in appendix **D**, where the fusion algebras (with and without discrimination of the bosonic/fermionic character) of the $N = 2$ supersymmetric vacua are explicitly reported.

8 Conclusions

In this paper we combined the results of [1], [18] and [19] in order to produce the classification of the linear, finite, irreducible representations of the N-extended one-dimensional supersymmetry algebras. The [18] algorithmic construction of Clifford algebras was used to compute, for each given value N, the irreducible representations. The complete classification has been here explicitly presented up to $N \leq 10$, while the length-4 irreps have been reported for the oxidized $N = 11^*, 12$ extended supersymmetries as well. We proved that the [19] results on oxidized Clifford algebras imply, as a corollary, that the $N = 3, 5 \quad mod \quad 8$ extended supersymmetries admit two classes of irreps, real and quaternionic, while the remaining values of N admit a unique class of irreps.

We further produced tensorings of irreps and showed how to use them in order to construct manifest multi-linear invariants of the N-extended supersymmetries with no need of introducing an associated superspace formalism. We pointed out that the invariants can be realized either in terms of *unconstrained* fields entering an irreducible multiplet or even, in specific cases, consistently *multilinearly constrained* fields.

We finally introduced the notion of the *fusion algebra* of the supersymmetric vacua and explicitly presented its simplest non-trivial (for $N = 2$) example in Appendix **D**.

In the Introduction we have already discussed several possible applications of the present results, for instance the classification of one-dimensional sigma-models, see e.g. [24], admitting N-extended supersymmetries. Another class of models which could be profitably investigated within this framework concerns the integrable systems in $1 + 1$ dimensions with extended number of supersymmetries, see [28] and [29].

It is worth mentioning another line of development which is outside the scope of the present paper and deserves further investigation. It concerns the systematic construction of the non-linear realizations of the extended supersymmetries. Some remarks about the relation between linear representations and non-linear realizations of the N-extended supersymmetry algebra can be found, e.g., in [14]. The most recent papers on non-linear Supersymmetric Mechanics are [30] (see also the references therein).

Let us finally conclude this paper by pointing out that it would be quite appealing to implement the algorithms here discussed in a computer algebra package. This would allow to explicitly produce for *arbitrarily* (in practice, the limit is due to the available computation time) large values of N the complete list of inequivalent irreps (the results explicitly reported here have been derived without any computer help). Such a package, once implemented, would allow to perform mathematical experiments which could lead to conjecture possible closed formulas satisfied by irreps for arbitrarily large values of N.

Appendix A

Supersymmetry irreps for $N \leq 8$

We focus here on the irreducible multiplets with length $l \leq 4$ since, applying the method described in Section **4**, we proved that the extended supersymmetries with $N \leq 9$ do not admit irreps with length higher than 4. We denote $l \leq 4$ irreps as $(d-p, d-q, p, q)$, where d is the total number of bosonic (fermionic) fields entering the multiplet. According to the results of Section **2**, the length 2 multiplets correspond to $p = q = 0$, while the length 3 multiplets are recovered from $q = 0$, $p \neq 0$ ($p = 1, 2, \ldots, d-1$).

As recalled in section **2**, irrep multiplets are either bosonic or fermionic according to the statistics of their leading component fields (the fields with lowest spin, whose total number is given by $d-p$). For our purposes it is convenient to denote as x_i, ($i = 1, \ldots, d-p$) such leading component fields. Their spin is conventionally assigned to be s. The $d-q$ fields of spin $s+\frac{1}{2}$ are here denoted as ψ_j ($j = 1, \ldots, d-q$), while the $p > 0$ fields of spin $s+1$ are expressed as g_k ($k = 1, \ldots, p$). The q fields of spin $s+\frac{3}{2}$ (for $q > 0$) are in the following denoted as ω_l, $l = 1, \ldots, q$. Therefore $(d-p, d-q, p, q) \equiv (x_i; \psi_j; g_k; \omega_l)$.

The fields x_i, g_k are all bosonic (fermionic) and the fields ψ_j, ω_l are all fermionic (bosonic) if the associated multiplet is bosonic (fermionic). It should be noticed that the transformation properties of the fields entering the multiplet do not depend on the statistics (either bosonic or fermionic) of the multiplet. Therefore, in the following we do not need to specify whether the irreducible multiplets under consideration are bosonic or fermionic.

All irreps of the N-extended supersymmetry can be systematically computed (for any arbitrary value N) with the algorithmic construction presented in Sections **2-4**. For completeness, it is convenient to explicitly present in this appendix all irreps up to $N = 8$, furnishing a representative in each irreducible class. For $N = 3, 5$ two classes of inequivalent irreps, real and quaternionic, have to be presented in accordance with the results of Section **3**.

We get the following list of irreps

i) **The $N = 1$ irrep**
We have only one irrep, $(1,1) \equiv (x;\psi)$, with transformation property

$$Q_1(1,1) = (\psi;\dot{x}). \tag{A.1}$$

ii) **The $N = 2$ irreps**
There are two inequivalent irreps

$$\begin{array}{rcl} (2,2) & \equiv & (x_1,x_2;\psi_1,\psi_2), \\ (1,2,1) & \equiv & (x;\psi_1,\psi_2;g), \end{array} \tag{A.2}$$

whose respective supersymmetry transformations are given by

$$\begin{array}{rcl} Q_1(2,2) & = & (\psi_2,\psi_1;\dot{x}_2,\dot{x}_1) \\ Q_2(2,2) & = & (\psi_1,-\psi_2;\dot{x}_1,-\dot{x}_2) \end{array} \tag{A.3}$$

and

$$\begin{array}{rcl} Q_1(1,2,1) & = & (\psi_2;g,\dot{x};\dot{\psi}_1) \\ Q_2(1,2,1) & = & (\psi_1;\dot{x},-g;-\dot{\psi}_2) \end{array} \tag{A.4}$$

iii) **The real $N = 3^{(*)}$ irreps**
They are recovered from the $Cl(4,3)$ Clifford algebra (see Section **3**). This case admits 4 inequivalent irreps, labeled, as in the quaternionic $N = 4$ case, as

$$\begin{array}{rcl} (4,4) & \equiv & (x_1,x_2,x_3,x_4;\psi_1,\psi_2,\psi_3,\psi_4), \\ (3,4,1) & \equiv & (x_1,x_2,x_3;\psi_1,\psi_2,\psi_3,\psi_4;g), \\ (2,4,2) & \equiv & (x_1,x_2;\psi_1,\psi_2,\psi_3,\psi_4;g_1,g_2), \\ (1,4,3) & \equiv & (x;\psi_1,\psi_2,\psi_3,\psi_4;g_1,g_2,g_3). \end{array} \tag{A.5}$$

Their supersymmetry transformations are respectively given by

$$\begin{array}{rcl} Q_1(4,4) & = & (\psi_4,\psi_3,\psi_2,\psi_1;\dot{x}_4,\dot{x}_3,\dot{x}_2,\dot{x}_1) \\ Q_2(4,4) & = & (\psi_3,-\psi_4,\psi_1,-\psi_2;\dot{x}_3,-\dot{x}_4,\dot{x}_1,-\dot{x}_2) \\ Q_3(4,4) & = & (\psi_1,\psi_2,-\psi_3,-\psi_4;\dot{x}_1,\dot{x}_2,-\dot{x}_3,-\dot{x}_4) \end{array} \tag{A.6}$$

$$\begin{array}{rcl} Q_1(3,4,1) & = & (\psi_4,\psi_3,\psi_2;g,\dot{x}_3,\dot{x}_2,\dot{x}_1;\dot{\psi}_1) \\ Q_2(3,4,1) & = & (\psi_3,-\psi_4,\psi_1;\dot{x}_3,-g,\dot{x}_1,-\dot{x}_2;-\dot{\psi}_2) \\ Q_3(3,4,1) & = & (\psi_1,\psi_2,-\psi_3;\dot{x}_1,\dot{x}_2,-\dot{x}_3,-g;-\dot{\psi}_4) \end{array} \tag{A.7}$$

$$\begin{array}{rcl} Q_1(2,4,2) & = & (\psi_4,\psi_3;g_2,g_1,\dot{x}_2,\dot{x}_1;\dot{\psi}_2,\dot{\psi}_1) \\ Q_2(2,4,2) & = & (\psi_3,-\psi_4;g_1,-g_2,\dot{x}_1,-\dot{x}_2;\dot{\psi}_1,-\dot{\psi}_2) \\ Q_3(2,4,2) & = & (\psi_1,\psi_2;\dot{x}_1,\dot{x}_2,-g_1,-g_2;-\dot{\psi}_3,-\dot{\psi}_4) \end{array} \tag{A.8}$$

and

$$
\begin{array}{rcl}
Q_1(1,3,4) & = & (\psi_4; g_3, g_2, g_1, \dot{x}; \dot{\psi}_3, \dot{\psi}_2, \dot{\psi}_1) \\
Q_2(1,3,4) & = & (\psi_3; g_2, -g_3, \dot{x}, -g_1; -\dot{\psi}_4, \dot{\psi}_1, -\dot{\psi}_2) \\
Q_3(1,3,4) & = & (\psi_1; \dot{x}, g_1, -g_2, -g_3; \dot{\psi}_2, -\dot{\psi}_3, -\dot{\psi}_4)
\end{array} \quad \text{(A.9)}
$$

iv) **The quaternionic $N = 3^{(**)}$ irreps**

The length 2 and 3 irreps of the quaternionic $N = 3$ supersymmetry can be directly read from the transformations of the $N = 4$ irreps (since $N = 4$ is the *oxidized* supersymmetry of the quaternionic $N = 3$), by restricting the supersymmetry transformations to be given by Q_1, Q_2, Q_3.

An extra, length-4, irrep, isomorphic to the $N = 3$ adjoint representation, is given by

$$
(1,3,3,1) \equiv (x; \psi_1, \psi_2, \psi_3; g_1, g_2, g_3; \omega). \quad \text{(A.10)}
$$

Its supersymmetry transformations are

$$
\begin{array}{rcl}
Q_1(1,3,3,1) & = & (\psi_1; \dot{x}, g_3, -g_2; -\omega, -\dot{\psi}_3, \dot{\psi}_2; -\dot{g}_1) \\
Q_2(1,3,3,1) & = & (\psi_3; g_2, -g_1, \dot{x}; -\dot{\psi}_2, \dot{\psi}_1, -\omega; -\dot{g}_3) \\
Q_3(1,3,3,1) & = & (\psi_2; -g_3, \dot{x}, g_1; \dot{\psi}_3, -\omega, -\dot{\psi}_1; -\dot{g}_2)
\end{array} \quad \text{(A.11)}
$$

v) **The $N = 4$ irreps**

This case admits 4 inequivalent irreps,

$$
\begin{array}{rcl}
(4,4) & \equiv & (x_1, x_2, x_3, x_4; \psi_1, \psi_2, \psi_3, \psi_4), \\
(3,4,1) & \equiv & (x_1, x_2, x_3; \psi_1, \psi_2, \psi_3, \psi_4; g), \\
(2,4,2) & \equiv & (x_1, x_2; \psi_1, \psi_2, \psi_3, \psi_4; g_1, g_2), \\
(1,4,3) & \equiv & (x; \psi_1, \psi_2, \psi_3, \psi_4; g_1, g_2, g_3),
\end{array} \quad \text{(A.12)}
$$

whose supersymmetry transformations are respectively given by

$$
\begin{array}{rcl}
Q_1(4,4) & = & (\psi_2, -\psi_1, -\psi_4, \psi_3; -\dot{x}_2, \dot{x}_1, \dot{x}_4, -\dot{x}_3) \\
Q_2(4,4) & = & (\psi_4, -\psi_3, \psi_2, -\psi_1; -\dot{x}_4, \dot{x}_3, -\dot{x}_2, \dot{x}_1) \\
Q_3(4,4) & = & (\psi_3, \psi_4, -\psi_1, -\psi_2; -\dot{x}_3, -\dot{x}_4, \dot{x}_1, \dot{x}_2) \\
Q_4(4,4) & = & (\psi_1, \psi_2, \psi_3, \psi_4; \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)
\end{array} \quad \text{(A.13)}
$$

$$
\begin{array}{rcl}
Q_1(3,4,1) & = & (\psi_2, -\psi_1, -\psi_4; -\dot{x}_2, \dot{x}_1, g, -\dot{x}_3; \dot{\psi}_3) \\
Q_2(3,4,1) & = & (\psi_4, -\psi_3, \psi_2; -g, \dot{x}_3, -\dot{x}_2, \dot{x}_1; -\dot{\psi}_1) \\
Q_3(3,4,1) & = & (\psi_3, \psi_4, -\psi_1; -\dot{x}_3, -g, \dot{x}_1, \dot{x}_2; -\dot{\psi}_2) \\
Q_4(3,4,1) & = & (\psi_1, \psi_2, \psi_3; \dot{x}_1, \dot{x}_2, \dot{x}_3, g; \dot{\psi}_4)
\end{array} \quad \text{(A.14)}
$$

$$\begin{aligned}
Q_1(2,4,2) &= (\psi_2,-\psi_1;-\dot{x}_2,\dot{x}_1,g_2,-g_1;-\dot{\psi}_4,\dot{\psi}_3)\\
Q_2(2,4,2) &= (\psi_4,-\psi_3;-g_2,g_1,-\dot{x}_2,\dot{x}_1;\dot{\psi}_2,-\dot{\psi}_1)\\
Q_3(2,4,2) &= (\psi_3,\psi_4;-g_1,-g_2,\dot{x}_1,\dot{x}_2;-\dot{\psi}_1,-\dot{\psi}_2)\\
Q_4(2,4,2) &= (\psi_1,\psi_2;\dot{x}_1,\dot{x}_2,g_1,g_2;\dot{\psi}_3,\dot{\psi}_4)
\end{aligned} \tag{A.15}$$

$$\begin{aligned}
Q_1(1,4,3) &= (\psi_2;-g_1,\dot{x},g_3,-g_2;-\dot{\psi}_1,-\dot{\psi}_4,\dot{\psi}_3)\\
Q_2(1,4,3) &= (\psi_4;-g_3,g_2,-g_1,\dot{x};-\dot{\psi}_3,\dot{\psi}_2,-\dot{\psi}_1)\\
Q_3(1,4,3) &= (\psi_3;-g_2,-g_3,\dot{x},g_1;\dot{\psi}_4,-\dot{\psi}_1,-\dot{\psi}_2)\\
Q_4(1,4,3) &= (\psi_1;\dot{x},g_1,g_2,g_3;\dot{\psi}_2,\dot{\psi}_3,\dot{\psi}_4)
\end{aligned} \tag{A.16}$$

vi) **The real $N = 5^{(*)}$ irreps**

The length 2 and 3 irreps are obtained from the $N = 8$ irreps by restricting the supersymmetry transformations to be given by Q_i, for $i = 1,\ldots,5$.

This case admits two, dually related (see Section **4**), length-4 irreps which cannot be oxidized to $N = 6,7$ irreps and four extra irreps, three of them oxidized to $N = 6$, while the last irrep is oxidized to $N = 7$. These extra four irreps are presented in the following. The two length-4 irreps with maximal number of $N = 5^{(*)}$ supersymmetries are given by

$$\begin{aligned}
(1,5,7,3) &= (x;\psi_1,\ldots,\psi_5;g_1,\ldots,g_7;\omega_1,\omega_2,\omega_3),\\
(3,7,5,1) &= (x_1,x_2,x_3;\psi_1,\ldots,\psi_7;g_1,\ldots,g_5;\omega).
\end{aligned} \tag{A.17}$$

Their respective supersymmetry transformations are

$$\begin{aligned}
Q_1(1,5,7,3) &= (\psi_4;g_5,-g_2,-g_3,\dot{x},g_1;\dot{\psi}_5,-\dot{\psi}_2,-\dot{\psi}_3,\omega_3,\dot{\psi}_1,-\omega_1,-\omega_2;-\dot{g}_6,-\dot{g}_7,\dot{g}_4)\\
Q_2(1,5,7,3) &= (\psi_1;\dot{x},-g_7,g_6,-g_5,g_4;-\omega_3,\omega_2,-\omega_1,\dot{\psi}_5,-\dot{\psi}_4,\dot{\psi}_3,-\dot{\psi}_2;-\dot{g}_3,\dot{g}_2,-\dot{g}_1)\\
Q_3(1,5,7,3) &= (\psi_2;g_7,\dot{x},-g_1,g_2,-g_3;-\dot{\psi}_3,\dot{\psi}_4,-\dot{\psi}_5,-\omega_1,\omega_2,-\omega_3,\dot{\psi}_1;-\dot{g}_4,\dot{g}_5,-\dot{g}_6)\\
Q_4(1,5,7,3) &= (\psi_3;-g_6,g_1,\dot{x},g_3,g_2;\dot{\psi}_2,\dot{\psi}_5,\dot{\psi}_4,-\omega_2,-\omega_1,-\dot{\psi}_1,-\omega_3;-\dot{g}_5,-\dot{g}_4,-\dot{g}_7)\\
Q_5(1,5,7,3) &= (\psi_5;-g_4,g_3,-g_2,-g_1,\dot{x};-\dot{\psi}_4,-\dot{\psi}_3,\dot{\psi}_2,-\dot{\psi}_1,\omega_3,\omega_2,-\omega_1;-\dot{g}_7,\dot{g}_6,\dot{g}_5)
\end{aligned} \tag{A.18}$$

and

$$\begin{aligned}
Q_1(3,7,5,1) &= (\psi_6,\psi_7,-\psi_4;-g_5,g_2,g_3,-\dot{x}_3,-g_1,\dot{x}_1,\dot{x}_2;-\dot{\psi}_5,\dot{\psi}_2,\dot{\psi}_3,-\omega,-\dot{\psi}_1;-\dot{g}_4)\\
Q_2(3,7,5,1) &= (\psi_3,-\psi_2,\psi_1;\dot{x}_3,-\dot{x}_2,\dot{x}_1,-g_5,g_4,-g_3,g_2;-\omega,\dot{\psi}_7,-\dot{\psi}_6,\dot{\psi}_5,-\dot{\psi}_4;-\dot{g}_1)\\
Q_3(3,7,5,1) &= (\psi_4,-\psi_5,\psi_6;g_3,-g_4,g_5,\dot{x}_1,-\dot{x}_2,\dot{x}_3,-g_1;-\dot{\psi}_7,-\omega,\dot{\psi}_1,-\dot{\psi}_2,\dot{\psi}_3;-\dot{g}_2)\\
Q_4(3,7,5,1) &= (\psi_5,\psi_4,\psi_7;-g_2,-g_5,-g_4,\dot{x}_2,\dot{x}_1,g_1,\dot{x}_3;\dot{\psi}_6,-\dot{\psi}_1,-\omega,-\dot{\psi}_3,-\dot{\psi}_2;-\dot{g}_3)\\
Q_5(3,7,5,1) &= (\psi_7,-\psi_6,-\psi_5;g_4,g_3,-g_2,g_1,-\dot{x}_3,-\dot{x}_2,\dot{x}_1;\dot{\psi}_4,-\dot{\psi}_3,\dot{\psi}_2,\dot{\psi}_1,-\omega;-\dot{g}_5)
\end{aligned} \tag{A.19}$$

vii) **The quaternionic $N = 5^{(**)}$ irreps.**

They are recovered from the $Cl(6,1)$ Clifford algebra (see Section **3**). This case admits 8 inequivalent irreps, labeled, as in the $N = 8$ case, as

$$\begin{array}{rcl} (8,8) & = & (x_1,\ldots x_8;\psi_1,\ldots,\psi_8),\\ (7,8,1) & = & (x_1,\ldots,x_7;\psi_1,\ldots,\psi_8;g),\\ (6,8,2) & = & (x_1,\ldots,x_6;\psi_1,\ldots,\psi_8;g_1,g_2),\\ (5,8,3) & = & (x_1,\ldots,x_5;\psi_1,\ldots,\psi_8;g_1,g_2,g_3),\\ (4,8,4) & = & (x_1,\ldots,x_4,;\psi_1,\ldots,\psi_8;g_1,\ldots,g_4),\\ (3,8,5) & = & (x_1,x_2,x_3;\psi_1,\ldots,\psi_8;g_1,\ldots,g_5),\\ (2,8,6) & = & (x_1,x_2;\psi_1,\ldots,\psi_8;g_1,\ldots,g_6),\\ (1,8,7) & = & (x;\psi_1,\ldots,\psi_8;g_1,\ldots,g_7), \end{array} \quad (A.20)$$

The length 2 $(8,8)$ multiplet admits the following supersymmetry transformations

$$\begin{array}{rcl} Q_1(8,8) & = & (\psi_6,-\psi_5,-\psi_8,\psi_7,-\psi_2,\psi_1,\psi_4,-\psi_3;\dot{x}_6,-\dot{x}_5,-\dot{x}_8,\dot{x}_7,-\dot{x}_2,\dot{x}_1,\dot{x}_4,-\dot{x}_3)\\ Q_2(8,8) & = & (\psi_8,-\psi_7,\psi_6,-\psi_5,-\psi_4,\psi_3,-\psi_2,\psi_1;\dot{x}_8,-\dot{x}_7,\dot{x}_6,-\dot{x}_5,-\dot{x}_4,\dot{x}_3,-\dot{x}_2,\dot{x}_1)\\ Q_3(8,8) & = & (\psi_7,\psi_8,-\psi_5,-\psi_6,-\psi_3,-\psi_4,\psi_1,\psi_2;\dot{x}_7,\dot{x}_8,-\dot{x}_5,-\dot{x}_6,-\dot{x}_3,-\dot{x}_4,\dot{x}_1,\dot{x}_2)\\ Q_4(8,8) & = & (\psi_5,\psi_6,\psi_7,\psi_8,\psi_1,\psi_2,\psi_3,\psi_4;\dot{x}_5,\dot{x}_6,\dot{x}_7,\dot{x}_8,\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4)\\ Q_5(8,8) & = & (\psi_1,\psi_2,\psi_3,\psi_4,-\psi_5,-\psi_6,-\psi_7,-\psi_8;\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4,-\dot{x}_5,-\dot{x}_6,-\dot{x}_7,-\dot{x}_8) \end{array}$$

$$(A.21)$$

The length 3 irreps are immediately read from the above transformations by setting, for any $p = 1,\ldots,d-1$ associated with the $(d-p,d,p)$ multiplet, $g_1 = \dot{x}_8$, $g_2 = \dot{x}_7$, ..., $g_p = \dot{x}_{9-p}$. To save space the length 3 irreps are not explicitly reported here.

viii) **The $N = 6$ irreps**

The length 2 and 3 irreps are obtained from the $N = 8$ irreps by restricting the supersymmetry transformations to be given by Q_i, for $i = 1,\ldots,6$.

This case admits 3 extra, length-4, irreps which cannot be oxidized to $N = 7$ and an extra length-4 irrep oxidized to $N = 7$. The three length-4 irreps with maximal number of $N = 6$ supersymmetries are given by

$$\begin{array}{rcl} (1,6,7,2) & = & (x;\psi_1,\ldots,\psi_6;g_1,\ldots,g_7;\omega_1,\omega_2),\\ (2,7,6,1) & = & (x_1,x_2;\psi_1,\ldots,\psi_7;g_1,\ldots,g_6;\omega_1),\\ (2,6,6,2) & = & (x_1,x_2;\psi_1,\ldots,\psi_6;g_1,\ldots,g_6;\omega_1,\omega_2), \end{array} \quad (A.22)$$

The $(2,6,6,2)$ irrep is self-dual, while $(1,6,7,2) \leftrightarrow (2,7,6,1)$ are dually related (see Section **4**). Their supersymmetry transformations are respectively given by

$$Q_1(1,6,7,2) = (\psi_1;\dot{x},g_1,g_6,g_7,-g_4,-g_5;\dot{\psi}_2,-\omega_1,-\omega_2,-\dot{\psi}_5,-\dot{\psi}_6,\dot{\psi}_3,\dot{\psi}_4;-\dot{g}_2,-\dot{g}_3)$$

$$\begin{array}{rcl}
Q_2(1,6,7,2) &=& (\psi_5; g_4, g_5, -g_2, -g_3, \dot{x}, g_1; \dot{\psi}_6, -\dot{\psi}_3, -\dot{\psi}_4, \dot{\psi}_1, \dot{\psi}_2, -\omega_1, -\omega_2; -\dot{g}_6, -\dot{g}_7)\\
Q_3(1,6,7,2) &=& (\psi_2; -g_1, \dot{x}, -g_7, g_6, -g_5, g_4; -\dot{\psi}_1, \omega_2, -\omega_1, \dot{\psi}_6, -\dot{\psi}_5, \dot{\psi}_4, -\dot{\psi}_3; -\dot{g}_3, \dot{g}_2)\\
Q_4(1,6,7,2) &=& (\psi_3; -g_6, g_7, \dot{x}, -g_1, g_2, -g_3; -\dot{\psi}_4, \dot{\psi}_5, -\dot{\psi}_6, -\omega_1, \omega_2, -\dot{\psi}_1, \dot{\psi}_2; -\dot{g}_4, \dot{g}_5)\\
Q_5(1,6,7,2) &=& (\psi_4; -g_7, -g_6, g_1, \dot{x}, g_3, g_2; \dot{\psi}_3, \dot{\psi}_6, \dot{\psi}_5, -\omega_2, -\omega_1, -\dot{\psi}_2, -\dot{\psi}_1; -\dot{g}_5, -\dot{g}_4)\\
Q_6(1,6,7,2) &=& (\psi_6; g_5, -g_4, g_3, -g_2, -g_1, \dot{x}; -\dot{\psi}_5, -\dot{\psi}_4, \dot{\psi}_3, -\dot{\psi}_2, \dot{\psi}_1, \omega_2, -\omega_1; -\dot{g}_7, \dot{g}_6)
\end{array} \tag{A.23}$$

$$\begin{array}{rcl}
Q_1(2,7,6,1) &=& (\psi_2, \psi_3; -g_2, \dot{x}_1, \dot{x}_2, g_5, g_6, -g_3, g_4; -\omega, -\dot{\psi}_1, -\dot{\psi}_6, -\dot{\psi}_7, \dot{\psi}_4, \dot{\psi}_5; -\dot{g}_1)\\
Q_2(2,7,6,1) &=& (\psi_6, \psi_7; -g_6, g_3, g_4, -g_1, -g_2, \dot{x}_1, \dot{x}_2; -\dot{\psi}_4, -\dot{\psi}_5, \dot{\psi}_2, \dot{\psi}_3, -\omega, -\dot{\psi}_1; -\dot{g}_5)\\
Q_3(2,7,6,1) &=& (\psi_3, -\psi_2; g_1, -\dot{x}_2, \dot{x}_1, -g_6, g_5, -g_4, g_3; \dot{\psi}_1, -\omega, \dot{\psi}_7, -\dot{\psi}_6, \dot{\psi}_5, -\dot{\psi}_4; -\dot{g}_2)\\
Q_4(2,7,6,1) &=& (\psi_4, -\psi_5; g_4, -g_5, g_6, \dot{x}_1, -\dot{x}_2, g_1, -g_2; \dot{\psi}_6, -\dot{\psi}_7, -\omega, \dot{\psi}_1, -\dot{\psi}_2, \dot{\psi}_3; -\dot{g}_3)\\
Q_5(2,7,6,1) &=& (\psi_5, \psi_4; -g_3, -g_6, -g_5, \dot{x}_2, \dot{x}_1, g_2, g_1; \dot{\psi}_7, \dot{\psi}_6, -\dot{\psi}_1, -\omega, -\dot{\psi}_3, -\dot{\psi}_2; -\dot{g}_4)\\
Q_6(2,7,6,1) &=& (\psi_7, -\psi_6; g_5, g_4, -g_3, g_2, -g_1, \dot{x}_2, \dot{x}_1; -\dot{\psi}_5, \dot{\psi}_4, -\dot{\psi}_3, \dot{\psi}_2, \dot{\psi}_1, -\omega; -\dot{g}_6)
\end{array} \tag{A.24}$$

and

$$\begin{array}{rcl}
Q_1(2,6,6,2) &=& (\psi_1, \psi_2; \dot{x}_1, \dot{x}_2, g_5, g_6, -g_3, -g_4; -\omega_1, -\omega_2, -\dot{\psi}_5, -\dot{\psi}_6, \dot{\psi}_3, \dot{\psi}_4; -\dot{g}_1, -\dot{g}_2)\\
Q_2(2,6,6,2) &=& (\psi_5, \psi_6; g_3, g_4, -g_1, -g_2, \dot{x}_1, \dot{x}_2; -\dot{\psi}_3, -\dot{\psi}_4, \dot{\psi}_1, \dot{\psi}_2, -\omega_1, -\omega_2; -\dot{g}_5, -\dot{g}_6)\\
Q_3(2,6,6,2) &=& (\psi_2, -\psi_1; -\dot{x}_2, \dot{x}_1, -g_6, g_5, -g_4, g_3; \omega_2, -\omega_1, \dot{\psi}_6, -\dot{\psi}_5, \dot{\psi}_4, -\dot{\psi}_3; -\dot{g}_2, \dot{g}_1)\\
Q_4(2,6,6,2) &=& (\psi_3, -\psi_4; -g_5, g_6, \dot{x}_1, -\dot{x}_2, g_1, -g_2; \dot{\psi}_5, -\dot{\psi}_6, -\omega_1, \omega_2, -\dot{\psi}_1, \dot{\psi}_2; -\dot{g}_3, \dot{g}_4)\\
Q_5(2,6,6,2) &=& (\psi_4, \psi_3; -g_6, -g_5, \dot{x}_2, \dot{x}_1, g_2, g_1; \dot{\psi}_6, \dot{\psi}_5, -\omega_2, -\omega_1, -\dot{\psi}_2, -\dot{\psi}_1; -\dot{g}_4, -\dot{g}_3)\\
Q_6(2,6,6,2) &=& (\psi_6, -\psi_5; g_4, -g_3, g_2, -g_1, -\dot{x}_2, \dot{x}_1; -\dot{\psi}_4, \dot{\psi}_3, -\dot{\psi}_2, \dot{\psi}_1, \omega_2, -\omega_1; -\dot{g}_6, \dot{g}_5)
\end{array} \tag{A.25}$$

ix) **The $N = 7$ irreps**

The length 2 and 3 irreps are obtained from the $N = 8$ irreps by restricting the supersymmetry transformations to be given by Q_i, for $i = 1, \ldots, 7$.

This case admits an extra, length-4, irrep

$$(1,7,7,1) \;=\; (x; \psi_1, \ldots, \psi_7; g_1, \ldots, g_7; \omega), \tag{A.26}$$

whose supersymmetry transformations are

$$\begin{array}{rcl}
Q_1(1,7,7,1) &=& (\psi_2; -g_3, \dot{x}, g_1, g_6, g_7, -g_4, -g_5; \dot{\psi}_3, -\omega, -\dot{\psi}_1, -\dot{\psi}_6, -\dot{\psi}_7, \dot{\psi}_4, \dot{\psi}_5; -\dot{g}_2)\\
Q_2(1,7,7,1) &=& (\psi_6; -g_7, g_4, g_5, -g_2, -g_3, \dot{x}, g_1; \dot{\psi}_7, -\dot{\psi}_4, -\dot{\psi}_5, \dot{\psi}_2, \dot{\psi}_3, -\omega, -\dot{\psi}_1; -\dot{g}_6)\\
Q_3(1,7,7,1) &=& (\psi_1; \dot{x}, g_3, -g_2, -g_5, g_4, g_7, -g_6; -\omega, -\dot{\psi}_3, \dot{\psi}_2, \dot{\psi}_5, -\dot{\psi}_4, -\dot{\psi}_7, \dot{\psi}_6; -\dot{g}_1)\\
Q_4(1,7,7,1) &=& (\psi_3; g_2, -g_1, \dot{x}, -g_7, g_6, -g_5, g_4; -\dot{\psi}_2, \dot{\psi}_1, -\omega, \dot{\psi}_7, -\dot{\psi}_6, \dot{\psi}_5, -\dot{\psi}_4; -\dot{g}_3)\\
Q_5(1,7,7,1) &=& (\psi_4; g_5, -g_6, g_7, \dot{x}, -g_1, g_2, -g_3; -\dot{\psi}_5, \dot{\psi}_6, -\dot{\psi}_7, -\omega, \dot{\psi}_1, -\dot{\psi}_2, \dot{\psi}_3; -\dot{g}_4)\\
Q_6(1,7,7,1) &=& (\psi_5; -g_4, -g_7, -g_6, g_1, \dot{x}, g_3, g_2; \dot{\psi}_4, \dot{\psi}_7, \dot{\psi}_6, -\dot{\psi}_1, -\omega, -\dot{\psi}_3, -\dot{\psi}_2; -\dot{g}_5)\\
Q_7(1,7,7,1) &=& (\psi_7; g_6, g_5, -g_4, g_3, -g_2, -g_1, \dot{x}; -\dot{\psi}_6, -\dot{\psi}_5, \dot{\psi}_4, -\dot{\psi}_3, \dot{\psi}_2, \dot{\psi}_1, -\omega; -\dot{g}_7)
\end{array}$$

(A.27)

x) **The $N = 8$ irreps**
The 8 inequivalent irreps are here listed

$$
\begin{array}{rcl}
(8,8) & = & (x_1,\ldots x_8;\psi_1,\ldots,\psi_8),\\
(7,8,1) & = & (x_1,\ldots,x_7;\psi_1,\ldots,\psi_8;g),\\
(6,8,2) & = & (x_1,\ldots,x_6;\psi_1,\ldots,\psi_8;g_1,g_2),\\
(5,8,3) & = & (x_1,\ldots,x_5;\psi_1,\ldots,\psi_8;g_1,g_2,g_3),\\
(4,8,4) & = & (x_1,\ldots,x_4,;\psi_1,\ldots,\psi_8;g_1,\ldots,g_4),\\
(3,8,5) & = & (x_1,x_2,x_3;\psi_1,\ldots,\psi_8;g_1,\ldots,g_5),\\
(2,8,6) & = & (x_1,x_2;\psi_1,\ldots,\psi_8;g_1,\ldots,g_6),\\
(1,8,7) & = & (x;\psi_1,\ldots,\psi_8;g_1,\ldots,g_7).
\end{array}
\qquad (A.28)
$$

The length-2 $(8,8)$ multiplet admits the following supersymmetry transformations

$$
\begin{array}{rcl}
Q_1(8,8) & = & (\psi_3,\psi_4,-\psi_1,-\psi_2,-\psi_7,-\psi_8,\psi_5,\psi_6;-\dot{x}_3,-\dot{x}_4,\dot{x}_1,\dot{x}_2,\dot{x}_7,\dot{x}_8,-\dot{x}_5,-\dot{x}_6)\\
Q_2(8,8) & = & (\psi_7,\psi_8,-\psi_5,-\psi_6,\psi_3,\psi_4,-\psi_1,-\psi_2;-\dot{x}_7,-\dot{x}_8,\dot{x}_5,\dot{x}_6,-\dot{x}_3,-\dot{x}_4,\dot{x}_1,\dot{x}_2)\\
Q_3(8,8) & = & (\psi_2,-\psi_1,-\psi_4,\psi_3,\psi_6,-\psi_5,-\psi_8,\psi_7;-\dot{x}_2,\dot{x}_1,\dot{x}_4,-\dot{x}_3,-\dot{x}_6,\dot{x}_5,\dot{x}_8,-\dot{x}_7)\\
Q_4(8,8) & = & (\psi_4,-\psi_3,\psi_2,-\psi_1,\psi_8,-\psi_7,\psi_6,-\psi_5;-\dot{x}_4,\dot{x}_3,-\dot{x}_2,\dot{x}_1,-\dot{x}_8,\dot{x}_7,-\dot{x}_6,\dot{x}_5)\\
Q_5(8,8) & = & (\psi_5,-\psi_6,\psi_7,-\psi_8,-\psi_1,\psi_2,-\psi_3,\psi_4;-\dot{x}_5,\dot{x}_6,-\dot{x}_7,\dot{x}_8,\dot{x}_1,-\dot{x}_2,\dot{x}_3,-\dot{x}_4)\\
Q_6(8,8) & = & (\psi_6,\psi_5,\psi_8,\psi_7,-\psi_2,-\psi_1,-\psi_4,-\psi_3;-\dot{x}_6,-\dot{x}_5,-\dot{x}_8,-\dot{x}_7,\dot{x}_2,\dot{x}_1,\dot{x}_4,\dot{x}_3)\\
Q_7(8,8) & = & (\psi_8,-\psi_7,-\psi_6,\psi_5,-\psi_4,\psi_3,\psi_2,-\psi_1;-\dot{x}_8,\dot{x}_7,\dot{x}_6,-\dot{x}_5,\dot{x}_4,-\dot{x}_3,-\dot{x}_2,\dot{x}_1)\\
Q_8(8,8) & = & (\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8;\dot{x}_1,\dot{x}_2,\dot{x}_3,\dot{x}_4,\dot{x}_5,\dot{x}_6,\dot{x}_7,\dot{x}_8)
\end{array}
\qquad (A.29)
$$

The length 3 irreps are immediately read from the above transformations by setting, for any $p = 1,\ldots,d-1$ associated with the $(d-p,d,p)$ multiplet, $g_1 = \dot{x}_8$, $g_2 = \dot{x}_7$, $\ldots$, $g_p = \dot{x}_{9-p}$. To save space the length 3 irreps are not explicitly reported here.

Appendix B

Classification of the $N = 9,10$ irreps and length-4 $N = 11^{(*)},12$ irreps

We present here the complete classification of irreps of the $N = 9,10$ supersymmetries producing (the length $l = 2$ and $l = 3$ irreps being already known) the whole list of length $l \geq 4$ inequivalent irreps. $N = 9$ does not admit length $l \geq 5$ irreps, while $N = 10$ is the lowest number of extended supersmmetries admitting irreps with length $l > 4$. For $N = 10$ the maximal length L of its irreps is $L = 5$.

We further produce the list of length-4 inequivalent irreps for the next two values of the oxidized supersymmetries, namely $N = 11^{(*)}$ and $N = 12$.

i) **Classification of the $N = 9$ irreps**

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 16 - h$, $d_2 = 16 - k$.

$N = 9$ presents 4 length-4 irreducible self-dual (under (4.3)) multiplets for

$$h = k = 1, 2, 3, 4 \tag{B.1}$$

and $2 \times (6+4+2) = 24$ non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{aligned} h = 1 \quad & \& \quad k = 2, \ldots, 7 \\ h = 2 \quad & \& \quad k = 3, \ldots, 6 \\ h = 3 \quad & \& \quad k = 4, 5 \end{aligned} \tag{B.2}$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

The previous results can be summarized as follows. Inequivalent length-4 irreps are in 1-to-1 correspondence with the ordered pair of positive integers h, k satisfying the constraint

$$h + k \quad \leq \quad 8. \tag{B.3}$$

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (4.4), the statistics of the multiplets) is

$$\overline{k}_4 \quad = \quad 28 \tag{B.4}$$

ii) **Classification of the $N = 10$ irreps**

$N = 10$ admits irreps up to length $l = 5$. We have

iia) *The length-*4 *classification.*

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 32 - h$, $d_2 = 32 - k$.

$N = 10$ presents 8 length-4 irreducible self-dual (under (4.3)) multiplets for

$$h = k = 1, 2, \ldots, 8 \tag{B.5}$$

and a set of $2 \times 3(\sum_{j=1}^{7} j) = 168$ non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{aligned} h = 1 \quad & \& \quad k = 2, \ldots, 22 \\ h = 2 \quad & \& \quad k = 3, \ldots, 20 \\ h = 3 \quad & \& \quad k = 4, \ldots, 18 \\ h = 4 \quad & \& \quad k = 5, \ldots, 16 \\ h = 5 \quad & \& \quad k = 6, \ldots, 14 \\ h = 6 \quad & \& \quad k = 7, \ldots, 12 \\ h = 7 \quad & \& \quad k = 8, 9, 10 \end{aligned} \tag{B.6}$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

If we set

$$r = min(h,k) \tag{B.7}$$

the previous results can be summarized as follows. Inequivalent length-4 irreps are in 1-to-1 correspondence with the ordered pair of positive integers h, k satisfying the constraint

$$h+k+r \leq 24. \tag{B.8}$$

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (4.4), the statistics of the multiplets) is

$$\overline{k}_4 = 176 \tag{B.9}$$

iib) The length-5 classification

A length-5 multiplet $(d_1, d_2, d_3, d_4, d_5)$ is characterized by three independent positive integers, let's say d_1, d_2, d_5, since $d_4 = 32 - d_2$ and $d_3 = 32 - d_1 - d_5$. The full list of length-5 irreps of the $N = 10$ supersymmetry can be listed according to the number d_5 of highest-spin auxiliary fields. The maximal number of auxiliary fields is 7. At any fixed $d_5 = 1, \ldots, 7$ the number of inequivalent irreps is $(8 - d_5)^2$. Therefore, the total number $\overline{k}_5$ of length-5 inequivalent irreps is given by

$$\overline{k}_5 = 1^2 + 2^2 + \ldots + 7^2 = 140 \tag{B.10}$$

The full list of irreps is here produced in terms, at any fixed d_5, of the ordered $\underline{d_1, d_2}$ pairs. We have

$$\begin{array}{lcl}
d_5 = 7 & : & \underline{1,10}. \\
d_5 = 6 & : & \underline{1,10}, \underline{1,11}, \underline{1,12}/\underline{2,12}. \\
d_5 = 5 & : & \underline{1,10}, \ldots, \underline{1,14}/\underline{2,12}, \ldots, \underline{2,14}/\underline{3,14}. \\
d_5 = 4 & : & \underline{1,10}, \ldots, \underline{1,16}/\underline{2,12}, \ldots, \underline{2,16}/\underline{3,14}, \ldots, \underline{3,16}/\underline{4,16}. \\
d_5 = 3 & : & \underline{1,10}, \ldots, \underline{1,18}/\underline{2,12}, \ldots, \underline{2,18}/\underline{3,14}, \ldots, \underline{3,18}/\underline{4,16}, \ldots, \underline{4,18}/\underline{5,18}. \\
d_5 = 2 & : & \underline{1,10}, \ldots, \underline{1,20}/\underline{2,12}, \ldots, \underline{2,20}/\underline{3,14}, \ldots, \underline{3,20}/\underline{4,16}, \ldots, \underline{4,20}/ \\
 & & \underline{5,18}, \ldots, \underline{5,20}/\underline{6,20}. \\
d_5 = 1 & : & \underline{1,10}, \ldots, \underline{1,22}/\underline{2,12}, \ldots, \underline{2,22}/\underline{3,14}, \ldots, \underline{3,22}/\underline{4,16}, \ldots, \underline{4,22}/ \\
 & & \underline{5,18}, \ldots, \underline{5,22}/\underline{6,20}, \ldots, \underline{6,22}/\underline{7,22}.
\end{array} \tag{B.11}$$

One can check that the above set of irreducible multiplets is indeed closed under the (4.3) *high* $\Leftrightarrow$ *low spin* duality transformations.

iii) **Classification of the length-4 $N = 11^{(*)}$ irreps**

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 64 - h$, $d_2 = 64 - k$.

$N = 11^{(*)}$ presents 16 length-4 irreducible self-dual (under (4.3)) multiplets for

$$h = k = 1, 2, \ldots, 16 \tag{B.12}$$

and 776 non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{array}{ll} h=1 \ \& \ k=2,\ldots,53 & h=9 \ \& \ k=10,\ldots,30 \\ h=2 \ \& \ k=3,\ldots,50 & h=10 \ \& \ k=11,\ldots,28 \\ h=3 \ \& \ k=4,\ldots,47 & h=11 \ \& \ k=12,\ldots,26 \\ h=4 \ \& \ k=5,\ldots,44 & h=12 \ \& \ k=13,\ldots,24 \\ h=5 \ \& \ k=6,\ldots,41 & h=13 \ \& \ k=14,\ldots,22 \\ h=6 \ \& \ k=7,\ldots,38 & h=14 \ \& \ k=15,\ldots,20 \\ h=7 \ \& \ k=8,\ldots,35 & h=15 \ \& \ k=16,\ldots,18 \\ h=8 \ \& \ k=9,\ldots,32 & \end{array} \tag{B.13}$$

together with the $(h \leftrightarrow k)$ dually interchanged multiplets.

The previous results can be summarized as follows. Let us set, as before (B.7), $r = min(h, k)$ and introduce the $s(r)$ function defined through

$$s(r) \;=\; \left\{ \begin{array}{ll} 8-r & \text{for } r = 1,\ldots,7 \\ 0 & \text{otherwise} \end{array} \right\} \tag{B.14}$$

Inequivalent length-4 irreps are in 1-to-1 correspondence with the ordered pair of positive integers h, k satisfying the constraint

$$h + k + r - s(r) \;\leq\; 48. \tag{B.15}$$

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (4.4), the statistics of the multiplets) is

$$\overline{k}_4 \;=\; 792. \tag{B.16}$$

iii) **Classification of the length-4 $N = 12$ irreps**

The length-4 irreducible multiplet (d_1, d_2, d_3, d_4) is for simplicity expressed in terms of the two positive integers $h \equiv d_1$, $k = d_4$, since $d_3 = 64 - h$, $d_2 = 64 - k$.

$N = 12$ presents 12 length-4 irreducible self-dual (under (4.3)) multiplets for

$$h = k = 1, 2, \ldots, 12 \tag{B.17}$$

and 584 non self-dual length-4 irreducible multiplets given by the series of coupled values

$$\begin{array}{ll} h=1 \ \& \ k=2,\ldots,52 & h=7 \ \& \ k=8,\ldots,28 \\ h=2 \ \& \ k=3,\ldots,48 & h=8 \ \& \ k=9,\ldots,24 \\ h=3 \ \& \ k=4,\ldots,44 & h=9 \ \& \ k=10,\ldots,21 \\ h=4 \ \& \ k=5,\ldots,40 & h=10 \ \& \ k=11,\ldots,18 \\ h=5 \ \& \ k=6,\ldots,36 & h=11 \ \& \ k=12,\ldots,15 \\ h=6 \ \& \ k=7,\ldots,32 & \end{array} \tag{B.18}$$

together with the ($h \leftrightarrow k$) dually interchanged multiplets.

The total number $\overline{k}_4$ of inequivalent length-4 irreps (without discriminating, see (4.4), the statistics of the multiplets) is

$$\overline{k}_4 = 596. \tag{B.19}$$

Appendix C

Irreps decompositions of multiplets tensor products

As discussed in Section **6**, multilinear terms entering a manifestly invariant N-extended supersymmetric action can be extracted by (multiple) tensor products of irreps. In our framework this procedure replaces the supersymmetric calculus. The advantage of our method consists in the fact that it can be systematically carried out for any arbitrary value of N (the limitations are only due to the increasing computational complexity and are not of conceptual nature), even when the superfield formalism is not available. For illustrative purposes it is convenient to present here some explicit examples which have been discussed in the main text.

i) **Tensoring the $N = 2$ bosonic irreps**

The two inequivalent $N = 2$ bosonic irreps, $(1, 2, 1)$ and $(2, 2)$, correspond in the superfield language to, respectively, the real and chiral $N = 2$ linear bosonic superfields. In our framework we get the following results, for their mutual tensoring:

ia) $(1, 2, 1)_{s=0} \times (1, 2, 1)_{s=0} = (1, 2, 1)_{\|s=0} + (\ldots)_{s>0}$

$(\ldots)_{s>0}$ on the r.h.s denotes the non-leading (i.e. of higher spin) bilinear irreps. The suffix "$\|$" on the $(1, 2, 1)_{\|s=0}$ multiplet on the r.h.s. means that it is *symmetric*, i.e. it is non-vanishing when the left and right multiplets in the l.h.s. are identified. Let us set $(x; \psi_1, \psi_2; g)$, $(y; \lambda_1, \lambda_2; f)$ the two multiplets on the l.h.s. . The bilinear multiplet $(1, 2, 1)_{\|s=0} \equiv (\tilde{x}; \tilde{\psi}_1, \tilde{\psi}_2; \tilde{g})$ on the r.h.s. is therefore given by

$$\begin{aligned}
\tilde{x} &= xy \\
\tilde{\psi}_1 &= \psi_1 y + x\lambda_1 \\
\tilde{\psi}_2 &= \psi_2 y + x\lambda_2 \\
\tilde{g} &= gy - \psi_1\lambda_2 + \psi_2\lambda_1 + xf
\end{aligned} \tag{C.1}$$

ib) $(1, 2, 1)_{s=0} \times (2, 2)_{s=0} = 2 \times (1, 2, 1)_{s=0} + (\ldots)_{s>0}$

The second multiplet on the l.h.s. is here given by $(y_1, y_2; \lambda_1, \lambda_2)$. The two leading bilinear multiplets in the r.h.s. are ${(1, 2, 1)_{\|s=0}}^{a,b} \equiv (\tilde{x}^{a,b}; \tilde{\psi}_1^{a,b}, \tilde{\psi}_2^{a,b}; \tilde{g}^{a,b})$, with

$$\tilde{x}^a = x_1 y$$

$$\begin{aligned}
\tilde{\psi}_1^a &= \psi_1 y + x_1\lambda_1 \\
\tilde{\psi}_2^a &= \psi_2 y + x_1\lambda_2 \\
\tilde{g}^a &= \dot{x}_2 y - \psi_1\lambda_2 + \psi_2\lambda_1 + x_1 f
\end{aligned} \qquad (C.2)$$

and

$$\begin{aligned}
\tilde{x}^b &= x_2 y \\
\tilde{\psi}_1^b &= -\psi_2 y + x_2\lambda_1 \\
\tilde{\psi}_2^b &= \psi_1 y + x_2\lambda_2 \\
\tilde{g}^b &= -\dot{x}_1 y + \psi_2\lambda_2 + \psi_1\lambda_1 + x_2 f
\end{aligned} \qquad (C.3)$$

ic) $(2,2)_{s=0} \times (2,2)_{s=0} = (2,2)_{\parallel s=0} + (1,2,1)_{\parallel s=0} + (1,2,1)_{\perp s=0} + (\ldots)_{s>0}$

Tensoring two length-2 multiplets, $(x_1, x_2; \psi_1, \psi_2)$, $(y_1, y_2; \lambda_1, \lambda_2)$, produces a bilinear *antisymmetric* leading $(s = 0)$ multiplet $(1,2,1)_{\perp s=0} \equiv (\overline{x}, \overline{\psi}_1, \overline{\psi}_2, \overline{g})$ which vanishes when $(x_1, x_2; \psi_1, \psi_2) \equiv (y_1, y_2; \lambda_1, \lambda_2)$ are identified, plus two *symmetric* leading bilinear multiplets, given by $(2,2)_{\parallel s=0} \equiv (\tilde{x}_1, \tilde{x}_2; \tilde{\psi}_1, \tilde{\psi}_2)$, $(1,2,1)_{\parallel s=0} \equiv (\hat{x}; \hat{\psi}_1, \hat{\psi}_2; \hat{g})$.

We have, explicitly,

$$\begin{aligned}
\overline{x} &= x_1 y_2 - x_2 y_1 \\
\overline{\psi}_1 &= \psi_1 y_2 - x_1\lambda_2 + \psi_2 y_1 - x_2\lambda_1 \\
\overline{\psi}_2 &= \psi_2 y_2 + x_1\lambda_1 - \psi_1 y_1 - x_2\lambda_2 \\
\overline{g} &= -2\psi_1\lambda_1 - 2\psi_2\lambda_2 + \dot{x}_2 y_2 - x_2\dot{y}_2 + \dot{x}_1 y_1 - x_1\dot{y}_1,
\end{aligned} \qquad (C.4)$$

$$\begin{aligned}
\tilde{x}_1 &= x_1 y_1 - x_2 y_2 \\
\tilde{x}_2 &= x_1 y_2 + x_2 y_1 \\
\tilde{\psi}_1 &= \psi_1 y_1 + x_1\lambda_1 + \psi_2 y_2 + x_2\lambda_2 \\
\tilde{\psi}_2 &= \psi_2 y_1 + x_1\lambda_2 - \psi_1 y_2 - x_2\lambda_1,
\end{aligned} \qquad (C.5)$$

and

$$\begin{aligned}
\hat{x} &= x_1 y_1 + x_2 y_2 \\
\hat{\psi}_1 &= \psi_1 y_1 - \psi_2 y_2 + x_1\lambda_1 - x_2\lambda_2 \\
\hat{\psi}_2 &= \psi_2 y_1 + \psi_1 y_2 + x_1\lambda_2 + x_2\lambda_1 \\
\hat{g} &= 2\psi_2\lambda_1 - 2\psi_1\lambda_2 + \dot{x}_2 y_1 - \dot{x}_1 y_2 + x_1\dot{y}_2 - x_2\dot{y}_1.
\end{aligned} \qquad (C.6)$$

ii) **Tensoring the $N = 4$ bosonic irreps (selected cases)**

We present here the mutual tensoring of the $(1,4,3)$ and the $(2,4,2)$ irreps.

iia) The $(1,4,3)_{s=0} \times (1,4,3)_{s=0}$ *case*

Let us denote the left multiplet as $(x;\psi_1,\psi_2,\psi_3,\psi_4;g_1,g_2,g_3)$ and the right multiplet as $(y;\lambda_1,\lambda_2,\lambda_3,\lambda_4;f_1,f_2,f_3)$. Their tensor product gives at the leading order $(s=0)$ the (reducible) adjoint representation of the $N=4$ supersymmetry, given by the $(1,4,6,4,1)$ multiplet with elements $\mathbf{1}$, $Q_i\mathbf{1}$ for $i=1,2,3,4$, $Q_jQ_j\mathbf{1}$ for $i<j$, $Q_iQ_jQ_k\mathbf{1}$ for $i<j<k$ and, finally, $Q_1Q_2Q_3Q_4\mathbf{1}$.

The following identifications hold

$$\begin{array}{rcl}
\mathbf{1} &=& xy,\\
Q_1\mathbf{1} &=& \psi_2y+x\lambda_2\\
Q_2\mathbf{1} &=& \psi_4y+x\lambda_4\\
Q_3\mathbf{1} &=& \psi_3y+x\lambda_3\\
Q_4\mathbf{1} &=& \psi_1y+x\lambda_1\\
Q_1Q_2\mathbf{1} &=& \psi_2\lambda_4-g_2y-\psi_4\lambda_2-xf_2\\
Q_1Q_3\mathbf{1} &=& \psi_2\lambda_3+g_3y-\psi_3\lambda_2+xf_3\\
Q_1Q_4\mathbf{1} &=& \psi_2\lambda_1-g_1y-\psi_1\lambda_2-xf_1\\
Q_2Q_3\mathbf{1} &=& \psi_4\lambda_3-g_1y-\psi_3\lambda_4-xf_1\\
Q_2Q_4\mathbf{1} &=& \psi_4\lambda_1-g_3y-\psi_1\lambda_4-xf_3\\
Q_3Q_4\mathbf{1} &=& \psi_3\lambda_1-g_2y-\psi_1\lambda_3-xf_2\\
Q_1Q_2Q_3\mathbf{1} &=& \dot{\psi}_1y-g_2\lambda_3-g_3\lambda_4-\psi_2f_1-g_1\lambda_2-\psi_4f_3-\psi_3f_2+x\dot{\lambda}_1\\
Q_1Q_2Q_4\mathbf{1} &=& -\dot{\psi}_3y-g_2\lambda_1+g_1\lambda_4-\psi_2f_3-g_3\lambda_2+\psi_4f_1-\psi_1f_2-x\dot{\lambda}_3\\
Q_1Q_3Q_4\mathbf{1} &=& \dot{\psi}_4y+g_3\lambda_1+g_1\lambda_3-\psi_2f_2-g_2\lambda_2+\psi_3f_1+\psi_1f_3+x\dot{\lambda}_4\\
Q_2Q_3Q_4\mathbf{1} &=& -\dot{\psi}_2y-g_1\lambda_1+g_3\lambda_3-\psi_4f_2-g_2\lambda_4+\psi_3f_3-\psi_1f_1-x\dot{\lambda}_2\\
Q_1Q_2Q_3Q_4\mathbf{1} &=& -\ddot{x}y-x\ddot{y}+\dot{\psi}_1\lambda_1+\dot{\psi}_2\lambda_2+\dot{\psi}_3\lambda_3+\dot{\psi}_4\lambda_4-\psi_1\dot{\lambda}_1-\psi_2\dot{\lambda}_2-\\
&& -\psi_3\dot{\lambda}_3-\psi_4\dot{\lambda}_4+2g_1f_1+2g_2f_2+2g_3f_3
\end{array}\qquad (C.7)$$

Since the identity $\mathbf{1}$ is bosonic, therefore $\Gamma^5\mathbf{1}=\mathbf{1}$, see (5.3).

$Q_1Q_2\mathbf{1}-Q_3Q_4\mathbf{1}$, $Q_2Q_3\mathbf{1}-Q_1Q_4\mathbf{1}$, $Q_3Q_1\mathbf{1}-Q_2Q_4\mathbf{1}$ are the three leading bosonic fields of a $(3,4,1)_{s=1}$ irrep contained in the adjoint representation as a subrepresentation. Quotienting out such an irrep from the adjoint representation (by consistently setting all eight corresponding fields identically equal to zero, see (5.1)) we obtain the $(1,4,3)_{s=0}$ irrep, in terms of a single spin 0 field (the identity $\mathbf{1}$), four spin $\frac{1}{2}$ fields (given by $Q_i\mathbf{1}$) and three spin 1 fields ($Q_1Q_2\mathbf{1}+Q_3Q_4\mathbf{1}$, $Q_2Q_3\mathbf{1}+Q_1Q_4\mathbf{1}$, $Q_3Q_1\mathbf{1}+Q_2Q_4\mathbf{1}$) which play the role of auxiliary fields.

iib) The $(2,4,2)_{s=0}\times(1,4,3)_{s=0}$ *case*

Let us denote now as $(x_1,x_2;\psi_1,\psi_2,\psi_3,\psi_4;g_1,g_2)$ the left multiplet and the right multiplet as $(y;\lambda_1,\lambda_2,\lambda_3,\lambda_4;f_1,f_2,f_3)$. Their tensor product produces at the leading $(s=0)$ order two (reducible) adjoint representations of the $N=4$ supersymmetry, given by two $(1,4,6,4,1)$ multiplets. They are respectively lead by $\mathbf{1}\equiv x_1y$ and by

$\mathbf{1} \equiv x_2 y$. Explicitly, we get in the first case

$$\begin{aligned}
\mathbf{1} &= x_1 y, \\
Q_1\mathbf{1} &= \psi_2 y + x_1\lambda_2 \\
Q_2\mathbf{1} &= \psi_4 y + x_1\lambda_4 \\
Q_3\mathbf{1} &= \psi_3 y + x_1\lambda_3 \\
Q_4\mathbf{1} &= \psi_1 y + x_1\lambda_1 \\
Q_1Q_2\mathbf{1} &= \psi_2\lambda_4 - g_1 y - \psi_4\lambda_2 - x_1 f_2 \\
Q_1Q_3\mathbf{1} &= \psi_2\lambda_3 + g_2 y - \psi_3\lambda_2 + x_1 f_3 \\
Q_1Q_4\mathbf{1} &= \psi_2\lambda_1 - \dot{x}_2 y - \psi_1\lambda_2 - x_1 f_1 \\
Q_2Q_3\mathbf{1} &= \psi_4\lambda_3 - \dot{x}_2 y - \psi_3\lambda_4 - x_1 f_1 \\
Q_2Q_4\mathbf{1} &= \psi_4\lambda_1 - g_2 y - \psi_1\lambda_4 - x_1 f_3 \\
Q_3Q_4\mathbf{1} &= \psi_3\lambda_1 - g_1 y - \psi_1\lambda_3 - x_1 f_2 \\
Q_1Q_2Q_3\mathbf{1} &= \dot{\psi}_1 y - g_1\lambda_3 - g_2\lambda_4 - \psi_2 f_1 - \dot{x}_2\lambda_2 - \psi_4 f_3 - \psi_3 f_2 + x_1\dot{\lambda}_1 \\
Q_1Q_2Q_4\mathbf{1} &= -\dot{\psi}_3 y - g_1\lambda_1 + \dot{x}_2\lambda_4 - \psi_2 f_3 - g_2\lambda_2 + \psi_4 f_1 - \psi_1 f_2 - x_1\dot{\lambda}_3 \\
Q_1Q_3Q_4\mathbf{1} &= \dot{\psi}_4 y + g_2\lambda_1 + \dot{x}_2\lambda_3 - \psi_2 f_2 - g_1\lambda_2 + \psi_3 f_1 + \psi_1 f_3 + x_1\dot{\lambda}_4 \\
Q_2Q_3Q_4\mathbf{1} &= -\dot{\psi}_2 y - \dot{x}_2\lambda_1 + g_2\lambda_3 - \psi_4 f_2 - g_1\lambda_4 + \psi_3 f_3 - \psi_1 f_1 - x_1\dot{\lambda}_2 \\
Q_1Q_2Q_3Q_4\mathbf{1} &= -\ddot{x}_1 y - x_1\ddot{y} + \dot{\psi}_1\lambda_1 + \dot{\psi}_2\lambda_2 + \dot{\psi}_3\lambda_3 + \dot{\psi}_4\lambda_4 - \psi_1\dot{\lambda}_1 - \psi_2\dot{\lambda}_2 - \\
&\quad -\psi_3\dot{\lambda}_3 - \psi_4\dot{\lambda}_4 + 2\dot{x}_2 f_1 + 2g_1 f_2 + 2g_2 f_3, \qquad \text{(C.8)}
\end{aligned}$$

while in the second case the corresponding formulae are

$$\begin{aligned}
\mathbf{1} &= x_2 y, \\
Q_1\mathbf{1} &= -\psi_1 y + x_2\lambda_2 \\
Q_2\mathbf{1} &= -\psi_3 y + x_2\lambda_4 \\
Q_3\mathbf{1} &= \psi_4 y + x_2\lambda_3 \\
Q_4\mathbf{1} &= \psi_2 y + x_2\lambda_1 \\
Q_1Q_2\mathbf{1} &= -\psi_1\lambda_4 - g_2 y + \psi_3\lambda_2 - x_2 f_2 \\
Q_1Q_3\mathbf{1} &= -\psi_1\lambda_3 - g_1 y - \psi_4\lambda_2 + x_2 f_3 \\
Q_1Q_4\mathbf{1} &= -\psi_1\lambda_1 + \dot{x}_1 y - \psi_2\lambda_4 - x_2 f_1 \\
Q_2Q_3\mathbf{1} &= -\psi_3\lambda_3 + \dot{x}_1 y - \psi_4\lambda_4 - x_2 f_1 \\
Q_2Q_4\mathbf{1} &= -\psi_3\lambda_1 + g_1 y - \psi_2\lambda_4 - x_2 f_3 \\
Q_3Q_4\mathbf{1} &= \psi_4\lambda_1 - g_2 y - \psi_2\lambda_3 - x_2 f_2 \\
Q_1Q_2Q_3\mathbf{1} &= \dot{\psi}_2 y - g_2\lambda_3 + g_1\lambda_4 + \psi_1 f_1 + \dot{x}_1\lambda_2 + \psi_3 f_3 - \psi_4 f_2 + x_2\dot{\lambda}_1 \\
Q_1Q_2Q_4\mathbf{1} &= -\dot{\psi}_4 y - g_2\lambda_1 - \dot{x}_1\lambda_4 + \psi_1 f_3 + g_1\lambda_2 - \psi_3 f_1 - \psi_2 f_2 + x_2\dot{\lambda}_3 \\
Q_1Q_3Q_4\mathbf{1} &= -\dot{\psi}_3 y - g_1\lambda_1 - \dot{x}_1\lambda_3 + \psi_1 f_2 - g_2\lambda_2 + \psi_4 f_1 + \psi_2 f_3 + x_2\dot{\lambda}_4 \\
Q_2Q_3Q_4\mathbf{1} &= \dot{\psi}_1 y + \dot{x}_1\lambda_1 - g_1\lambda_3 + \psi_3 f_2 - g_2\lambda_4 + \psi_4 f_3 - \psi_2 f_1 - x_2\dot{\lambda}_2 \\
Q_1Q_2Q_3Q_4\mathbf{1} &= -\ddot{x}_2 y - x_2\ddot{y} + \dot{\psi}_2\lambda_1 - \dot{\psi}_1\lambda_2 + \dot{\psi}_4\lambda_3 - \dot{\psi}_3\lambda_4 - \psi_2\dot{\lambda}_1 + \psi_1\dot{\lambda}_2 - \\
&\quad -\psi_4\dot{\lambda}_3 + \psi_3\dot{\lambda}_4 - 2\dot{x}_1 f_1 - 2g_1 f_3 + 2g_2 f_2 \qquad \text{(C.9)}
\end{aligned}$$

For both these adjoint representations, the reduction into its irreducible components has to be performed as in the *iia*) case discussed above.

iic) *The* $(2,4,2)_{s=0} \times (2,4,2)_{s=0}$ *case*

Let us express as $(x_1, x_2; \psi_1, \psi_2, \psi_3, \psi_4; g_1, g_2)$ the left multiplet and the right multiplet as $(y_1, y_2; \lambda_1, \lambda_2, \lambda_3, \lambda_4; f_1, f_2)$. At the leading (spin $s = 0$) order, their tensor product produces, as *symmetric* (see the discussion above at the point *ia*) of the present appendix) bilinear multiplets, a $(2,4,2)$ irrep lead by $x_1y_1 - x_2y_2$, $x_1y_2 + x_2y_1$, plus an adjoint $(1,4,6,4,1)$ reducible multiplet of $N = 4$ lead by $x_1y_1 + x_2y_2$. Explicitly, the bilinear multiplet $(2,4,2) \equiv (\tilde{x}_1, \tilde{x}_2; \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4; \tilde{g}_1, \tilde{g}_2)$ is given by

$$
\begin{aligned}
\tilde{x}_1 &= x_1y_1 - x_2y_2 \\
\tilde{x}_2 &= x_1y_2 + x_2y_1 \\
\tilde{\psi}_1 &= \psi_1y_1 + x_1\lambda_1 - \psi_2y_2 - x_2\lambda_2 \\
\tilde{\psi}_2 &= \psi_2y_1 + x_1\lambda_2 + \psi_1y_2 + x_2\lambda_1 \\
\tilde{\psi}_3 &= \psi_3y_1 + x_1\lambda_3 - \psi_4y_2 - x_2\lambda_4 \\
\tilde{\psi}_4 &= \psi_4y_1 + x_1\lambda_4 + \psi_3y_2 + x_2\lambda_3 \\
\tilde{g}_1 &= g_1y_1 - \psi_2\lambda_4 + \psi_4\lambda_2 + x_1f_1 - g_2y_2 + \psi_1\lambda_3 - \psi_3\lambda_1 - x_2f_2 \\
\tilde{g}_2 &= g_2y_1 - \psi_4\lambda_1 + \psi_1\lambda_4 + x_1f_2 + g_1y_2 - \psi_3\lambda_2 + \psi_2\lambda_3 + x_2f_1
\end{aligned}
\qquad \text{(C.10)}
$$

Its supersymmetric transformations are reported in Appendix **A** (third case of *v*)).

The adjoint multiplet is

$$
\begin{aligned}
\mathbf{1} &= x_1y_1 + x_2y_2, \\
Q_1\mathbf{1} &= \psi_2y_1 + x_1\lambda_2 - \psi_1y_2 - x_2\lambda_1 \\
Q_2\mathbf{1} &= \psi_4y_1 + x_1\lambda_4 - \psi_3y_2 - x_2\lambda_3 \\
Q_3\mathbf{1} &= \psi_3y_1 + x_1\lambda_3 + \psi_4y_2 + x_2\lambda_4 \\
Q_4\mathbf{1} &= \psi_1y_1 + x_1\lambda_1 + \psi_2y_2 + x_2\lambda_2 \\
Q_1Q_2\mathbf{1} &= \psi_2\lambda_4 - g_1y_1 - \psi_4\lambda_2 - x_1f_1 + \psi_1\lambda_3 - g_2y_2 - \psi_3\lambda_1 - x_2f_2 \\
Q_1Q_3\mathbf{1} &= \psi_2\lambda_3 + g_2y_1 - \psi_3\lambda_2 + x_1f_2 - \psi_1\lambda_4 - g_1y_2 + \psi_4\lambda_1 - x_2f_1 \\
Q_1Q_4\mathbf{1} &= 2\psi_2\lambda_1 - 2\psi_1\lambda_2 - \dot{x}_2y_1 - x_1\dot{y}_2 + \dot{x}_1y_2 + x_2\dot{y}_1 \\
Q_2Q_3\mathbf{1} &= 2\psi_4\lambda_3 - 2\psi_3\lambda_4 - \dot{x}_2y_1 - x_1\dot{y}_2 + \dot{x}_1y_2 + x_2\dot{y}_1 \\
Q_2Q_4\mathbf{1} &= \psi_4\lambda_1 - g_2y_1 - \psi_1\lambda_4 - x_1f_2 - \psi_3\lambda_2 + g_1y_2 + \psi_2\lambda_3 + x_2f_1 \\
Q_3Q_4\mathbf{1} &= \psi_3\lambda_1 - g_1y_1 - \psi_1\lambda_3 - x_1f_1 + \psi_4\lambda_2 - g_2y_2 - \psi_2\lambda_4 - x_2f_2 \\
Q_1Q_2Q_3\mathbf{1} &= \dot{\psi}_1y_1 + \dot{\psi}_2y_2 + x_1\dot{\lambda}_1 + x_2\dot{\lambda}_2 - \psi_1\dot{y}_1 - \psi_2\dot{y}_2 - \dot{x}_1\lambda_1 - \dot{x}_2\lambda_2 - \\
&\quad -2g_1\lambda_3 - 2g_2\lambda_4 - 2\psi_3f_1 - 2\psi_4f_2 \\
Q_1Q_2Q_4\mathbf{1} &= -\dot{\psi}_3y_1 - \dot{\psi}_4y_2 - x_1\dot{\lambda}_3 - x_2\dot{\lambda}_4 + \psi_3\dot{y}_1 + \psi_4\dot{y}_2 + \dot{x}_1\lambda_3 + \dot{x}_2\lambda_4 - \\
&\quad -2g_1\lambda_1 - 2g_2\lambda_2 - 2\psi_1f_1 - 2\psi_2f_2 \\
Q_1Q_3Q_4\mathbf{1} &= -\dot{\psi}_3y_2 + \dot{\psi}_4y_1 - x_2\dot{\lambda}_3 + x_1\dot{\lambda}_4 + \psi_3\dot{y}_2 - \psi_4\dot{y}_1 + \dot{x}_2\lambda_3 - \dot{x}_1\lambda_4 - \\
&\quad -2g_1\lambda_2 + 2g_2\lambda_1 - 2\psi_2f_1 + 2\psi_1f_2
\end{aligned}
$$

$$\begin{aligned}
Q_2Q_3Q_4\mathbf{1} &= -\dot{\psi}_2 y_1 + \dot{\psi}_1 y_2 - x_1\dot{\lambda}_2 + x_2\dot{\lambda}_1 - \psi_1\dot{y}_2 + \psi_2\dot{y}_1 - \dot{x}_2\lambda_1 + \dot{x}_1\lambda_2 - \\
&\quad -2g_1\lambda_4 + 2g_2\lambda_3 - 2\psi_4 f_1 + 2\psi_3 f_2 \\
Q_1Q_2Q_3Q_4\mathbf{1} &= -\ddot{x}_1 y_1 - \ddot{x}_2 y_2 - x_1\ddot{y}_1 - x_2\ddot{y}_2 + 2\dot{x}_1\dot{y}_1 + 2\dot{x}_2\dot{y}_2 + \\
&\quad 2\dot{\psi}_1\lambda_1 + 2\dot{\psi}_2\lambda_2 + 2\dot{\psi}_3\lambda_3 + 2\dot{\psi}_4\lambda_4 - 2\psi_1\dot{\lambda}_1 - 2\psi_2\dot{\lambda}_2 - \\
&\quad -2\psi_3\dot{\lambda}_3 - 2\psi_4\dot{\lambda}_4 + 4g_1 f_1 + 4g_2 f_2
\end{aligned} \tag{C.11}$$

The reduction of this adjoint multiplet into its irreps constituents can be carried on as in the case *iia*) examined above.

Appendix D

The $N = 2$ supersymmetric vacua fusion algebra

We present here the simplest non-trivial example of a supersymmetric vacuum fusion algebra, giving explicit results for the $N = 2$ extended supersymmetry.

According to the discussion of section **7**, two different $N = 2$ cases can be considered. At first we can label the $N = 2$ irreps as,

$$\begin{aligned}
[1] &\equiv (2,2) \\
[2] &\equiv (1,2,1)
\end{aligned} \tag{D.1}$$

without distinguishing their character (bosonic or fermionic).

In this case it can be easily proven that the two 2×2 fusion matrices N_1, N_2 are given by

$$\begin{aligned}
N_1 &= \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix} \\
N_2 &= \begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix}
\end{aligned} \tag{D.2}$$

The two fusion matrices indeed commute, as they should do, according to the property 3) of section **7**. It is worth mentioning the usefulness of the commutativity property of the fusion algebra to explicitly determine the fusion matrices. Already at this level in fact the fusion associated to the irrep decomposition of $[1] \times [2]$ can be written down, without explicitly carrying out the actual computation, just by the knowledge of the $[1] \times [1]$ and $[2] \times [2]$ fusions (which are easier to compute than the "mixed" $[1] \times [2]$ fusion) and of the $(2,2) \times (1,2,1) = (2,6,6,2)$ spin content of the reducible tensored representation. This r.h.s. admits in principle several decompositions into irreps. It can be easily proven, by checking the leading order terms, that the r.h.s. is decomposed according to $2 \cdot (1,2,1)_{s=0} + (2,4,2)_{s=\frac{1}{2}}$. The $(2,4,2)_{s=\frac{1}{2}}$ representation is reducible. It admits in principle two decompositions into irreps, either $(2,2)_{s=\frac{1}{2}} + (2,2)_{s=1}$ or $2 \cdot (1,2,1)_{s=\frac{1}{2}}$. The first case, however, would produce a set of fusion matrices N_1, N_2 which *do not commute*. The second case, leading to the two commuting matrices above, is verified.

If we discriminate between bosonic and fermionic representations, the inequivalent $N = 2$ irreps can be labeled as follows

$$\begin{aligned} [1] &\equiv (2,2)_{Bos} \\ [2] &\equiv (1,2,1)_{Bos} \\ [3] &\equiv (2,2)_{Fer} \\ [4] &\equiv (1,2,1)_{Fer} \end{aligned} \tag{D.3}$$

Under this assumption we obtain an $N = 2$ fusion algebra realized in terms of four 4×4, mutually commuting, matrices. They are explicitly given by

$$\begin{aligned} N_1 &= \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \\ N_2 &= \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix} \\ N_3 &= \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix} \\ N_4 &= N_2 \end{aligned} \tag{D.4}$$

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