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Higher-Derivative Wave Equations in (1+1)D: Exact Solutions with External Sources*

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Abstract

Two space-time dimensions exhibit the relevant property of Lorentz-invariant left-right-mover factorisation. This special result is crucial for the attainment of exact analytic solutions to higher-derivative wave equations such as $\Box^m \Phi = 0$ and $i\gamma^\mu \partial_\mu \Box^n \Psi = 0$, with $m \geq 2$ and $n \geq 1$ and Φ and Ψ being scalar and fermionic functions, respectively. The exact (classical) field solutions can be worked out for an arbitrary set of initial conditions that specify the funtions and their required time derivatives at t=0. In this work, we take into account external sources and the method previously developped is extended to include the effect of the sources. The whole idea underneath the approach consists in suitably shifting the source contributions to the initial value problem of the corresponding sourceless problem. To ilustrate the general procedure, we present explicit solutions to the equation $\Box^2 \Phi = J(t; x)$, where the source J is left arbitrary. We discuss the possibility that the external source be so chosen that it suppresses the unphysical modes present in Φ by virtue of the higher space and time derivatives that control the dynamics.

Key-words: Higher-derivative equations; Non-physical modes; General initial conditions.

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The main idea of the work is to present a method that extends the D'Alembert solution[1] for wave equations of the forms

$$\Box^m \Phi = J(t;x)$$

for an arbitrary set of initial conditions and a general external source.

A general method has been worked out and explicit solutions are known for m=2,3 and 4.

Main point: light-cone coordinates,

$$\xi \equiv x - t, \qquad \eta \equiv x + t;$$

left- and right-mover decomposition is crucial for the attainment of the explicit solutions.

$$\Box^2 \Phi = J(t;x)$$

with the initial conditions:

$$\Phi(0;x) = F(x), \quad \dot{\Phi}(0;x) = G(x), \quad \ddot{\Phi}(0;x) = H(x), \quad \dddot{\Phi}(0;x) = R(x).$$

and the sources,

$$\Phi(t;x) = \tilde{\Phi}(\xi;\eta)$$

and

$$J(t;x) = \tilde{J}(\xi;\eta).$$

The initial conditions are taken into account to define auxiliary functions, $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{R}$:

$$\mathcal{F}(x) \equiv F(x) + \int_0^x d\alpha \int_0^x d\beta \tilde{J}(\alpha;\beta);$$

$$\mathcal{G}(x) \equiv G(x) - \int_0^x d\beta \tilde{J}(x;\beta) + \int_0^x d\alpha \tilde{J}(\alpha;x);$$

$$\mathcal{H}(x) \equiv H(x) + \int_0^\xi d\alpha \frac{\partial \tilde{J}}{\partial \eta}(\alpha;\eta) + \int_0^\eta d\beta \frac{\partial \tilde{J}}{\partial \xi}(\xi;\beta),$$

taken at t=0,i.e., $\xi = \eta = x$.

$$\mathcal{R}(x) \equiv R(x) + \int_0^\eta d\beta \frac{\partial^2 \tilde{J}(\xi;\beta)}{\partial \xi^2} + \int_0^\xi d\alpha \frac{\partial^2 \tilde{J}(\alpha;\eta)}{\partial \eta^2} - \frac{\partial \tilde{J}(\xi;\eta)}{\partial \xi} - \frac{\partial \tilde{J}(\xi;\eta)}{\partial \eta},$$

also for t=0 at the R.H.S. of $\mathcal{R}$.

The whole idea is to introduce the effects of the source J at t=0 into the initial-condition funtions[2]; this yields the complete solution to the eq.(1):

$$\Phi = \Phi_{hom} + \Phi_J,$$

where Φ_{hom} is solution of

$$\Box^2 \Phi_{hom} = 0$$

and Φ_J is the particular solution associated to J.

Hence,

$$\begin{aligned}\Phi(t;x) &= \tilde{\Phi}(\xi;\eta) = -\int_0^{\xi} d\alpha \int_0^{\eta} d\beta \tilde{J}(\alpha;\beta) + \frac{1}{2}\mathcal{F}(\xi) + \frac{1}{2}\mathcal{F}(\eta) - \frac{1}{8}(\xi-\eta)\mathcal{F}'(\xi) + \\ &+ \frac{1}{8}(\xi-\eta)\mathcal{F}'(\eta) + \frac{1}{8}(\xi-\eta)\mathcal{G}(\xi) + \frac{1}{8}(\xi-\eta)\mathcal{G}(\eta) + \frac{3}{4}\int_{\xi}^{\eta} dy\mathcal{G}(y) - \\ &- \frac{1}{8}(\xi-\eta)\int_{\xi}^{\eta} dy\mathcal{H}(y)y + \frac{1}{8}\eta\int_0^{\xi} dy \int_0^{y} dz\mathcal{R}(z) - \frac{1}{8}\xi\int_0^{\eta} dy\int_0^{y} dz\mathcal{R}(z) + \\ &+ \frac{1}{8}\int_{\xi}^{\eta} dy \int_0^{y} dz\mathcal{R}(z)z, \qquad (1)\end{aligned}$$

where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{R}$ are given above.

This explicit solution is just the illustration of how the method applies to any power m. The whole idea underneath the treatment is to solve the homogeneous equation, $\Box^m$, with the $t = 0 - source$ effects included into the inital conditions.

Next, we present plots of solutions for different situations. The 3D-case is under study and numerical solutions have already been found by means of finite differences. The higher-derivative Diracs equation in 2D has also been solved for n=1 and 2 with general initial conditions and arbitrary sources.

The external source contributes additional terms to the field solution and these contributions may be arranged, by means of some special choice for J, so that the linear-raising time behaviour be suppressed. This, at the classical level, eliminates the divergent piece of Φ and this ensures a good interpretation if one wishes to propose a particle interpretation for Φ at the second-quantized level. This would mean that the model could describe some physically acceptable excitations whenever the Φ-field is coupled to this external source especially chosen to kill off the linear dependence in t.

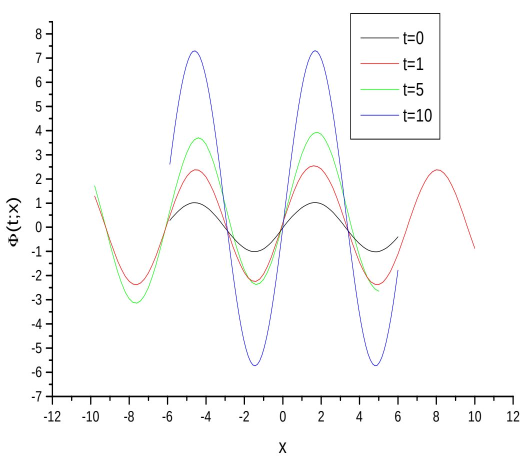

Figure 1: Φ-plots for periodic initial conditions

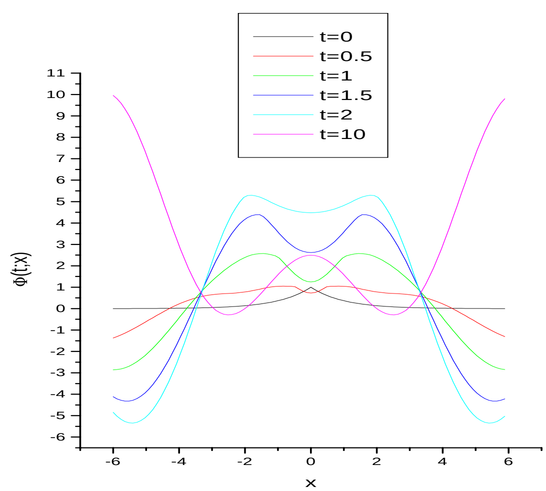

Figure 2: Φ-plots for exponentially decreasing initial conditions

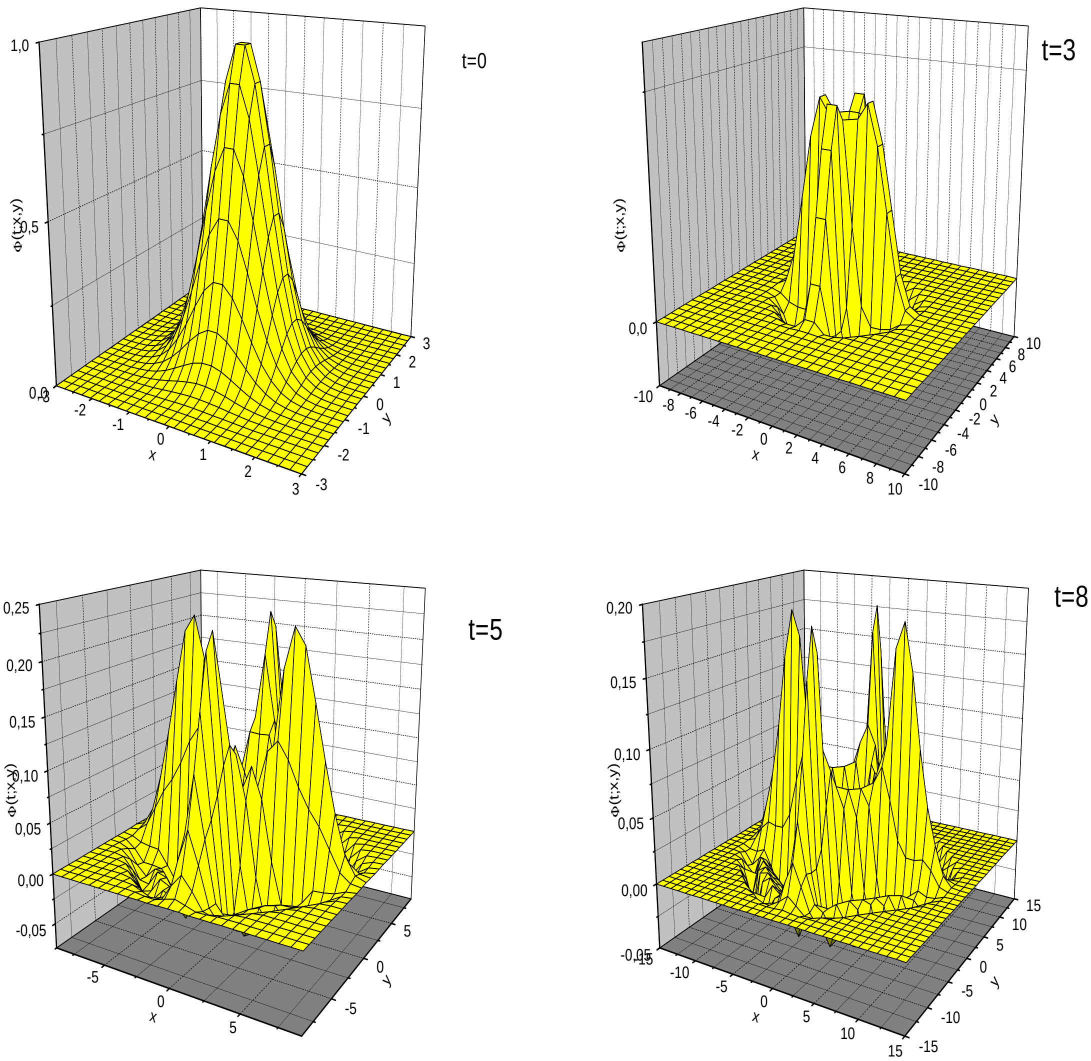

Figure 3:Time evolution of Φ in the 2D-case.

References

[1] R. Courant and D. Hilbert, Methods of Mathematical Physics - vol. 2, Interscience Publishers, New York, 1962.

[2] R. S. Simões, Soluções de D'Alembert Estendidas para Equações de Onda com Derivadas de Ordens Superiores - Tese de Mestrado, CBPF, abril/2001.